# THE HORMONES

PHYSIOLOGY, CHEMISTRY AND APPLICATIONS

VOLUME III

# THE HORMONES

## Physiology, Chemistry and Applications

*Edited by*

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VOLUME III



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## Preface to Volume III

Seven years have passed since the appearance of Volume I of The Hormones At that time it was suggested that the subject might have completed its first unfolding—virtually all the major hormones especially in the mammals and man having been discovered bioassayed and at least partially identified chemically The subsequent period it was felt would be one first of consolidation, then of increasing emphasis upon physiological problems and the inner mechanism of the action of hormones This prognostication, it now appears was true only in part It was true for the hormones of the thyroid anterior pituitary and gonads which have always occupied so central a position in research and application To a lesser extent it was true in other areas But as regards the hormones of the invertebrates and insects and in the realms of the posterior pituitary and the nervous systems the appearance of new facts and ideas has opened new horizons The concept of nerve as secreting and transmitting hormones as well as the long awaited isolation of the pure posterior pituitary hormones themselves are both notable Even in the plants the discovery that auxin plays a role in flowering brings what was thought to be a growth hormone into relationship with the reproduction process Not less to be regarded as new departures are the discoveries concerning the pituitary adrenal axis and the widespread medical applications just touched on in Volume II which have followed

The expected emphasis upon mechanism of action on the contrary has not yet developed to the degree expected and the major discoveries in this direction are still probably for the future although there have been interesting and suggestive developments In the case of the animal hormones particularly intensive investigations of their metabolic fate have been considered necessary preludes to studies of their specific roles in biochemical reactions in several instances it has been indicated that a metabolite of the secretory product rather than the secreted substance itself may be responsible for the presumed specific effects Furthermore with the broadening of our knowledge of hormone interaction the need for distinction between primary and secondary effects of a given hormone has become more evident For example the adrenocorticomimetic effects of estrogens are cited in the text the effects are presumably exerted through stimulating the secretion of ACTH from the anterior pituitary Another complication

which has emerged with clarity is that mammals differ more widely in their physiology than was formerly thought For this reason, any attempt to study mechanisms of action as problems in general physiology must recognize species differences in response to hormones, and the past septennium has unearthed additional remarkable hormonal responses occurring in one species and completely absent in another

For all these reasons the present volume is not merely a supplement to Volumes I and II that is, it is not merely a chronicle of recent experiments, extending in detail what has already been laid down in outline Instead some parts of the volume supplant their predecessors and certainly the majority of the chapters at least modify or recast the picture which had been presented While part of its information is certainly supplementary an important part must be regarded as revision or perhaps as reassessment

The partition of subject matter between the authors and the planned content of the individual chapters has been somewhat revised from that of the previous volumes to allow of more unification and changed emphasis Some of the authors are those who already reviewed their subjects in Volumes I and II, but the regrouping of the subject matter as well as death retirement, or preoccupation with other affairs has necessitated a number of changes In any event the chapters reflect individual viewpoints as much as ever It is hoped that the elements both of variety and of uniformity will combine to make the book a useful tool in the difficult task of integrating modern biology

G PINCUS
K V THIMANN

*June 1955*

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# Chapter I

## Plant Growth Hormones

By Kenneth V. Thimann and A. Carl Leopold

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## I Introduction

The progress of any field of science is highly irregular some parts may make a major break through whereas other parts—apparently no more complex—seem blocked by some difficulty of procedure or of concepts so that ideas and experiments proceed in spirals without advancing The study of plant hormones has shown the expected irregularity of advance during the six years since the appearance of Volume I of *The Hormones* Some aspects like that of the inhibition of the growth of buds have not undergone any really fundamental change whereas the role of auxin in flowering and the mechanism of auxin action in growth have developed with great rapidity Recent evidence for an *in vitro* action of auxin and

But $C$ is also equal to $AD$

Therefore each of the straight lines $AE$, $C$ is equal to $AD$,

so that $AE$ is also equal to $C$ [C N 1]

Therefore given the two straight lines $AB$, $C$, from $AB$ the greater $AE$ has been cut off equal to $C$ the less

(Being) what it was required to do

PROPOSITION 4

*If two triangles have the two sides equal to two sides respectively and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend*

Let $ABC$, $DEF$ be two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$ $DF$ respectively, namely $AB$ to $DE$ and $AC$ to $DF$, and the angle $BAC$ equal to the angle $EDF$

I say that the base $BC$ is also equal to the base $EF$ the triangle $ABC$ will be equal to the triangle $DEF$, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend that is, the angle $ABC$ to the angle $DEF$ and the angle $ACB$ to the angle $DFE$

For, if the triangle $ABC$ be applied to the triangle $DEF$

and if the point $A$ be placed on the point $D$

and the straight line $AB$ on $DE$

then the point $B$ will also coincide with $E$ because $AB$ is equal to $DE$

Again, $AB$ coinciding with $DE$

the straight line $AC$ will also coincide with $DF$ because the angle $BAC$ is equal to the angle $EDF$

hence the point $C$ will also coincide with the point $F$, because $AC$ is again equal to $DF$

But $B$ also coincided with $E$

hence the base $BC$ will coincide with the base $EF$

[For if when $B$ coincides with $E$ and $C$ with $F$ the base $BC$ does not coincide with the base $EF$ two straight lines will enclose a space which is impossible

Therefore the base $BC$ will coincide with $EF$] and will be equal to it [C N 4]

Thus the whole triangle $ABC$ will coincide with the whole triangle $DEF$, and will be equal to it

And the remaining angles will also coincide with the remaining angles and will be equal to them

the angle $ABC$ to the angle $DEF$

and the angle $ACB$ to the angle $DFE$

Therefore etc

(Being) what it was required to prove

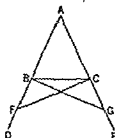
PROPOSITION 5

*In isosceles triangles the angles at the base are equal to one another and if the equal straight lines be produced further the angles under the base will be equal to one another*

Let $ABC$ be an isosceles triangle having the side $AB$ equal to the side $AC$,

and let the straight lines $BD$, $CE$ be produced further in a straight line with $AB$, $AC$ [Post 2]

I say that the angle $ABC$ is equal to the angle $ACB$ and the angle $CBD$ to the angle $BCE$




Let a point $F$ be taken at random on $BD$,
from $AE$ the greater let $AG$ be cut off equal to $AF$ the less [I 3]
and let the straight lines $FC$, $GB$ be joined [Post 1]

Then since $AF$ is equal to $AG$ and $AB$ to $AC$,
the two sides $FA$, $AC$ are equal to the two sides $GA$ $AB$ respectively,
and they contain a common angle, the angle $FAG$
Therefore the base $FC$ is equal to the base $GB$
and the triangle $AFC$ is equal to the triangle $AGB$
and the remaining angles will be equal to the remaining angles respectively namely those which the equal sides subtend
that is the angle $ACF$ to the angle $ABG$
and the angle $AFC$ to the angle $AGB$ [I 4]

And, since the whole $AF$ is equal to the whole $AG$
*and in these $AB$ is equal to $AC$*
the remainder $BF$ is equal to the remainder $CG$

But $FC$ was also proved equal to $GB$
therefore the two sides $BF$, $FC$ are equal to the two sides $CG$ $GB$ respectively,
and the angle $BFC$ is equal to the angle $CGB$
while the base $BC$ is common to them
therefore the triangle $BFC$ is also equal to the triangle $CGB$ and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend
therefore the angle $FBC$ is equal to the angle $GCB$,
and the angle $BCF$ to the angle $CBG$

*Accordingly since the whole angle $ABG$ was proved equal to the angle $ACF$*
and in these the angle $CBG$ is equal to the angle $BCF$
the remaining angle $ABC$ is equal to the remaining angle $ACB$
and they are at the base of the triangle $ABC$
But the angle $FBC$ was also proved equal to the angle $GCB$
and they are under the base

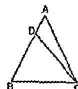
Therefore etc Q E D

## Proposition 6

*If in a triangle two angles be equal to one another the sides which subtend the equal angles will also be equal to one another*

Let $ABC$ be a triangle having the angle $ABC$ equal to the angle $ACB$

I say that the side $AB$ is also equal to the side $AC$




*For if $AB$ is unequal to $AC$ one of them is greater*

Let $AB$ be greater and from $AB$ the greater let $DB$ be cut off equal to $AC$ the less
let $DC$ be joined

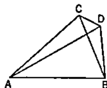
Then since $DB$ is equal to $AC$
and $BC$ is common,

the two sides $DB$ $BC$ are equal to the two sides $AC$, $CB$ respectively,
and the angle $DBC$ is equal to the angle $ACB$,
therefore the base $DC$ is equal to the base $AB$,
and the triangle $DBC$ will be equal to the triangle $ACB$
the less to the greater
which is absurd
Therefore $AB$ is not unequal to $AC$,
it is therefore equal to it
Therefore etc Q E D

## Proposition 7

*Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities) and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it*

For, if possible given two straight lines $AC$, $CB$ constructed on the straight line $AB$ and meeting at the point $C$, let two other straight lines $AD$ $DB$ be constructed on the same straight line $AB$, on the same side of it meeting in another point $D$ and equal to the former two respectively namely each to that which has the same extremity with it so that $CA$ is equal to $DA$ which has the same extremity $A$ with it and $CB$ to $DB$ which has the same extremity $B$ with it, and let $CD$ be joined

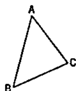


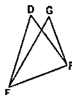

Then since $AC$ is equal to $AD$,
the angle $ACD$ is also equal to the angle $ADC$, [I 5]
therefore the angle $ADC$ is greater than the angle $DCB$
therefore the angle $CDB$ is much greater than the angle $DCB$
Again since $CB$ is equal to $DB$
the angle $CDB$ is also equal to the angle $DCB$
But it was also proved much greater than it
which is impossible
Therefore etc Q E D

## Proposition 8

*If two triangles have the two sides equal to two sides respectively and have also the base equal to the base they will also have the angles equal which are contained by the equal straight lines*

Let $ABC$, $DEF$ be two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$ $DF$ respectively namely $AB$ to $DE$, and $AC$ to $DF$ and let them have the base $BC$ equal to the base $EF$




I say that the angle $BAC$ is also equal to the angle $EDF$

For if the triangle $ABC$ be applied to the triangle $DEF$ and if the point $B$ be placed on the point $E$ and the straight line $BC$ on $EF$,
the point $C$ will also coincide with $F$,

because $BC$ is equal to $EF$

Then $BC$ coinciding with $EF$,

$BA$ $AC$ will also coincide with $ED$ $DF$,

for if the base $BC$ coincides with the base $EF$ and the sides $BA$, $AC$ do not coincide with $ED$ $DF$ but fall beside them as $EG$ $GF$

then given two straight lines constructed on a straight line (from its extremities) and meeting in a point there will have been constructed on the same straight line (from its extremities) and on the same side of it two other straight lines meeting in another point and equal to the former two respectively namely each to that which has the same extremity with it

But they cannot be so constructed [I 7]

Therefore it is not possible that if the base $BC$ be applied to the base $EF$, the sides $BA$, $AC$ should not coincide with $ED$ $DF$

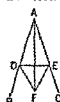
*they will therefore coincide,*

so that the angle $BAC$ will also coincide with the angle $EDF$ and will be equal to it

If therefore etc Q E D

### Proposition 9

*To bisect a given rectilineal angle*




Let the angle $BAC$ be the given rectilineal angle

Thus it is required to bisect it

Let a point $D$ be taken at random on $AB$

let $AE$ be cut off from $AC$ equal to $AD$ [I 3]

let $DE$ be joined and on $DE$ let the equilateral triangle $DEF$ be constructed

let $AF$ be joined

I say that the angle $BAC$ has been bisected by the straight line $AF$

For, since $AD$ is equal to $AE$

and $AF$ is common

the two sides $DA$ $AF$ are equal to the two sides $EA$ $AF$ respectively

And the base $DF$ is equal to the base $EF$

therefore the angle $DAF$ is equal to the angle $EAF$ [I 8]

Therefore the given rectilineal angle $BAC$ has been bisected by the straight line $AF$ Q E F

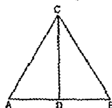
### Proposition 10

*To bisect a given finite straight line*

Let $AB$ be the given finite straight line

Thus it is required to bisect the finite straight line $AB$

Let the equilateral triangle $ABC$ be constructed on it [I 1]




and let the angle $ACB$ be bisected by the straight line $CD$ [I 9]

I say that the straight line $AB$ has been bisected at the point $D$

For since $AC$ is equal to $CB$

and $CD$ is common

the two sides $AC$ $CD$ are equal to the two sides $BC$, $CD$ respectively

and the angle $ACD$ is equal to the angle $BCD$

therefore the base $AD$ is equal to the base $BD$ [I 4]

Therefore the given finite straight line $AB$ has been bisected at $D$ Q E F

### Proposition 11

*To draw a straight line at right angles to a given straight line from a given point on it*

Let $AB$ be the given straight line and $C$ the given point on it

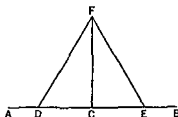
Thus it is required to draw from the point $C$ a straight line at right angles to the straight line $AB$

Let a point $D$ be taken at random on $AC$,

let $CE$ be made equal to $CD$, [I 3]

on $DE$ let the equilateral triangle $FDE$ be constructed, [I 1]

and let $FC$ be joined




I say that the straight line $FC$ has been drawn at right angles to the given straight line $AB$ from $C$ the given point on it

For, since $DC$ is equal to $CE$,

and $CF$ is common,

the two sides $DC$ $CF$ are equal to the two sides $EC$ $CF$ respectively,

and the base $DF$ is equal to the base $FE$,

therefore the angle $DCF$ is equal to the angle $ECF$ [I 8]

and they are adjacent angles

But when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right [Def 10]

therefore each of the angles $DCF$, $FCE$ is right

Therefore the straight line $CF$ has been drawn at right angles to the given straight line $AB$ from the given point $C$ on it Q E F

### Proposition 12

*To a given infinite straight line, from a given point which is not on it to draw a perpendicular straight line*

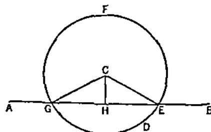
Let $AB$ be the given infinite straight line and $C$ the given point which is not on it

thus it is required to draw to the given infinite straight line $AB$, from the given point $C$ which is not on it, a perpendicular straight line

For let a point $D$ be taken at random on the other side of the straight line $AB$ and with centre $C$ and distance $CD$ let the circle $EFG$ be described [Post 3]

let the straight line $EG$ be bisected at $H$ [I 10]

and let the straight lines $CG$, $CH$ $CE$ be joined [Post 1]




I say that $CH$ has been drawn perpendicular to the given infinite straight line $AB$ from the given point $C$ which is not on it

For since $GH$ is equal to $HE$

and $HC$ is common

the two sides $GH$, $HC$ are equal to the two sides $EH$ $HC$ respectively
and the base $CG$ is equal to the base $CE$
therefore the angle $CHG$ is equal to the angle $EHC$ [I 8]
And they are adjacent angles

But when a straight line set up on a straight line makes the adjacent angles equal to one another each of the equal angles is right and the straight line standing on the other is called a perpendicular to that on which it stands [Def 10]

Therefore $CH$ has been drawn perpendicular to the given infinite straight line $AB$ from the given point $C$ which is not on it Q E F

## Proposition 13

*If a straight line set up on a straight line make angles it will make either two right angles or angles equal to two right angles*

For let any straight line $AB$ set up on the straight line $CD$ make the angles $CBA$ $ABD$

I say that the angles $CBA$ $ABD$ are either two rights angles or equal to two right angles

Now if the angle $CBA$ is equal to the angle $ABD$ they are two right angles [Def 10]

But if not let $BE$ be drawn from the point $B$ at right angles to $CD$ [I 11]

therefore the angles $CBE$ $EBD$ are two right angles

Then since the angle $CBE$ is equal to the two angles $CBA$ $ABE$,
let the angle $EBD$ be added to each
therefore the angles $CBE$ $EBD$ are equal to the three angles $CBA$ $ABE$, $EBD$ [C N 2]

Again since the angle $DBA$ is equal to the two angles $DBE$ $EBA$
let the angle $ABC$ be added to each
therefore the angles $DBA$ $ABC$ are equal to the three angles $DBE$ $EBA$ $ABC$ [C N 2]

But the angles $CBE$ $EBD$ were also proved equal to the same three angles and things which are equal to the same thing are also equal to one another [C N 1]

therefore the angles $CBE$ $EBD$ are also equal to the angles $DBA$ $ABC$
But the angles $CBE$ $EBD$ are two right angles
therefore the angles $DBA$ $ABC$ are also equal to two right angles
Therefore etc Q E D

## Proposition 14

*If with any straight line and at a point on it two straight lines not lying on the same side make the adjacent angles equal to two right angles the two straight lines will be in a straight line with one another*

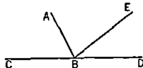
For with any straight line $AB$ and at the point $B$ on it let the two straight lines $BC$ $BD$ not lying on the same side make the adjacent angles $ABC$ $ABD$ equal to two right angles

I say that $BD$ is in a straight line with $CB$

For if $BD$ is not in a straight line with $BC$, let $BE$ be in a straight line with $CB$

Then since the straight line $AB$ stands on the straight line $CBE$,
the angles $ABC$, $ABE$ are equal to two right angles [I 13]

But the angles $ABC$ $ABD$ are also equal to two right angles




therefore the angles $CBA$, $ABE$ are equal to the angles $CBA$, $ABD$ [Post 4 and C N 1]

Let the angle $CBA$ be subtracted from each,

therefore the remaining angle $ABE$ is equal to the remaining angle $ABD$ [C N 3]

the less to the greater which is impossible

Therefore $BE$ is not in a straight line with $CB$

Similarly we can prove that neither is any other straight line except $BD$

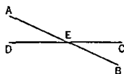
Therefore $CB$ is in a straight line with $BD$

Therefore etc Q E D

PROPOSITION 15

*If two straight lines cut one another, they make the vertical angles equal to one another*

For let the straight lines $AB$ $CD$ cut one another at the point $E$,




I say that the angle $AEC$ is equal to the angle $DEB$,
and the angle $CEB$ to the angle $AED$

For since the straight line $AE$ stands on the straight line $CD$, making the angles $CEA$, $AED$,
the angles $CEA$ $AED$ are equal to two right angles [I 13]

Again since the straight line $DE$ stands on the straight line $AB$, making the angles $AED$ $DEB$
the angles $AED$ $DEB$ are equal to two right angles [I 13]

But the angles $CEA$ $AED$ were also proved equal to two right angles,
therefore the angles $CEA$, $AED$ are equal to the angles $AED$ $DEB$ [Post 4 and C N 1]

Let the angle $AED$ be subtracted from each
therefore the remaining angle $CEA$ is equal to the remaining angle $BED$ [C N 3]

Similarly it can be proved that the angles $CEB$ $DEA$ are also equal

Therefore etc Q E D

[PORISM From this it is manifest that if two straight lines cut one another they will make the angles at the point of section equal to four right angles ]

PROPOSITION 16

*In any triangle if one of the sides be produced the exterior angle is greater than either of the interior and opposite angles*

Let $ABC$ be a triangle and let one side of it $BC$ be produced to $D$,

I say that the exterior angle $ACD$ is greater than either of the interior and opposite angles $CBA$ $BAC$

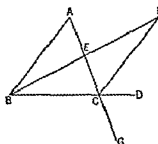
Let $AC$ be bisected at $E$ [I 10] and let $BE$ be joined and produced in a straight line to $F$

let $EF$ be made equal to $BE$ [I 3]

let $FC$ be joined [Post 1] and let $AC$ be drawn through to $G$ [Post 2]

Then since $AE$ is equal to $EC$ and $BE$ to $EF$,

the two sides $AE$ $EB$ are equal to the two sides $CE$ $EF$ respectively




and the angle $AEB$ is equal to the angle $FEC$ for they are vertical angles [I 15]

Therefore the base $AB$ is equal to the base $FC$ and the triangle $ABE$ is equal to the triangle $CFE$ and the remaining angles are equal to the remaining angles respectively namely, those which the equal sides subtend [I 4]

therefore the angle $BAE$ is equal to the angle $ECF$

But the angle $ECD$ is greater than the angle $ECF$ [C N 5]

therefore the angle $ACD$ is greater than the angle $BAE$

Similarly also if $BC$ be bisected the angle $BCG$ that is the angle $ACD$ [I 15] can be proved greater than the angle $ABC$ as well

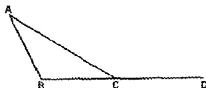
Therefore etc Q E D

## Proposition 17

*In any triangle two angles taken together in any manner are less than two right angles*

Let $ABC$ be a triangle

I say that two angles of the triangle $ABC$ taken together in any manner are less than two right angles




For let $BC$ be produced to $D$ [Post 2]

Then since the angle $ACD$ is an exterior angle of the triangle $ABC$ it is greater than the interior and opposite angle $ABC$ [I 16]

Let the angle $ACB$ be added to each

therefore the angles $ACD$ $ACB$ are greater than the angles $ABC$, $BCA$

But the angles $ACD$ $ACB$ are equal to two right angles [I 13]

Therefore the angles $ABC$ $BCA$ are less than two right angles

Similarly we can prove that the angles $BAC$ $ACB$ are also less than two right angles and so are the angles $CAB$ $ABC$ as well

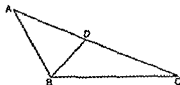
Therefore etc Q E D

## Proposition 18

*In any triangle the greater side subtends the greater angle*

For let $ABC$ be a triangle having the side $AC$ greater than $AB$

I say that the angle $ABC$ is also greater than the angle $BCA$




For since $AC$ is greater than $AB$ let $AD$ be made equal to $AB$ [I 3] and let $BD$ be joined

Then since the angle $ADB$ is an exterior angle of the triangle $BCD$ it is greater than the interior and opposite angle $DCB$ [I 16]

But the angle $ADB$ is equal to the angle $ABD$ since the side $AB$ is equal to $AD$

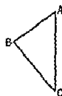
therefore the angle $ABD$ is also greater than the angle $ACB$
therefore the angle $ABC$ is much greater than the angle $ACB$
Therefore etc Q E D

PROPOSITION 19

*In any triangle the greater angle is subtended by the greater side*

Let $ABC$ be a triangle having the angle $ABC$ greater than the angle $BCA$,
I say that the side $AC$ is also greater than the side $AB$

For, if not, $AC$ is either equal to $AB$ or less




Now $AC$ is not equal to $AB$
for then the angle $ABC$ would also have been equal to the angle $ACB$ [I 5]

but it is not,
therefore $AC$ is not equal to $AB$

Neither is $AC$ less than $AB$,
for then the angle $ABC$ would also have been less than the angle $ACB$ [I 18]

but it is not
therefore $AC$ is not less than $AB$

And it was proved that it is not equal either
Therefore $AC$ is greater than $AB$
Therefore etc Q E D

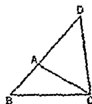
PROPOSITION 20

*In any triangle two sides taken together in any manner are greater than the remaining one*

For let $ABC$ be a triangle
I say that in the triangle $ABC$ two sides taken together in any manner are greater than the remaining one namely

$BA$ $AC$ greater than $BC$
$AB$, $BC$ greater than $AC$,
$BC$ $CA$ greater than $AB$

For let $BA$ be drawn through to the point $D$, let $DA$ be made equal to $CA$, and let $DC$ be joined




Then since $DA$ is equal to $AC$
the angle $ADC$ is also equal to the angle $ACD$ [I 5]
therefore the angle $BCD$ is greater than the angle $ADC$ [C N 5]

And since $DCB$ is a triangle having the angle $BCD$ greater than the angle $BDC$
and the greater angle is subtended by the greater side [I 19]
therefore $DB$ is greater than $BC$

But $DA$ is equal to $AC$
therefore $BA$ $AC$ are greater than $BC$

Similarly we can prove that $AB$ $BC$ are also greater than $CA$ and $BC$ $CA$ than $AB$
Therefore etc Q E D

PROPOSITION 21

*If on one of the sides of a triangle from its extremities there be constructed two*

*straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle*

On $BC$, one of the sides of the triangle $ABC$ from its extremities $B$ $C$, let the two straight lines $BD$ $DC$ be constructed meeting within the triangle

I say that $BD$ $DC$ are less than the remaining two sides of the triangle $BA$, $AC$, but contain an angle $BDC$ greater than the angle $BAC$

For let $BD$ be drawn through to $E$

Then, since in any triangle two sides are greater than the remaining one [I 20]

therefore in the triangle $ABE$, the two sides $AB$, $AE$ are greater than $BE$

Let $EC$ be added to each

therefore $BA$ $AC$ are greater than $BE$ $EC$

Again since in the triangle $CED$

the two sides $CE$ $ED$ are greater than $CD$

let $DB$ be added to each

therefore $CE$, $EB$ are greater than $CD$ $DB$

But $BA$ $AC$ were proved greater than $BE$ $EC$

therefore $BA$ $AC$ are much greater than $BD$ $DC$

Again since in any triangle the exterior angle is greater than the interior and opposite angle [I 16]

therefore in the triangle $CDE$,

the exterior angle $BDC$ is greater than the angle $CED$

For the same reason moreover in the triangle $ABE$ also

the exterior angle $CEB$ is greater than the angle $BAC$

But the angle $BDC$ was proved greater than the angle $CEB$

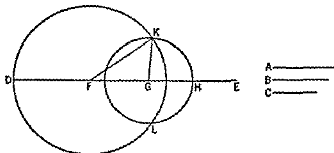
therefore the angle $BDC$ is much greater than the angle $BAC$

Therefore etc Q E D

## Proposition 22

*Out of three straight lines which are equal to three given straight lines to construct a triangle thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one* [I 20]

Let the three given straight lines be $A$ $B$ $C$ and of these let two taken together in any manner be greater than the remaining one




namely $A$ $B$ greater than $C$

$A$ $C$ greater than $B$

and $B$ $C$ greater than $A$

thus it is required to construct a triangle out of straight lines equal to $A$, $B$, $C$

Let there be set out a straight line $DE$, terminated at $D$ but of infinite length in the direction of $E$

and let $DF$ be made equal to $A$ $FG$ equal to $B$, and $GH$ equal to C [I 3]

With centre $F$ and distance $FD$ let the circle $DKL$ be described

again with centre $G$ and distance $GH$ let the circle $KLH$ be described

and let $KF$, $KG$ be joined

*I say that the triangle $KFG$ has been constructed out of three straight lines equal to $A$ $B$ $C$*

For since the point $F$ is the centre of the circle $DKL$

$FD$ is equal to $FK$

But $FD$ is equal to $A$

therefore $KF$ is also equal to $A$

Again since the point $G$ is the centre of the circle $LKH$,

$GH$ is equal to $GK$

But $GH$ is equal to $C$

therefore $KG$ is also equal to $C$

And $FG$ is also equal to $B$,

therefore the three straight lines $KF$ $FG$, $GK$ are equal to the three straight lines $A$ $B$, $C$

Therefore out of the three straight lines $KF$, $FG$, $GK$ which are equal to the three given straight lines $A$ $B$, $C$, the triangle $KFG$ has been constructed

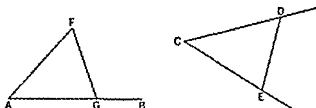
Q E F

PROPOSITION 23

*On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle*

Let $AB$ be the given straight line $A$ the point on it and the angle $DCE$ the given rectilineal angle

thus it is required to construct on the given straight line $AB$ and at the point $A$ on it, a rectilineal angle equal to the given rectilineal angle $DCE$




On the straight lines $CD$, $CE$ respectively let the points $D$ $E$ be taken at random,

let $DE$ be joined

and out of three straight lines which are equal to the three straight lines $CD$, $DE$, $CE$ let the triangle $AFG$ be constructed in such a way that $CD$ is equal to $AF$, $CE$ to $AG$ and further $DE$ to $FG$ [I 22]

Then, since the two sides $DC$, $CE$ are equal to the two sides $FA$, $AG$ respectively,

and the base $DE$ is equal to the base $FG$

the angle $DCE$ is equal to the angle $FAG$ [I 8]

Therefore on the given straight line $AB$ and at the point $A$ on it the rectilineal angle $FAG$ has been constructed equal to the given rectilineal angle $DCE$

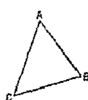
Q E F

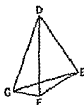
PROPOSITION 24

*If two triangles have the two sides equal to two sides respectively but have the one of the angles contained by the equal straight lines greater than the other they will also have the base greater than the base*

Let $ABC$, $DEF$ be two triangles having the two sides $AB$, $AC$ equal to the two sides $DE$, $DF$ respectively namely $AB$ to $DE$, and $AC$ to $DF$, and let the angle at $A$ be greater than the angle at $D$

I say that the base $BC$ is also greater than the base $EF$






For since the angle $BAC$ is greater than the angle $EDF$ let there be constructed on the straight line $DE$, and at the point $D$ on it, the angle $EDG$ equal to the angle $BAC$, [I 23]

let $DG$ be made equal to either of the two straight lines $AC$, $DF$, and let $EG$ $FG$ be joined

Then since $AB$ is equal to $DE$ and $AC$ to $DG$

the two sides $BA$ $AC$ are equal to the two sides $ED$ $DG$, respectively

and the angle $BAC$ is equal to the angle $EDG$

therefore the base $BC$ is equal to the base $EG$ [I 4]

Again since $DF$ is equal to $DG$

the angle $DGF$ is also equal to the angle $DFG$, [I 5]

therefore the angle $DFG$ is greater than the angle $EGF$

Therefore the angle $EFG$ is much greater than the angle $EGF$

And since $EFG$ is a triangle having the angle $EFG$ greater than the angle $EGF$

and the greater angle is subtended by the greater side [I 19]

the side $EG$ is also greater than $EF$

But $EG$ is equal to $BC$

Therefore $BC$ is also greater than $EF$

*Therefore etc* Q E D

PROPOSITION 25

*If two triangles have the two sides equal to two sides respectively but have the base greater than the base they will also have the one of the angles contained by the equal straight lines greater than the other*

Let $ABC$ $DEF$ be two triangles having the two sides $AB$ $AC$ equal to the two sides $DE$ $DF$ respectively namely $AB$ to $DE$ and $AC$ to $DF$ and let the base $BC$ be greater than the base $EF$

I say that the angle $BAC$ is also greater than the angle $EDF$

For if not it is either equal to it or less

Now the angle $BAC$ is not equal to the angle $EDF$ for then the base $BC$ would also have been equal to the base $EF$ [I 4]

but it is not,

therefore the angle $BAC$ is not equal to the angle $EDF$

Neither again is the angle $BAC$ less than the angle $EDF$ for then the base $BC$ would also have been less than the base $EF$, [I 24]




but it is not therefore the angle $BAC$ is not less than the angle $EDF$
But it was proved that it is not equal either,
therefore the angle $BAC$ is greater than the angle $EDF$
Therefore etc Q E D

### Proposition 26

*If two triangles have the two angles equal to two angles respectively and one side equal to one side namely either the side adjoining the equal angles or that subtending one of the equal angles they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle*

Let $ABC$ $DEF$ be two triangles having the two angles $ABC$ $BCA$ equal to the two angles $DEF$ $EFD$ respectively namely the angle $ABC$ to the angle $DEF$ and the angle $BCA$ to the angle $EFD$, and let them also have one side equal to one side first that adjoining the equal angles namely $BC$ to $EF$

I say that they will also have the remaining sides equal to the remaining sides respectively namely $AB$ to $DE$ and $AC$ to $DF$ and the remaining angle to the remaining angle namely the angle $BAC$ to the angle $EDF$




For if $AB$ is unequal to $DE$ one of them is greater
Let $AB$ be greater and let $BG$ be made equal to $DE$ and let $GC$ be joined
Then since $BG$ is equal to $DE$ and $BC$ to $EF$
the two sides $GB$ $BC$ are equal to the two sides $DE$ $EF$ respectively,
and the angle $GBC$ is equal to the angle $DEF$
therefore the base $GC$ is equal to the base $DF$
and the triangle $GBC$ is equal to the triangle $DEF$
and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]
therefore the angle $GCB$ is equal to the angle $DFE$
But the angle $DFE$ is by hypothesis equal to the angle $BCA$
therefore the angle $BCG$ is equal to the angle $BCA$
the less to the greater which is impossible
Therefore $AB$ is not unequal to $DE$
and is therefore equal to it
But $BC$ is also equal to $EF$

therefore the two sides $AB$, $BC$ are equal to the two sides $DE$, $EF$ respectively,

and the angle $ABC$ is equal to the angle $DEF$,
therefore the base $AC$ is equal to the base $DF$,
and the remaining angle $BAC$ is equal to the remaining angle $EDF$ [I 4]

Again, let sides subtending equal angles be equal as $AB$ to $DE$,

I say again that the remaining sides will be equal to the remaining sides namely $AC$ to $DF$ and $BC$ to $EF$, and further the remaining angle $BAC$ is equal to the remaining angle $EDF$

For, if $BC$ is unequal to $EF$, one of them is greater

Let $BC$ be greater if possible and let $BH$ be made equal to $EF$, let $AH$ be joined

*Then since $BH$ is equal to $EF$, and $AB$ to $DE$*
the two sides $AB$ $BH$ are equal to the two sides $DE$ $EF$ respectively, and they contain equal angles

therefore the base $AH$ is equal to the base $DF$
and the triangle $ABH$ is equal to the triangle $DEF$
and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]

therefore the angle $BHA$ is equal to the angle $EFD$

But the angle $EFD$ is equal to the angle $BCA$
therefore, in the triangle $AHC$ the exterior angle $BHA$ is equal to the interior and opposite angle $BCA$

which is impossible [I 16]

Therefore $BC$ is not unequal to $EF$
and is therefore equal to it

But $AB$ is also equal to $DE$
therefore the two sides $AB$ $BC$ are equal to the two sides $DE$ $EF$ respectively, and they contain equal angles,

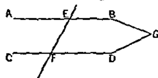
therefore the base $AC$ is equal to the base $DF$
the triangle $ABC$ equal to the triangle $DEF$
and the remaining angle $BAC$ equal to the remaining angle $EDF$ [I 4]

Therefore etc Q E D

## Proposition 27

*If a straight line falling on two straight lines make the alternate angles equal to one another the straight lines will be parallel to one another*

For let the straight line $EF$ falling on the two straight lines $AB$ $CD$ make the alternate angles $AEF$ $EFD$ equal to one another




I say that $AB$ is parallel to $CD$

For if not $AB$ $CD$ when produced will meet either in the direction of $B$ $D$ or towards $A$, $C$

Let them be produced and meet, in the direction of $B$ $D$ at $G$

Then in the triangle $GEF$
the exterior angle $AEF$ is equal to the interior and opposite angle $EFG$
which is impossible [I 16]

Therefore $AB$ $CD$ when produced will not meet in the direction of $B$ $D$
Similarly it can be proved that neither will they meet towards $A$ $C$

But straight lines which do not meet in either direction are parallel [Def 23]

therefore $AB$ is parallel to $CD$

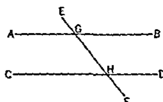
Therefore etc Q E D

PROPOSITION 28

*If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side or the interior angles on the same side equal to two right angles the straight lines will be parallel to one another*

For let the straight line $EF$ falling on the two straight lines $AB$ $CD$ make the exterior angle $EGB$ equal to the interior and opposite angle $CHD$ or the interior angles on the same side namely $BGH$, $GHD$, equal to two right angles,

I say that $AB$ is parallel to $CD$




For, since the angle $EGB$ is equal to the angle $GHD$

while the angle $EGB$ is equal to the angle $AGH$ [I 15]

the angle $AGH$ is also equal to the angle $GHD$

and they are alternate

therefore $AB$ is parallel to $CD$ [I 27]

Again since the angles $BGH$ $GHD$ are equal to two right angles and the angles $AGH$, $BGH$ are also equal to two right angles [I 13]

the angles $AGH$ $BGH$ are equal to the angles $BGH$ $GHD$

Let the angle $BGH$ be subtracted from each

therefore the remaining angle $AGH$ is equal to the remaining angle $GHD$,

and they are alternate

therefore $AB$ is parallel to $CD$ [I 27]

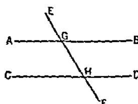
Therefore etc Q E D]

PROPOSITION 29

*A straight line falling on parallel straight lines makes the alternate angles equal to one another the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles*

For let the straight line $EF$ fall on the parallel straight lines $AB$ $CD$

I say that it makes the alternate angles $AGH$ $GHD$ equal, the exterior angle $EGB$ equal to the interior and opposite angle $GHD$ and the interior angles on the same side, namely $BGH$ $GHD$ equal to two right angles




For if the angle $AGH$ is unequal to the angle $GHD$ one of them is greater

Let the angle $AGH$ be greater

Let the angle $BGH$ be added to each

therefore the angles $AGH$ $BGH$ are greater than the angles $BGH$, $GHD$

But the angles $AGH$, $BGH$ are equal to two right angles, [I 13]

therefore the angles $BGH$ $GHD$ are less than two right angles

But straight lines produced indefinitely from angles less than two right angles meet [Post 5]

therefore $AB$ $CD$ if produced indefinitely will meet,

but they do not meet, because they are by hypothesis parallel
Therefore the angle $AGH$ is not unequal to the angle $GHD$,
and is therefore equal to it
Again, the angle $AGH$ is equal to the angle $EGB$, [I 15]
therefore the angle $EGB$ is also equal to the angle $GHD$ [C.N 1]
Let the angle $BGH$ be added to each,
therefore the angles $EGB$, $BGH$ are equal to the angles $BGH$ $GHD$ [C N 2]
But the angles $EGB$ $BGH$ are equal to two right angles [I 13]
therefore the angles $BGH$, $GHD$ are also equal to two right angles
Therefore etc Q E D

PROPOSITION 30

*Straight lines parallel to the same straight line are also parallel to one another*

Let each of the straight lines $AB$ $CD$ be parallel to $EF$ I say that $AB$ is also parallel to $CD$

For let the straight line $GK$ fall upon them
Then, since the straight line $GK$ has fallen on the parallel straight lines $AB$, $EF$,
the angle $AGK$ is equal to the angle $GHF$ [I 29]
Again since the straight line $GK$ has fallen on the parallel straight lines $EF$, $CD$
the angle $GHF$ is equal to the angle $GKD$ [I 29]
But the angle $AGK$ was also proved equal to the angle $GHF$
therefore the angle $AGK$ is also equal to the angle $GKD$ [C N 1]
and they are alternate
Therefore $AB$ is parallel to $CD$ Q E D

PROPOSITION 31

*Through a given point to draw a straight line parallel to a given straight line*

Let $A$ be the given point and $BC$ the given straight line
thus it is required to draw through the point $A$ a straight line parallel to the straight line $BC$

Let a point $D$ be taken at random on $BC$ and let $AD$ be joined, on the straight line $DA$, and at the point $A$ on it let the angle $DAE$ be constructed equal to the angle $ADC$ [I 23] and let the straight line $AF$ be produced in a straight line with $EA$

Then since the straight line $AD$ falling on the two straight lines $BC$ $EF$ has made the alternate angles $EAD$ $ADC$ equal to one another
therefore $EAF$ is parallel to $BC$ [I 27]
Therefore through the given point $A$ the straight line $EAF$ has been drawn parallel to the given straight line $BC$ Q E F

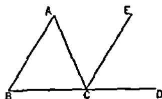
PROPOSITION 32

*In any triangle, if one of the sides be produced the exterior angle is equal to the two interior and opposite angles and the three interior angles of the triangle are equal to two right angles*

Let $ABC$ be a triangle, and let one side of it $BC$ be produced to $D$

I say that the exterior angle $ACD$ is equal to the two interior and opposite angles $CAB$, $ABC$, and the three interior angles of the triangle $ABC$, $BCA$, $CAB$ are equal to two right angles

For let $CE$ be drawn through the point $C$ parallel to the straight line $AB$ [I 31]




Then since $AB$ is parallel to $CE$,

and $AC$ has fallen upon them

the alternate angles $BAC$, $ACE$ are equal to one another [I 29]

Again, since $AB$ is parallel to $CE$

and the straight line $BD$ has fallen upon them

the exterior angle $ECD$ is equal to the interior and opposite angle $ABC$ [I 29]

But the angle $ACE$ was also proved equal to the angle $BAC$,

therefore the whole angle $ACD$ is equal to the two interior and opposite angles $BAC$ $ABC$

Let the angle $ACB$ be added to each

therefore the angles $ACD$ $ACB$ are equal to the three angles $ABC$, $BCA$, $CAB$

But the angles $ACD$ $ACB$ are equal to two right angles [I 13]

therefore the angles $ABC$, $BCA$, $CAB$ are also equal to two right angles

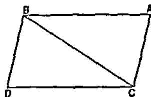
Therefore etc Q E D

PROPOSITION 33

*The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel*

Let $AB$ $CD$ be equal and parallel and let the straight lines $AC$, $BD$ join them (at the extremities which are) in the same directions (respectively),

I say that $AC$ $BD$ are also equal and parallel




Let $BC$ be joined

Then since $AB$ is parallel to $CD$ and $BC$ has fallen upon them

the alternate angles $ABC$, $BCD$ are equal to one another [I 29]

And since $AB$ is equal to $CD$

and $BC$ is common

the two sides $AB$ $BC$ are equal to the two sides $DC$ $CB$

and the angle $ABC$ is equal to the angle $BCD$

therefore the base $AC$ is equal to the base $BD$

and the triangle $ABC$ is equal to the triangle $DCB$,

and the remaining angles will be equal to the remaining angles respectively namely those which the equal sides subtend [I 4]

therefore the angle $ACB$ is equal to the angle $CBD$

And since the straight line $BC$ falling on the two straight lines $AC$, $BD$ has made the alternate angles equal to one another

$AC$ is parallel to $BD$ [I 27]

And it was also proved equal to it

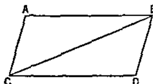
Therefore etc Q E D

PROPOSITION 34

*In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas*

Let $ACDB$ be a parallelogrammic area, and $BC$ its diameter

I say that the opposite sides and angles of the parallelogram $ACDB$ are equal to one another and the diameter $BC$ bisects it




For since $AB$ is parallel to $CD$,

and the straight line $BC$ has fallen upon them the alternate angles $ABC$ $BCD$ are equal to one another [I 29]

Again, since $AC$ is parallel to $BD$ and $BC$ has fallen upon them,

the alternate angles $ACB$ $CBD$ are equal to one another [I 29]

Therefore $ABC$ $DCB$ are two triangles having the two angles $ABC$, $BCA$ equal to the two angles $DCB$, $CBD$ respectively, and one side equal to one side namely that adjoining the equal angles and common to both of them, $BC$, therefore they will also have the remaining sides equal to the remaining sides respectively, and the remaining angle to the remaining angle [I 26]

therefore the side $AB$ is equal to $CD$,

and $AC$ to $BD$,

and further the angle $BAC$ is equal to the angle $CDB$

And, since the angle $ABC$ is equal to the angle $BCD$

and the angle $CBD$ to the angle $ACB$

the whole angle $ABD$ is equal to the whole angle $ACD$ [C.N 2]

And the angle $BAC$ was also proved equal to the angle $CDB$

Therefore in parallelogrammic areas the opposite sides and angles are equal to one another

I say, next, that the diameter also bisects the areas

For, since $AB$ is equal to $CD$,

and $BC$ is common,

the two sides $AB$ $BC$ are equal to the two sides $DC$, $CB$ respectively,

and the angle $ABC$ is equal to the angle $BCD$,

therefore the base $AC$ is also equal to $DB$

and the triangle $ABC$ is equal to the triangle $DCB$ [I 4]

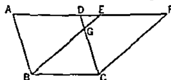
Therefore the diameter $BC$ bisects the parallelogram $ACDB$ Q E D

PROPOSITION 35

*Parallelograms which are on the same base and in the same parallels are equal to one another*

Let $ABCD$ $EBCF$ be parallelograms on the same base $BC$ and in the same parallels $AF$ $BC$,

I say that $ABCD$ is equal to the parallelogram $EBCF$




For since $ABCD$ is a parallelogram

$AD$ is equal to $BC$ [I 34]

For the same reason also

$EF$ is equal to $BC$

so that $AD$ is also equal to $EF$ [C.N 1]

and $DE$ is common,

therefore the whole $AF$ is equal to the whole $DF$. [C.N. 2]

But $AB$ is also equal to $DC$, [I. 34]

therefore the two sides $EA$, $AB$ are equal to the two sides $FD$, $DC$ respectively,

and the angle $FDC$ is equal to the angle $EAB$,

the exterior to the interior; [I. 29]

therefore the base $EB$ is equal to the base $FC$,

and the triangle $EAB$ will be equal to the triangle $FDC$. [I. 4]

Let $DGE$ be subtracted from each;

therefore the trapezium $ABGD$ which remains is equal to the trapezium $EGCF$ which remains. [C.N. 3]

Let the triangle $GBC$ be added to each;

therefore the whole parallelogram $ABCD$ is equal to the whole parallelogram $EBCF$. [C.N. 2]

Therefore etc. Q. E. D.

PROPOSITION 36.

*Parallelograms which are on equal bases and in the same parallels are equal to one another.*

Let $ABCD$, $EFGH$ be parallelograms which are on equal bases $BC$, $FG$ and in the same parallels $AH$, $BG$;

I say that the parallelogram $ABCD$ is equal to $EFGH$.

For let $BE$, $CH$ be joined.

Then, since $BC$ is equal to $FG$,

while

$FG$ is equal to $EH$,

$BC$ is also equal to $EH$. [C.N. 1]

But they are also parallel.

And $EB$, $HC$ join them;

but straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are equal and parallel. [I. 33]

Therefore $EBCH$ is a parallelogram. [I. 34]

And it is equal to $ABCD$;

for it has the same base $BC$ with it, and is in the same parallels $BC$, $AH$ with it. [I. 35]

For the same reason also $EFGH$ is equal to the same $EBCH$; [I. 35]

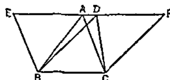
so that the parallelogram $ABCD$ is also equal to $EFGH$. [C.N. 1]

Therefore etc. Q. E. D.

PROPOSITION 37.

*Triangles which are on the same base and in the same parallels are equal to one another.*

Let $ABC$, $DBC$ be triangles on the same base $BC$ and in the same parallels $AD$, $BC$;




I say that the triangle $ABC$ is equal to the triangle $DBC$.

Let $AD$ be produced in both directions to $E$, $F$;

through $B$ let $BE$ be drawn parallel to $CA$, [I. 31]

and through *C* let *CF* be drawn parallel to *BD* [I 31]

Then each of the figures *EBCA*, *DBCF* is a parallelogram, and they are equal

for they are on the same base *BC* and in the same parallels *BC EF* [I 35]

Moreover the triangle *ABC* is half of the parallelogram *EBCA*, for the diameter *AB* bisects it [I 34]

And the triangle *DBC* is half of the parallelogram *DBCF* for the diameter *DC* bisects it [I 34]

[But the halves of equal things are equal to one another ]

Therefore the triangle *ABC* is equal to the triangle *DBC*

Therefore etc Q E D

PROPOSITION 38

*Triangles which are on equal bases and in the same parallels are equal to one another*

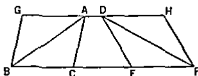
Let *ABC DEF* be triangles on equal bases *BC EF* and in the same parallels *BC AD*,

I say that the triangle *ABC* is equal to the triangle *DEF*

For let *AD* be produced in both directions to *G H*

through *B* let *BG* be drawn parallel to *CA*, [I 31]

and through *F* let *FH* be drawn parallel to *DE*




Then each of the figures *GBCA*, *DEFH* is a parallelogram

and *GBCA* is equal to *DEFH*,

for they are on equal bases *BC*, *EF* and in the same parallels *BF GH* [I 36]

Moreover the triangle *ABC* is half of the parallelogram *GBCA* for the diameter *AB* bisects it [I 34]

And the triangle *FED* is half of the parallelogram *DEFH* for the diameter *DF* bisects it [I 34]

[But the halves of equal things are equal to one another ]

Therefore the triangle *ABC* is equal to the triangle *DEF*

Therefore etc Q E D

PROPOSITION 39

*Equal triangles which are on the same base and on the same side are also in the same parallels*

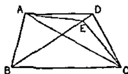
Let *ABC DBC* be equal triangles which are on the same base *BC* and on the same side of it

[I say that they are also in the same parallels ]

And [For] let *AD* be joined I say that *AD* is parallel to *BC*

For if not let *AE* be drawn through the point *A* parallel to the straight line *BC* [I 31]

and let *EC* be joined




Therefore the triangle *ABC* is equal to the triangle *EBC*,

for it is on the same base *BC* with it and in the same parallels [I 37]

But *ABC* is equal to *DBC*

therefore $DBC$ is also equal to $FBC$ [C.N. 1]

the greater to the less which is impossible

Therefore $AF$ is not parallel to $BC$

Similarly we can prove that neither is any other straight line except $AD$, therefore $AD$ is parallel to $BC$

Therefore etc Q E D

### Proposition 40

*Equal triangles which are on equal bases and on the same side are also in the same parallels*

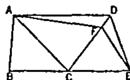
Let $ABC$, $CDE$ be equal triangles on equal bases $BC$, $CE$ and on the same side

I say that they are also in the same parallels

For let $AD$ be joined

I say that $AD$ is parallel to $BE$

For if not let $AF$ be drawn through $A$ parallel to $BE$ [I 31] and let $FE$ be joined




Therefore the triangle $ABC$ is equal to the triangle $FCE$,

for they are on equal bases $BC$, $CE$ and in the same parallels $BE$, $AF$ [I 38]

But the triangle $ABC$ is equal to the triangle $DCE$,

therefore the triangle $DCE$ is also equal to the triangle $FCE$, [C.N. 1]

the greater to the less which is impossible

Therefore $AF$ is not parallel to $BE$

Similarly we can prove that neither is any other straight line except $AD$, therefore $AD$ is parallel to $BE$

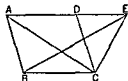
Therefore etc Q E D

### Proposition 41

*If a parallelogram have the same base with a triangle and be in the same parallels the parallelogram is double of the triangle*

For let the parallelogram $ABCD$ have the same base $BC$ with the triangle $EBC$ and let it be in the same parallels $BC$, $AE$

I say that the parallelogram $ABCD$ is double of the triangle $BEC$




For let $AC$ be joined

Then the triangle $ABC$ is equal to the triangle $EBC$, for it is on the same base $BC$ with it and in the same parallels $BC$, $AE$ [I 37]

But the parallelogram $ABCD$ is double of the triangle $ABC$ for the diameter $AC$ bisects it [I 34]

so that the parallelogram $ABCD$ is also double of the triangle $EBC$

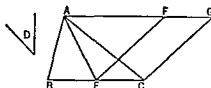
Therefore etc Q E D

### Proposition 42

*To construct in a given rectilineal angle a parallelogram equal to a given triangle*

Let $ABC$ be the given triangle and $D$ the given rectilineal angle

thus it is required to construct in the rectilineal angle $D$ a parallelogram equal




to the triangle $ABC$

Let $BC$ be bisected at $E$, and let $AE$ be joined,

on the straight line $EC$, and at the point $E$ on it let the angle $CEF$ be constructed equal to the angle $D$, [I 23]

through $A$ let $AG$ be drawn parallel to $EC$ and [I 31]

through $C$ let $CG$ be drawn parallel to $EF$

Then $FECG$ is a parallelogram

And since $BE$ is equal to $EC$,

the triangle $ABE$ is also equal to the triangle $AEC$

for they are on equal bases $BE$, $EC$ and in the same parallels $BC$ $AG$, [I 38]

therefore the triangle $ABC$ is double of the triangle $AEC$

But the parallelogram $FECG$ is also double of the triangle $AEC$, for it has the same base with it and is in the same parallels with it, [I 41]

therefore the parallelogram $FECG$ is equal to the triangle $ABC$

And it has the angle $CEF$ equal to the given angle $D$

Therefore the parallelogram $FECG$ has been constructed equal to the given triangle $ABC$ in the angle $CEF$ which is equal to $D$ Q E F

## Proposition 43

*In any parallelogram the complements of the parallelograms about the diameter are equal to one another*

Let $ABCD$ be a parallelogram and $AC$ its diameter

and about $AC$ let $EH$, $FG$ be parallelograms and $BK$, $KD$ the so-called complements,

I say that the complement $BK$ is equal to the complement $KD$

For since $ABCD$ is a parallelogram, and $AC$ its diameter,

the triangle $ABC$ is equal to the triangle $ACD$ [I 34]

Again since $EH$ is a parallelogram, and $AK$ is its diameter,

the triangle $AEK$ is equal to the triangle $AHK$

For the same reason

the triangle $KFC$ is also equal to $KGC$

Now since the triangle $AEK$ is equal to the triangle $AHK$,

and $KFC$ to $KGC$

the triangle $AEK$ together with $KGC$ is equal to the triangle $AHK$ together with $KFC$ [C.N 2]

And the whole triangle $ABC$ is also equal to the whole $ADC$,

therefore the complement $BK$ which remains is equal to the complement $KD$ which remains [C.N 3]

Therefore etc Q E D

## Proposition 44

*To a given straight line to apply in a given rectilineal angle a parallelogram equal to a given triangle*

Let $AB$ be the given straight line $C$ the given triangle and $D$ the given rectilineal angle,

thus it is required to apply to the given straight line $AB$, in an angle equal to the angle $D$, a parallelogram equal to the given triangle $C$.

Let the parallelogram $BEFG$ be constructed equal to the triangle $C$ in the angle $EBG$ which is equal to $D$ [I. 42]; let it be placed so that $BE$ is in a straight line with $AB$; let $FG$ be drawn through to $H$, and let $AH$ be drawn through $A$ parallel to either $BG$ or $EF$. [I. 31]

Let $HB$ be joined.

Then, since the straight line $HF$ falls upon the parallels $AH$, $EF$, the angles $AHF$, $HFE$ are equal to two right angles. [I. 29]

Therefore the angles $BHG$, $GFE$ are less than two right angles; and straight lines produced indefinitely from angles less than two right angles meet; [Post. 5]

therefore $HB$, $FE$, when produced, will meet.

Let them be produced and meet at $K$; through the point $K$ let $KL$ be drawn parallel to either $EA$ or $FH$, [I. 31]

and let $HA$, $GB$ be produced to the points $L$, $M$.

Then $HLKF$ is a parallelogram, $HK$ is its diameter, and $AG$, $ME$ are parallelograms, and $LB$, $BF$ the so-called complements about $HK$;

therefore $LB$ is equal to $BF$. [I. 43]

But $BF$ is equal to the triangle $C$;

therefore $LB$ is also equal to $C$. [C.N. 1]

And, since the angle $GBE$ is equal to the angle $ABM$, [I. 15]

while the angle $GBE$ is equal to $D$,

the angle $ABM$ is also equal to the angle $D$.

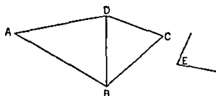
Therefore the parallelogram $LB$ equal to the given triangle $C$ has been applied to the given straight line $AB$, in the angle $ABM$ which is equal to $D$.

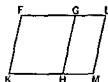
Q. E. F.

PROPOSITION 45

*To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.*

Let $ABCD$ be the given rectilineal figure and $E$ the given rectilineal angle; thus it is required to construct, in the given angle $E$, a parallelogram equal to the rectilineal figure $ABCD$.




Let $DB$ be joined, and let the parallelogram $FH$ be constructed equal to the triangle $ABD$ in the angle $HKF$ which is equal to $E$; [I. 42]

let the parallelogram $GM$ equal to the triangle $DBC$ be applied to the straight

line $GH$ in the angle $GHM$ which is equal to $E$ [I 44]

Then since the angle $E$ is equal to each of the angles $HKF$, $GHM$,

the angle $HKF$ is also equal to the angle $GHM$ [C.N 1]

Let the angle $KHG$ be added to each,

therefore the angles $FKH$, $KHG$ are equal to the angles $KHG$, $GHM$

But the angles $FKH$, $KHG$ are equal to two right angles, [I 29]

therefore the angles $KHG$, $GHM$ are also equal to two right angles

Thus, with a straight line $GH$, and at the point $H$ on it two straight lines $KH$, $HM$ not lying on the same side make the adjacent angles equal to two right angles,

therefore $KH$ is in a straight line with $HM$ [I 14]

And, since the straight line $HG$ falls upon the parallels $KM$, $FG$ the alternate angles $MHG$, $HGF$ are equal to one another [I 29]

Let the angle $HGL$ be added to each

*therefore the angles $MHG$, $HGL$ are equal to the angles $HGF$, $HGL$* [C.N 2]

But the angles $MHG$, $HGL$ are equal to two right angles, [I 29]

therefore the angles $HGF$, $HGL$ are also equal to two right angles [C.N 1]

Therefore $FG$ is in a straight line with $GL$ [I 14]

And since $FK$ is equal and parallel to $HG$, [I 34]

and $HG$ to $ML$ also

$KF$ is also equal and parallel to $ML$, [C.N 1, I 30]

and the straight lines $KM$, $FL$ join them (at their extremities), therefore $KM$, $FL$ are also equal and parallel [I 33]

Therefore $KFLM$ is a parallelogram

And since the triangle $ABD$ is equal to the parallelogram $FH$

and $DBC$ to $GM$

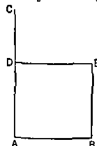
the whole rectilineal figure $ABCD$ is equal to the whole parallelogram $KFLM$

Therefore the parallelogram $KFLM$ has been constructed equal to the given rectilineal figure $ABCD$ in the angle $FKM$ which is equal to the given angle $E$

Q E F

## Proposition 46

*On a given straight line to describe a square*




Let $AB$ be the given straight line thus it is required to describe a square on the straight line $AB$

Let $AC$ be drawn at right angles to the straight line $AB$ from the point $A$ on it [I 11] and let $AD$ be made equal to $AB$

through the point $D$ let $DE$ be drawn parallel to $AB$ and through the point $B$ let $BE$ be drawn parallel to $AD$ [I 31]

Therefore $ADEB$ is a parallelogram

therefore $AB$ is equal to $DE$ and $AD$ to $BE$ [I 34]

But $AB$ is equal to $AD$

therefore the four straight lines $BA$, $AD$, $DE$, $EB$ are equal to one another,

therefore the parallelogram $ADEB$ is equilateral

I say next that it is also right angled

For since the straight line $AD$ falls upon the parallels $AB$, $DE$

the angles $BAD$, $ADE$ are equal to two right angles

But the angle $BAD$ is right,

therefore the angle $ADE$ is also right

And in parallelogrammic areas the opposite sides and angles are equal to one another, [I 34]

therefore each of the opposite angles $ABE$, $BED$ is also right

Therefore $ADEB$ is right-angled

And it was also proved equilateral

Therefore it is a square, and it is described on the straight line $AB$ Q E F

PROPOSITION 47

*In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle*

Let $ABC$ be a right angled triangle having the angle $BAC$ right,

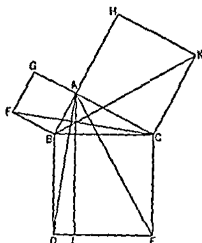
I say that the square on $BC$ is equal to the squares on $BA$, $AC$

For let there be described on $BC$ the square $BDEC$, and on $BA$, $AC$ the squares $GB$, $HC$ [I 46]

through $A$ let $AL$ be drawn parallel to either $BD$ or $CE$ and let $AD$ $FC$ be joined

Then since each of the angles $BAC$, $BAG$ is right it follows that with a straight line $BA$ and at the point $A$ on it the two straight lines $AC$ $AG$ not lying on the same side make the adjacent angles equal to two right angles,

therefore $CA$ is in a straight line with $AG$ [I 14]




For the same reason

$BA$ is also in a straight line with $AH$

And since the angle $DBC$ is equal to the angle $FBA$ for each is right

let the angle $ABC$ be added to each

therefore the whole angle $DBA$ is equal to the whole angle $FBC$ [C.N 2]

And since $DB$ is equal to $BC$ and $FB$ to $BA$

the two sides $AB$ $BD$ are equal to the two sides $FB$ $BC$ respectively,

and the angle $ABD$ is equal to the angle $FBC$

therefore the base $AD$ is equal to the base $FC$,

and the triangle $ABD$ is equal to the triangle $FBC$ [I 4]

Now the parallelogram $BL$ is double of the triangle $ABD$ for they have the same base $BD$ and are in the same parallels $BD$ $AL$ [I 41]

And the square $GB$ is double of the triangle $FBC$

for they again have the same base $FB$ and are in the same parallels $FB$ $GC$ [I 41]

[But the doubles of equals are equal to one another ]

Therefore the parallelogram $BL$ is also equal to the square $GB$

Similarly if $AE$ $BK$ be joined

the parallelogram $CL$ can also be proved equal to the square $HC$,

therefore the whole square $BDEC$ is equal to the two squares $GB$, $HC$ [C.N 2]

And the square $BDEC$ is described on $BC$,

and the squares $GB$, $HC$ on $BA$, $AC$

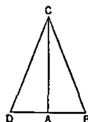
Therefore the square on the side $BC$ is equal to the squares on the sides $BA$, $AC$

Therefore etc Q E D

## Proposition 48

*If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right*

For in the triangle $ABC$ let the square on one side $BC$ be equal to the squares on the sides $BA$, $AC$,




I say that the angle $BAC$ is right

For let $AD$ be drawn from the point $A$ at right angles to the straight line $AC$, let $AD$ be made equal to $BA$, and let $DC$ be joined

Since $DA$ is equal to $AB$,

the square on $DA$ is also equal to the square on $AB$

Let the square on $AC$ be added to each

therefore the squares on $DA$ $AC$ are equal to the squares on $BA$ $AC$

But the square on $DC$ is equal to the squares on $DA$ $AC$, for the angle $DAC$ is right, [I 47]

and the square on $BC$ is equal to the squares on $BA$, $AC$, for this is the hypothesis

therefore the square on $DC$ is equal to the square on $BC$

so that the side $DC$ is also equal to $BC$

And since $DA$ is equal to $AB$,

and $AC$ is common,

the two sides $DA$ $AC$ are equal to the two sides $BA$ $AC$

and the base $DC$ is equal to the base $BC$

therefore the angle $DAC$ is equal to the angle $BAC$ [I 8]

But the angle $DAC$ is right,

therefore the angle $BAC$ is also right

Therefore etc Q E D

# BOOK TWO

## DEFINITIONS

1 Any rectangular parallelogram is said to be *contained* by the two straight lines containing the right angle

2 And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a *gnomon*

## BOOK II PROPOSITIONS

### Proposition 1

*If there be two straight lines and one of them be cut into any number of segments whatever the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments*

Let *A* *BC* be two straight lines, and let *BC* be cut at random at the points *D* *E*

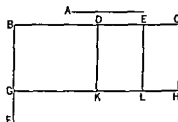
I say that the rectangle contained by *A* *BC* is equal to the rectangle contained by *A* *BD* that contained by *A* *DE* and that contained by *A* *EC*

For let *BF* be drawn from *B* at right angles to *BC*, [I 11]

let *BG* be made equal to *A* [I 3]

through *G* let *GH* be drawn parallel to *BC* [I 31]

and through *D* *E* *C* let *DK* *EL*, *CH* be drawn parallel to *BG*




Then *BH* is equal to *BK* *DL* *EH*

Now *BH* is the rectangle *A* *BC* for it is contained by *GB* *BC* and *BG* is equal to *A*

*BK* is the rectangle *A* *BD* for it is contained by *GB* *BD* and *BG* is equal to *A*

and *DL* is the rectangle *A* *DE* for *DK* that is *BG* is equal to *A* [I 34]

Similarly also *EH* is the rectangle *A* *EC*

Therefore the rectangle *A* *BC* is equal to the rectangle *A*, *BD* the rectangle *A*, *DE* and the rectangle *A* *EC*

Therefore etc Q E D

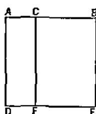
### Proposition 2

*If a straight line be cut at random the rectangle contained by the whole and both of the segments is equal to the square on the whole*

For let the straight line *AB* be cut at random at the point *C*

I say that the rectangle contained by *AB* *BC* together with the rectangle contained by *BA* *AC* is equal to the square on *AB*

For let the square $ADEB$ be described on $AB$ [I 46] and let $CF$ be drawn through $C$ parallel to either $AD$ or $BE$ [I 31]




Then $AE$ is equal to $AF$, $CE$

Now $AE$ is the square on $AB$,

$AF$ is the rectangle contained by $BA$, $AC$ for it is contained by $DA$, $AC$, and $AD$ is equal to $AB$,

and $CE$ is the rectangle $AB$ $BC$ for $BE$ is equal to $AB$

Therefore the rectangle $BA$, $AC$ together with the rectangle $AB$, $BC$ is equal to the square on $AB$

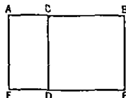
Therefore etc Q E D

### Proposition 3

*If a straight line be cut at random the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment*

For let the straight line $AB$ be cut at random at $C$,

I say that the rectangle contained by $AB$, $BC$ is equal to the rectangle contained by $AC$, $CB$ together with the square on $BC$




For let the square $CDEB$ be described on $CB$, [I 46] let $ED$ be drawn through to $F$,

and through $A$ let $AF$ be drawn parallel to either $CD$ or $BE$ [I 31]

Then $AE$ is equal to $AD$, $CE$

Now $AE$ is the rectangle contained by $AB$ $BC$ for it is contained by $AB$ $BE$ and $BE$ is equal to $BC$

$AD$ is the rectangle $AC$, $CB$ for $DC$ is equal to $CB$,

and $DB$ is the square on $CB$

Therefore the rectangle contained by $AB$, $BC$ is equal to the rectangle contained by $AC$, $CB$ together with the square on $BC$

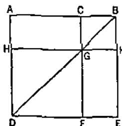
Therefore etc Q E D

### Proposition 4

*If a straight line be cut at random the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments*

For let the straight line $AB$ be cut at random at $C$,

I say that the square on $AB$ is equal to the squares on $AC$ $CB$ and twice the rectangle contained by $AC$, $CB$




For let the square $ADEB$ be described on $AB$ [I 46]

let $BD$ be joined

through $C$ let $CF$ be drawn parallel to either $AD$ or $EB$,

and through $G$ let $HK$ be drawn parallel to either $AB$ or $DE$ [I 31]

Then since $CF$ is parallel to $AD$ and $BD$ has fallen on them

the exterior angle $CGB$ is equal to the interior and opposite angle $ADB$ [I 29]

But the angle $ADB$ is equal to the angle $ABD$

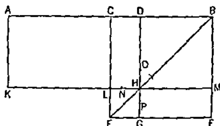
since the side $BA$ is also equal to $AD$, [I 5]

therefore the angle $CGB$ is also equal to the angle $GBC$,
so that the side $BC$ is also equal to the side $CG$ [I 6]
But $CB$ is equal to $GK$, and $CG$ to $KB$, [I 34]
therefore $GK$ is also equal to $KB$,
therefore $CGKB$ is equilateral
I say next that it is also right-angled
For since $CG$ is parallel to $BK$
the angles $KBC$, $GCB$ are equal to two right angles [I 29]
But the angle $KBC$ is right
therefore the angle $BCG$ is also right
so that the opposite angles $CGK$, $GKB$ are also right [I 34]
Therefore $CGKB$ is right angled,
and it was also proved equilateral,
therefore it is a square
and it is described on $CB$
For the same reason
$HF$ is also a square,
and it is described on $HG$, that is $AC$ [I 34]
Therefore the squares $HF$, $KC$ are the squares on $AC$, $CB$
Now, since $AG$ is equal to $GE$
and $AG$ is the rectangle $AC$, $CB$, for $GC$ is equal to $CB$,
therefore $GE$ is also equal to the rectangle $AC$, $CB$
Therefore $AG$, $GE$ are equal to twice the rectangle $AC$, $CB$
But the squares $HF$, $CK$ are also the squares on $AC$, $CB$, therefore the four areas $HF$, $CK$, $AG$, $GE$ are equal to the squares on $AC$, $CB$ and twice the rectangle contained by $AC$, $CB$
But $HF$, $CK$, $AG$, $GE$ are the whole $ADEB$
which is the square on $AB$
Therefore the square on $AB$ is equal to the squares on $AC$, $CB$ and twice the rectangle contained by $AC$, $CB$
Therefore etc Q E D

PROPOSITION 5

*If a straight line be cut into equal and unequal segments the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half*

For let a straight line $AB$ be cut into equal segments at $C$ and into unequal segments at $D$,




I say that the rectangle contained by $AD$, $DB$ together with the square on $CD$ is equal to the square on $CB$

For let the square $CEFB$ be described on $CB$, [I 46]
and let $BE$ be joined
through $D$ let $DG$ be drawn parallel to either $CE$ or $BF$
through $H$ again let $KM$ be drawn parallel to either $AB$ or $EF$
and again through $A$ let $AK$ be drawn parallel to either $CL$ or $BM$ [I 31]
Then, since the complement $CH$ is equal to the complement $HF$, [I 43]

let $DM$ be added to each,
therefore the whole $CM$ is equal to the whole $DF$
But $CM$ is equal to $AL$,
since $AC$ is also equal to $CB$ [I 36]
therefore $AL$ is also equal to $DF$
Let $CH$ be added to each,
therefore the whole $AH$ is equal to the gnomon $NOP$
But $AH$ is the rectangle $AD$ $DB$, for $DH$ is equal to $DB$,
therefore the gnomon $NOP$ is also equal to the rectangle $AD$ $DB$
Let $LG$, which is equal to the square on $CD$, be added to each
therefore the gnomon $NOP$ and $LG$ are equal to the rectangle contained by $AD$, $DB$ and the square on $CD$

But the gnomon $NOP$ and $LG$ are the whole square $CEFB$ which is described on $CB$,
therefore the rectangle contained by $AD$ $DB$ together with the square on $CD$ is equal to the square on $CB$

Therefore etc Q E D

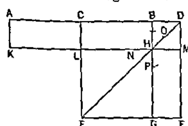
PROPOSITION 6

*If a straight line be bisected and a straight line be added to it in a straight line the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line*

For let a straight line $AB$ be bisected at the point $C$, and let a straight line $BD$ be added to it in a straight line,

I say that the rectangle contained by $AD$, $DB$ together with the square on $CB$ is equal to the square on $CD$

For let the square $CEFD$ be described on $CD$ [I 46]
and let $DE$ be joined,
through the point $B$ let $BG$ be drawn parallel to either $EC$ or $DF$
through the point $H$ let $KM$ be drawn parallel to either $AB$ or $EF$
and further through $A$ let $AK$ be drawn parallel to either $CL$ or $DM$ [I 31]




Then since $AC$ is equal to $CB$,
$AL$ is also equal to $CH$ [I 36]
But $CH$ is equal to $HF$ [I 43]
Therefore $AL$ is also equal to $HF$
Let $CM$ be added to each,
therefore the whole $AM$ is equal to the gnomon $NOP$
But $AM$ is the rectangle $AD$ $DB$,
for $DM$ is equal to $DB$,
therefore the gnomon $NOP$ is also equal to the rectangle $AD$ $DB$
Let $LG$ which is equal to the square on $BC$ be added to each,
therefore the rectangle contained by $AD$ $DB$ together with the square on $CB$ is equal to the gnomon $NOP$ and $LG$

But the gnomon $NOP$ and $LG$ are the whole square $CEFD$ which is described on $CD$,
therefore the rectangle contained by $AD$, $DB$ together with the square on $CB$ is equal to the square on $CD$

Therefore etc Q E D

PROPOSITION 7

*If a straight line be cut at random the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment*

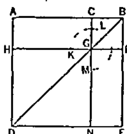
For let a straight line $AB$ be cut at random at the point $C$,

I say that the squares on $AB$, $BC$ are equal to twice the rectangle contained by $AB$ $BC$ and the square on $CA$

For let the square $ADEB$ be described on $AB$, [I 46]

and let the figure be drawn

Then since $AG$ is equal to $GE$ [I 43] let $CF$ be added to each, therefore the whole $AF$ is equal to the whole $CE$




Therefore $AF$, $CE$ are double of $AF$

But $AF$, $CE$ are the gnomon $KLM$ and the square $CF$ therefore the gnomon $KLM$ and the square $CF$ are double of $AF$

But twice the rectangle $AB$, $BC$ is also double of $AF$ for $BF$ is equal to $BC$,

therefore the gnomon $KLM$ and the square $CF$ are equal to twice the rectangle $AB$, $BC$

Let $DG$ which is the square on $AC$, be added to each, therefore the gnomon $KLM$ and the squares $BG$, $GD$ are equal to twice the rectangle contained by $AB$, $BC$ and the square on $AC$

But the gnomon $KLM$ and the squares $BG$, $GD$ are the whole $ADEB$ and $CF$,

which are squares described on $AB$ $BC$,

therefore the squares on $AB$ $BC$ are equal to twice the rectangle contained by $AB$ $BC$ together with the square on $AC$

Therefore etc Q E D

PROPOSITION 8

*If a straight line be cut at random four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line*

For let a straight line $AB$ be cut at random at the point $C$

I say that four times the rectangle contained by $AB$ $BC$ together with the square on $AC$ is equal to the square described on $AB$ $BC$ as on one straight line

For let [the straight line] $BD$ be produced in a straight line [with $AB$] and let $BD$ be made equal to $CB$,

let the square $AEFD$ be described on $AD$ and let the figure be drawn double

Then since $CB$ is equal to $BD$ while $CB$ is equal to $GK$, and $BD$ to $KN$,

therefore $GK$ is also equal to $KN$

For the same reason

$QR$ is also equal to $RP$

And, since $BC$ is equal to $BD$ and $GK$ to $KN$

therefore $CK$ is also equal to $KD$ and $GR$ to $RN$ [I 36]

But $CK$ is equal to $RN$ for they are complements of the parallelogram $CP$ [I 43]

therefore $KD$ is also equal to $GR$

therefore the four areas $DK$ $CK$ $GR$ $RN$ are equal to one another

Therefore the four are quadruple of $CK$

Again since $CB$ is equal to $BD$

while $BD$ is equal to $BK$ that is $CG$

and $CB$ is equal to $GK$ that is $GQ$

therefore $CG$ is also equal to $GQ$

And since $CG$ is equal to $GQ$ and $QR$ to $RP$,

$AG$ is also equal to $MQ$ and $QL$ to $RF$ [I 36]

But $MQ$ is equal to $QL$ for they are complements of the parallelogram $ML$ [I 43]

therefore $AG$ is also equal to $RF$,

therefore the four areas $AG$ $MQ$ $QL$ $RF$ are equal to one another

Therefore the four are quadruple of $AG$

But the four areas $CK$ $KD$ $GR$ $RN$ were proved to be quadruple of $CK$ therefore the eight areas which contain the gnomon $STU$, are quadruple of $AK$

Now, since $AK$ is the rectangle $AB$ $BD$ for $BK$ is equal to $BD$

therefore four times the rectangle $AB$ $BD$ is quadruple of $AK$

But the gnomon $STU$ was also proved to be quadruple of $AK$

therefore four times the rectangle $AB$, $BD$ is equal to the gnomon $STU$

Let $OH$ which is equal to the square on $AC$ be added to each

therefore four times the rectangle $AB$ $BD$ together with the square on $AC$ is equal to the gnomon $STU$ and $OH$

But the gnomon $STU$ and $OH$ are the whole square $AEFD$

which is described on $AD$

therefore four times the rectangle $AB$ $BD$ together with the square on $AC$ is equal to the square on $AD$

But $BD$ is equal to $BC$

therefore four times the rectangle contained by $AB$ $BC$ together with the square on $AC$ is equal to the square on $AD$ that is to the square described on $AB$ and $BC$ as on one straight line

Therefore etc Q E D

## Proposition 9

*If a straight line be cut into equal and unequal segments the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section*

For let a straight line $AB$ be cut into equal segments at $C$ and into unequal segments at $D$

I say that the squares on $AD$ $DB$ are double of the squares on $AC$ $CD$

For let $CE$ be drawn from $C$ at right angles to $AB$ and let it be made equal to either $AC$ or $CB$

let $EA$ $EB$ be joined

let $DF$ be drawn through $D$ parallel to $EC$

and $FG$ through $F$ parallel to $AB$

and let $AF$ be joined

Then, since $AC$ is equal to $CF$

the angle $EAC$ is also equal to the angle $AEC$

And since the angle at $C$ is right

the remaining angles $EAC$ $AEC$ are equal to one right angle [I 32]

And they are equal,

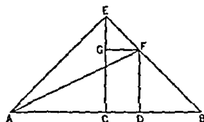
therefore each of the angles $CEA$, $CAE$ is half a right angle

For the same reason

each of the angles $CEB$, $EBC$ is also half a right angle

therefore the whole angle $AEB$ is right

And since the angle $GEF$ is half a right angle




and the angle $EGF$ is right for it is equal to the interior and opposite angle $ECB$ [I 29]

the remaining angle $EFG$ is half a right angle [I 32]

therefore the angle $GEF$ is equal to the angle $EFG$,

so that the side $FG$ is also equal to $GF$ [I 6]

Again since the angle at $B$ is half a right angle

and the angle $FDB$ is right, for it is again equal to the interior and opposite angle $ECB$, [I 29]

the remaining angle $BFD$ is half a right angle [I 32]

therefore the angle at $B$ is equal to the angle $DFB$,

so that the side $FD$ is also equal to the side $DB$ [I 6]

Now since $AC$ is equal to $CE$

the square on $AC$ is also equal to the square on $CE$

therefore the squares on $AC$ $CE$ are double of the square on $AC$

But the square on $EA$ is equal to the squares on $AC$ $CE$ for the angle $ACE$ is right [I 47]

therefore the square on $EA$ is double of the square on $AC$

Again since $EG$ is equal to $GF$

the square on $EG$ is also equal to the square on $GF$,

therefore the squares on $EG$ $GF$ are double of the square on $GF$

But the square on $EF$ is equal to the squares on $EG$ $GF$

therefore the square on $EF$ is double of the square on $GF$

But $GF$ is equal to $CD$ [I 34]

therefore the square on $EF$ is double of the square on $CD$

But the square on $EA$ is also double of the square on $AC$

therefore the squares on $AE$, $EF$ are double of the squares on $AC$, $CD$

And the square on $AF$ is equal to the squares on $AE$ $EF$ for the angle $AEF$ is right [I 47]

therefore the square on $AF$ is double of the squares on $AC$ $CD$

But the squares on $AD$, $DF$ are equal to the square on $AF$ for the angle at $D$ is right, [I 47]

therefore the squares on $AD$ $DF$ are double of the squares on $AC$ $CD$

And $DF$ is equal to $DB$

therefore the squares on $AD$, $DB$ are double of the squares on $AC$ $CD$

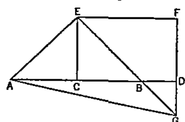
Therefore etc Q E D

PROPOSITION 10

*If a straight line be bisected and a straight line be added to it in a straight line the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line*

For let a straight line $AB$ be bisected at $C$, and let a straight line $BD$ be added to it in a straight line,

I say that the squares on $AD$, $DB$ are double of the squares on $AC$ $CD$




For let $CE$ be drawn from the point $C$ at right angles to $AB$ [I 11] and let it be made equal to either $AC$ or $CB$ [I 3],

let $EA$ $EB$ be joined

through $E$ let $EF$ be drawn parallel to $AD$ and through $D$ let $FD$ be drawn parallel to $CE$ [I 31]

Then since a straight line $EF$ falls on the parallel straight lines $EC$ $FD$

the angles $CEF$ $EFD$ are equal to two right angles [I 29]

therefore the angles $FEB$ $EFD$ are less than two right angles

But straight lines produced from angles less then two right angles meet [I Post 5]

therefore $EB$ $FD$ if produced in the direction $B$, $D$ will meet

Let them be produced and meet at $G$

and let $AG$ be joined

Then since $AC$ is equal to $CE$,

the angle $EAC$ is also equal to the angle $AEC$, [I 5]

and the angle at $C$ is right

therefore each of the angles $EAC$ $AEC$ is half a right angle [I 32]

For the same reason

each of the angles $CEB$ $EBC$ is also half a right angle

therefore the angle $AEB$ is right

And since the angle $EBC$ is half a right angle

the angle $DBG$ is also half a right angle [I 15]

But the angle $BDG$ is also right

for it is equal to the angle $DCE$ they being alternate, [I 29]

therefore the remaining angle $DGB$ is half a right angle, [I 32]

therefore the angle $DGB$ is equal to the angle $DBG$

so that the side $BD$ is also equal to the side $GD$ [I 6]

Again, since the angle $EGF$ is half a right angle

and the angle at $F$ is right for it is equal to the opposite angle the angle at $C$ [I 34]

the remaining angle $FEG$ is half a right angle [I 32]

therefore the angle $EGF$ is equal to the angle $FEG$

so that the side $GF$ is also equal to the side $EF$ [I 6]

Now since the square on $EC$ is equal to the square on $CA$

the squares on $EC$ $CA$ are double of the square on $CA$

But the square on $EA$ is equal to the squares on $EC$ $CA$, [I 47]

therefore the square on $EA$ is double of the square on $AC$ [C.N. 1]

Again since $FG$ is equal to $EF$,

the square on $FG$ is also equal to the square on $FE$,

therefore the squares on $GF$, $FE$ are double of the square on $EF$

But the square on $EG$ is equal to the squares on $GF$, $FE$, [I 47]

therefore the square on $EG$ is double of the square on $EF$

And $EF$ is equal to $CD$ [I 34]

therefore the square on $EG$ is double of the square on $CD$

But the square on $EA$ was also proved double of the square on $AC$,

therefore the squares on $AE$, $EG$ are double of the squares on $AC$, $CD$

And the square on $AG$ is equal to the squares on $AE$, $EG$ [I 47]

therefore the square on $AG$ is double of the squares on $AC$, $CD$

But the squares on $AD$, $DG$ are equal to the square on $AG$, [I 47]

therefore the squares on $AD$, $DG$ are double of the squares on $AC$, $CD$

And $DG$ is equal to $DB$

therefore the squares on $AD$, $DB$ are double of the squares on $AC$, $CD$

Therefore etc Q E D

## Proposition 11

*To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment*

Let $AB$ be the given straight line,

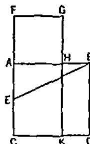
thus it is required to cut $AB$ so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment

For let the square $ABDC$ be described on $AB$, [I 46]

let $AC$ be bisected at the point $E$ and let $BF$ be joined

let $CA$ be drawn through to $F$ and let $EF$ be made equal to $BE$

let the square $FH$ be described on $AF$ and let $GH$ be drawn through to $K$




I say that $AB$ has been cut at $H$ so as to make the rectangle contained by $AB$, $BH$ equal to the square on $AH$

For since the straight line $AC$ has been bisected at $E$, and $FA$ is added to it

the rectangle contained by $CF$, $FA$ together with the square on $AE$ is equal to the square on $EF$ [II 6]

But $EF$ is equal to $EB$

therefore the rectangle $CF$, $FA$ together with the square on $AE$ is equal to the square on $EB$

But the squares on $BA$, $AE$ are equal to the square on $EB$ for the angle at $A$ is right [I 47]

therefore the rectangle $CF$, $FA$ together with the square on $AE$ is equal to the squares on $BA$, $AE$

Let the square on $AE$ be subtracted from each

therefore the rectangle $CF$, $FA$ which remains is equal to the square on $AB$

Now the rectangle $CF$, $FA$ is $FK$ for $AF$ is equal to $FG$,

and the square on $AB$ is $AD$

therefore $FK$ is equal to $AD$

Let $AK$ be subtracted from each,

therefore $FH$ which remains is equal to $HD$

And $HD$ is the rectangle $AB$ $BH$, for $AB$ is equal to $BD$,

and $FH$ is the square on $AH$

therefore the rectangle contained by $AB$ $BH$ is equal to the square on $HA$ therefore the given straight line $AB$ has been cut at $H$ so as to make the rectangle contained by $AB$, $BH$ equal to the square on $HA$ Q E F

PROPOSITION 12

*In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle*

Let $ABC$ be an obtuse-angled triangle having the angle $BAC$ obtuse, and let $BD$ be drawn from the point $B$ perpendicular to $CA$ produced,

I say that the square on $BC$ is greater than the squares on $BA$, $AC$ by twice the rectangle contained by $CA$ $AD$

For, since the straight line $CD$ has been cut at random at the point $A$,

the square on $DC$ is equal to the squares on $CA$ $AD$ and twice the rectangle contained by $CA$, $AD$ [II 4]

Let the square on $DB$ be added to each

therefore the squares on $CD$, $DB$ are equal to the squares on $CA$ $AD$ $DB$ and twice the rectangle $CA$, $AD$

But the square on $CB$ is equal to the squares on $CD$, $DB$, for the angle at $D$ is right, [I 47]

and the square on $AB$ is equal to the squares on $AD$ $DB$ [I 47]

therefore the square on $CB$ is equal to the squares on $CA$ $AB$ and twice the rectangle contained by $CA$ $AD$,

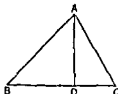
so that the square on $CB$ is greater than the squares on $CA$ $AB$ by twice the rectangle contained by $CA$ $AD$

Therefore etc Q E D

PROPOSITION 13

*In acute angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle*

Let $ABC$ be an acute-angled triangle having the angle at $B$ acute, and let $AD$ be drawn from the point $A$ perpendicular to $BC$,




I say that the square on $AC$ is less than the squares on $CB$ $BA$ by twice the rectangle contained by $CB$ $BD$

For since the straight line $CB$ has been cut at random at $D$

the squares on $CB$ $BD$ are equal to twice the rectangle contained by $CB$ $BD$ and the square on $DC$

Let the square on $DA$ be added to each,
therefore the squares on $CB$ $BD$ $DA$ are equal to twice the rectangle contained by $CB$, $BD$ and the squares on $AD$ $DC$

But the square on $AB$ is equal to the squares on $BD$, $DA$, for the angle at $D$ is right, [I 47]

and the square on $AC$ is equal to the squares on $AD$, $DC$,
therefore the squares on $CB$, $BA$ are equal to the square on $AC$ and twice the rectangle $CB$, $BD$,
so that the square on $AC$ alone is less than the squares on $CB$, $BA$ by twice the rectangle contained by $CB$ $BD$

Therefore etc Q E D

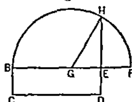
PROPOSITION 14

*To construct a square equal to a given rectilineal figure*

Let $A$ be the given rectilineal figure,
thus it is required to construct a square equal to the rectilineal figure $A$

For let there be constructed the rectangular parallelogram $BD$ equal to the rectilineal figure $A$ [I 45]




Then, if $BE$ is equal to $ED$, that which was enjoined will have been done for a square $BD$ has been constructed equal to the rectilineal figure $A$

But if not one of the straight lines $BE$ $ED$ is greater

Let $BE$ be greater, and let it be produced to $F$,

let $EF$ be made equal to $ED$ and let $BF$ be bisected at $G$

With centre $G$ and distance one of the straight lines $GB$ $GF$ let the semicircle $BHF$ be described let $DE$ be produced to $H$ and let $GH$ be joined

Then since the straight line $BF$ has been cut into equal segments at $G$ and into unequal segments at $E$
the rectangle contained by $BE$ $EF$ together with the square on $EG$ is equal to the square on $GF$ [II 5]

But $GF$ is equal to $GH$,
therefore the rectangle $BE$, $EF$ together with the square on $GE$ is equal to the square on $GH$

But the squares on $HE$ $EG$ are equal to the square on $GH$, [I 47]
therefore the rectangle $BE$, $EF$ together with the square on $GE$ is equal to the squares on $HE$, $EG$

Let the square on $GE$ be subtracted from each
therefore the rectangle contained by $BE$, $EF$ which remains is equal to the square on $EH$

But the rectangle $BE$ $EF$ is $BD$ for $EF$ is equal to $ED$

therefore the parallelogram $BD$ is equal to the square on $HE$

And $BD$ is equal to the rectilineal figure $A$

Therefore the rectilineal figure $A$ is also equal to the square which can be described on $EH$

Therefore a square namely that which can be described on $EH$, has been constructed equal to the given rectilineal figure $A$ Q E F

# BOOK THREE

## DEFINITIONS

1 *Equal circles* are those the diameters of which are equal or the radii of which are equal

2 A straight line is said to *touch a circle* which, meeting the circle and being produced, does not cut the circle

3 *Circles* are said to *touch one another* which meeting one another do not cut one another

4 In a circle straight lines are said *to be equally distant from the centre* when the perpendiculars drawn to them from the centre are equal

5 And that straight line is said to be *at a greater distance* on which the greater perpendicular falls

6 A *segment of a circle* is the figure contained by a straight line and a circumference of a circle

7 An *angle of a segment* is that contained by a straight line and a circumference of a circle

8 An *angle in a segment* is the angle which when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is *the base of the segment*, is contained by the straight lines so joined

9 And when the straight lines containing the angle cut off a circumference, the angle is said to *stand upon* that circumference

10 A *sector of a circle* is the figure which when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them

11 *Similar segments of circles* are those which admit equal angles or in which the angles are equal to one another

## BOOK III PROPOSITIONS

### Proposition 1

*To find the centre of a given circle*

Let $ABC$ be the given circle

thus it is required to find the centre of the circle $ABC$

*Let a straight line $AB$ be drawn through it at random*, and let it be bisected at the point $D$

from $D$ let $DC$ be drawn at right angles to $AB$ and let it be drawn through to $E$ let $CE$ be bisected at $F$

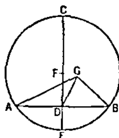
I say that $F$ is the centre of the circle $ABC$

For suppose it is not but if possible let $G$ be the centre

and let $GA$, $GD$, $GB$ be joined

Then since $AD$ is equal to $DB$ and $DG$ is common

the two sides $AD$, $DG$ are equal to the two sides $BD$, $DG$ respectively, and the base $GA$ is equal to the base $GB$, for they are radii,




therefore the angle $ADG$ is equal to the angle $GDB$ [I 8]

But when a straight line set up on a straight line makes the adjacent angles equal to one another each of the equal angles is right, [I Def 10]

therefore the angle $GDB$ is right

But the angle $FDB$ is also right

therefore the angle $FDB$ is equal to the angle $GDB$, the greater to the less which is impossible

Therefore $G$ is not the centre of the circle $ABC$

Similarly we can prove that neither is any other point except $F$

Therefore the point $F$ is the centre of the circle $ABC$

PORISM From this it is manifest that if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line Q E F

## PROPOSITION 2

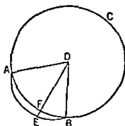
*If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle*

Let $ABC$ be a circle and let two points $A$, $B$ be taken at random on its circumference

I say that the straight line joined from $A$ to $B$ will fall within the circle

For suppose it does not but if possible let it fall outside as $AEB$, let the centre of the circle $ABC$ be taken [III 1], and let it be $D$, let $DA$, $DB$ be joined and let $DFE$ be drawn through

Then since $DA$ is equal to $DB$




the angle $DAE$ is also equal to the angle $DBE$ [I 5]

And since one side $AEB$ of the triangle $DAE$ is produced

the angle $DEB$ is greater than the angle $DAE$ [I 16]

But the angle $DAE$ is equal to the angle $DBE$

therefore the angle $DEB$ is greater than the angle $DBE$

And the greater angle is subtended by the greater side, [I 19]

therefore $DB$ is greater than $DE$

But $DB$ is equal to $DF$

therefore $DF$ is greater than $DE$,

the less than the greater which is impossible

Therefore the straight line joined from $A$ to $B$ will not fall outside the circle

Similarly we can prove that neither will it fall on the circumference itself

therefore it will fall within

Therefore etc Q E D

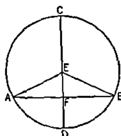
## Proposition 3

*If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles and if it cut it at right angles it also bisects it*

Let $ABC$ be a circle and in it let a straight line $CD$ through the centre bisect a straight line $AB$ not through the centre at the point $F$,

I say that it also cuts it at right angles

For let the centre of the circle $ABC$ be taken and let it be $E$ let $EA$, $EB$ be joined




Then, since $AF$ is equal to $FB$ and $FE$ is common, two sides are equal to two sides,
and the base $EA$ is equal to the base $EB$,
therefore the angle $AFE$
is equal to the angle $BFE$ [I 8]

But when a straight line set up on a straight line makes the adjacent angles equal to one another each of the equal angles is right, [I Def 10]

therefore each of the angles $AFE$, $BFE$ is right

Therefore $CD$ which is through the centre and bisects $AB$ *which is not through the centre also cuts it at right angles*

Again let $CD$ cut $AB$ at right angles

I say that it also bisects it that is that $AF$ is equal to $FB$

For with the same construction

since $EA$ is equal to $EB$,

the angle $EAF$ is also equal to the angle $EBF$ [I 5]

But the right angle $AFE$ is equal to the right angle $BFE$ therefore $EAF$ $EBF$ are two triangles having two angles equal to two angles and one side equal to one side namely $EF$ which is common to them and subtends one of the equal angles,

therefore they will also have the remaining sides equal to the remaining sides [I 26]

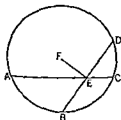
therefore $AF$ is equal to $FB$

Therefore etc Q E D

## Proposition 4

*If in a circle two straight lines cut one another which are not through the centre they do not bisect one another*

Let $ABCD$ be a circle and in it let the two straight lines $AC$ $BD$ which are not through the centre cut one another at $E$




I say that they do not bisect one another

For if possible let them bisect one another so that $AE$ is equal to $EC$ and $BE$ to $ED$,

let the centre of the circle $ABCD$ be taken [III 1] and let it be $F$ let $FE$ be joined

Then since a straight line $FE$ through the centre bisects a straight line $AC$ not through the centre

it also cuts it at right angles [III 3]

therefore the angle $FEA$ is right

Again, since a straight line *FE* bisects a straight line *BD*,

it also cuts it at right angles, [III 3]

therefore the angle *FEB* is right

But the angle *FEA* was also proved right,

therefore the angle *FEA* is equal to the angle *FEB* the less to the greater which is impossible

Therefore *AC*, *BD* do not bisect one another

Therefore etc Q E D

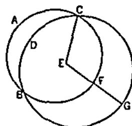
## Proposition 5

*If two circles cut one another they will not have the same centre*

For let the circles *ABC*, *CDG* cut one another at the points *B*, *C*,

I say that they will not have the same centre

For if possible let it be *E*, let *EC* be joined, and let *EFG* be drawn through at random




Then, since the point *E* is the centre of the circle *ABC*,

*EC* is equal to *EF* [I Def 15]

Again, since the point *E* is the centre of the circle *CDG*,

*EC* is equal to *EG*

But *EC* was proved equal to *EF* also

therefore *EF* is also equal to *EG*, the less to the greater which is impossible

Therefore the point E is not the centre of the circles *ABC*, *CDG*

Therefore etc Q E D

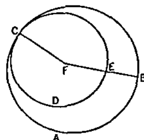
## Proposition 6

*If two circles touch one another they will not have the same centre*

For let the two circles *ABC* *CDE* touch one another at the point *C*

I say that they will not have the same centre

For if possible let it be *F* let *FC* be joined and let *FEB* be drawn through at random




Then since the point *F* is the centre of the circle *ABC*

*FC* is equal to *FB*

Again since the point *F* is the centre of the circle *CDE*,

*FC* is equal to *FE*

But *FC* was proved equal to *FB*

therefore *FE* is also equal to *FB* the less to the greater which is impossible

Therefore *F* is not the centre of the circles *ABC*, *CDE*

Therefore etc Q E D

## Proposition 7

*If on the diameter of a circle a point be taken which is not the centre of the circle and from the point straight lines fall upon the circle that will be greatest on which the centre is the remainder of the same diameter will be least and of the rest the*

*nearer to the straight line through the centre is always greater than the more remote and only two equal straight lines will fall from the point on the circle one on each side of the least straight line*

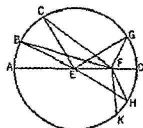
Let $ABCD$ be a circle and let $AD$ be a diameter of it, on $AD$ let a point $F$ be taken which is not the centre of the circle let $E$ be the centre of the circle

and from $F$ let straight lines $FB$, $FC$, $FG$ fall upon the circle $ABCD$,

I say that $FA$ is greatest $FD$ is least, and of the rest $FB$ is greater than $FC$, and $FC$ than $FG$

For let $BE$ $CE$ $GE$ be joined

Then, since in any triangle two sides are greater than the remaining one [I 20]




$EB$ $EF$ are greater than $BF$

But $AE$ is equal to $BE$

therefore $AF$ is greater than $BF$

Again since $BE$ is equal to $CE$ and $FE$ is common

the two sides $BE$ $EF$ are equal to the two sides $CE$ $EF$

But the angle $BEF$ is also greater than the angle $CEF$

therefore the base $BF$ is greater than the base $CF$ [I 24]

For the same reason

$CF$ is also greater than $FG$

Again since $GF$, $FE$ are greater than $EG$

and $EG$ is equal to $ED$

$GF$, $FE$ are greater than $ED$

Let $EF$ be subtracted from each,

therefore the remainder $GF$ is greater than the remainder $FD$

Therefore $FA$ is greatest $FD$ is least, and $FB$ is greater than $FC$ and $FC$ than $FG$

I say also that from the point $F$ only two equal straight lines will fall on the circle $ABCD$, one on each side of the least $FD$

For on the straight line $EF$ and at the point $E$ on it let the angle $FEH$ be constructed equal to the angle $GEF$ [I 23] and let $FH$ be joined

Then since $GE$ is equal to $EH$

and $EF$ is common

the two sides $GE$ $EF$ are equal to the two sides $HE$ $EF$

*and the angle $GEF$ is equal to the angle $HEF$*

therefore the base $FG$ is equal to the base $FH$ [I 4]

I say again that another straight line equal to $FG$ will not fall on the circle from the point $F$

For if possible let $FK$ so fall

Then since $FK$ is equal to $FG$ and $FH$ to $FG$

$FK$ is also equal to $FH$

the nearer to the straight line through the centre being thus equal to the more remote which is impossible

Therefore another straight line equal to $GF$ will not fall from the point $F$ upon the circle

therefore only one straight line will so fall

Therefore etc Q. E. D.

### Proposition 8

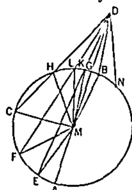
*If a point be taken outside a circle and from the point straight lines be drawn through to the circle one of which is through the centre and the others are drawn at random, then of the straight lines which fall on the concave circumference that through the centre is greatest while of the rest the nearer to that through the centre is always greater than the more remote but of the straight lines falling on the convex circumference that between the point and the diameter is least while of the rest the nearer to the least is always less than the more remote and only two equal straight lines will fall on the circle from the point one on each side of the least*

Let $ABC$ be a circle and let a point $D$ be taken outside $ABC$, let there be drawn through from it straight lines $DA$, $DE$, $DF$, $DC$, and let $DA$ be through the centre,

I say that, of the straight lines falling on the concave circumference $AEFC$ the straight line $DA$ through the centre is greatest

while $DE$ is greater than $DF$ and $DF$ than $DC$,

but of the straight lines falling on the convex circumference $HLKG$ the straight line $DG$ between the point and the diameter $AG$ is least and the nearer to the least $DG$ is always less than the more remote namely $DK$ than $DL$ and $DL$ than $DH$




For let the centre of the circle $ABC$ be taken [III 1] and let it be $M$, let $ME$ $MF$ $MC$, $MK$ $ML$ $MH$ be joined

Then since $AM$ is equal to $EM$, let $MD$ be added to each

therefore $AD$ is equal to $EM$ $MD$

But $EM$, $MD$ are greater than $ED$

therefore $AD$ is also greater than $ED$

Again since $ME$ is equal to $MF$

and $MD$ is common

therefore $EM$ $MD$ are equal to $FM$ $MD$

and the angle $EMD$ is greater than the angle $FMD$

therefore the base $ED$ is greater than the base $FD$ [I 24]

Similarly we can prove that $FD$ is greater than $CD$ therefore $DA$ is greatest while $DE$ is greater than $DF$ and $DF$ than $DC$

Next since $MK$ $KD$ are greater than $MD$ [I 20]

and $MG$ is equal to $MK$

therefore the remainder $KD$ is greater than the remainder $GD$

so that $GD$ is less than $KD$

And since on $MD$ one of the sides of the triangle $MLD$ two straight lines $MK$ $KD$ were constructed meeting within the triangle

therefore $MK$ $KD$ are less than $ML$ $LD$ [I 21]

and $MK$ is equal to $ML$,

therefore the remainder $DK$ is less than the remainder $DL$

Similarly we can prove that $DL$ is also less than $DH$

therefore $DG$ is least while $DK$ is less than $DL$ and $DL$ than $DH$

I say also that only two equal straight lines will fall from the point $D$ on the

circle, one on each side of the least $DG$

On the straight line $MD$, and at the point $M$ on it, let the angle $DMB$ be constructed equal to the angle $KMD$, and let $DB$ be joined

Then, since $MK$ is equal to $MB$,

and $MD$ is common

the two sides $KM$, $MD$ are equal to the two sides $BM$ $MD$ respectively,

*and the angle $KMD$ is equal to the angle $BMD$,*

therefore the base $DK$ is equal to the base $DB$ [I 4]

I say that no other straight line equal to the straight line $DK$ will fall on the circle from the point $D$

For, if possible let a straight line so fall and let it be $DN$

Then, since $DK$ is equal to $DN$,

while $DK$ is equal to $DB$

$DB$ is also equal to $DN$,

that is, the nearer to the least $DG$ equal to the more remote which was proved impossible

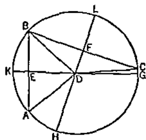
Therefore no more than two equal straight lines will fall on the circle $ABC$ from the point $D$, one on each side of $DG$ the least

Therefore etc Q E D

PROPOSITION 9

*If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle*

Let $ABC$ be a circle and $D$ a point within it, and from $D$ let more than two equal straight lines, namely $DA$ $DB$ $DC$, fall on the circle $ABC$,




I say that the point $D$ is the centre of the circle $ABC$

For let $AB$ $BC$ be joined and bisected at the points $E$ $F$, and let $ED$, $FD$ be joined and drawn through to the points $G$ $K$, $H$, $L$

Then since $AE$ is equal to $EB$ and $ED$ is common

the two sides $AE$ $ED$ are equal to the two sides $BE$ $ED$

and the base $DA$ is equal to the base $DB$

therefore the angle $AED$ is equal to the angle $BED$ [I 8]

Therefore each of the angles $AED$, $BED$ is right [I Def 10]

therefore $GK$ cuts $AB$ into two equal parts and at right angles

And since if in a circle a straight line cut a straight line into two equal parts and at right angles the centre of the circle is on the cutting straight line [III 1 Por]

the centre of the circle is on $GK$

For the same reason

the centre of the circle $ABC$ is also on $HL$

And the straight lines $GK$, $HL$ have no other point common but the point $D$

therefore the point $D$ is the centre of the circle $ABC$

Therefore etc Q E

## PROPOSITION 10

*A circle does not cut a circle at more points than two*

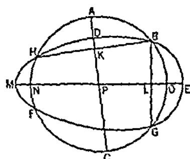
For if possible let the circle $ABC$ cut the circle $DEF$ at more points than two namely $B$ $C$ $F$, $H$,

let $BH$ $BG$ be joined and bisected at the points $K$, $L$,
and from $K$ $L$ let $KC$, $LM$ be drawn at right angles to $BH$, $BG$ and carried through to the points $A$ $E$

Then since in the circle $ABC$ a straight line $AC$ cuts a straight line $BH$ into two equal parts and at right angles

the centre of the circle $ABC$ is on $AC$

[III 1, Por]




Again since in the same circle $ABC$ a straight line $NO$ cuts a straight line $BG$ into two equal parts and at right angles

the centre of the circle $ABC$ is on $NO$

But it was also proved to be on $AC$, and *the straight lines $AC$ $NO$ meet at no point* except at $P$,

therefore the point $P$ is the centre of the circle $ABC$

Similarly we can prove that $P$ is also the centre of the circle $DEF$,
therefore the two circles $ABC$, $DEF$ which cut one another have the same centre $P$ which is impossible [III 5]

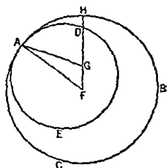
Therefore etc Q E D

## PROPOSITION 11

*If two circles touch one another internally and their centres be taken the straight line joining their centres, if it be also produced will fall on the point of contact of the circles*

For let the two circles $ABC$ $ADE$ touch one another internally at the point $A$ and let the centre $F$ of the circle $ABC$ and the centre $G$ of $ADE$, be taken,

I say that the straight line joined from $G$ to $F$ and produced will fall on $A$




For suppose it does not but if possible let it fall as $FGH$ and let $AF$ $AG$ be joined

Then since $AG$ $GF$ are greater than $FA$ that is than $FH$

let $FG$ be subtracted from each

therefore the remainder $AG$ is greater than the remainder $GH$

But $AG$ is equal to $GD$,

therefore $GD$ is also greater than $GH$

the less than the greater which is impossible

Therefore the straight line joined from $F$ to $G$ will not fall outside,

therefore it will fall at $A$ on the point of contact

Therefore etc Q E D

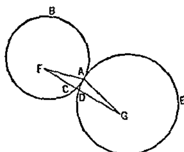
PROPOSITION 12

*If two circles touch one another externally, the straight line joining their centres will pass through the point of contact*

For let the two circles $ABC$, $ADE$ touch one another externally at the point $A$, and let the centre $F$ of $ABC$ and the centre $G$ of $ADE$, be taken

I say that the straight line joined from $F$ to $G$ will pass through the point of contact at $A$

For suppose it does not but, if possible, let it pass as $FCDG$, and let $AF$, $AG$ be joined




Then since the point $F$ is the centre of the circle $ABC$,

$FA$ is equal to $FC$

Again, since the point $G$ is the centre of the circle $ADE$

$GA$ is equal to $GD$

But $FA$ was also proved equal to $FC$, therefore $FA$, $AG$ are equal to $FC$, $GD$ so that the whole $FG$ is greater than $FA$, $AG$,

but it is also less [I 20] which is impossible

Therefore the straight line joined from $F$ to $G$ will not fail to pass through the point of contact at $A$,

therefore it will pass through it

Therefore etc Q E D

PROPOSITION 13

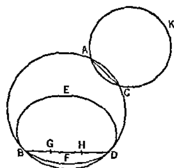
*A circle does not touch a circle at more points than one whether it touch it internally or externally*

For, if possible let the circle $ABDC$ touch the circle $EBFD$ first internally, at more points than one namely $D$ $B$

Let the centre $G$ of the circle $ABDC$ and the centre $H$ of $EBFD$ be taken

Therefore the straight line joined from $G$ to $H$ will fall on $B$, $D$ [III 11]

Let it so fall as $BGHD$




Then since the point $G$ is the centre of the circle $ABCD$

$BG$ is equal to $GD$,

therefore $BG$ is greater than $HD$

therefore $BH$ is much greater than $HD$

Again since the point $H$ is the centre of the circle $EBFD$

$BH$ is equal to $HD$,

but it was also proved much greater than it which is impossible

Therefore a circle does not touch a circle internally at more points than one

I say further that neither does it so touch it externally

For if possible let the circle $ACK$ touch the circle $ABDC$ at more points than one namely $A$, $C$,

and let $AC$ be joined

Then, since on the circumference of each of the circles $ABDC$, $ACK$ two points $A$, $C$ have been taken at random, the straight line joining the points will fall within each circle, [III 2]
but it fell within the circle $ABDC$ and outside $ACK$ [III Def 3] which is absurd

Therefore a circle does not touch a circle externally at more points than one

And it was proved that neither does it so touch it internally

Therefore etc Q E D

### Proposition 14

*In a circle equal straight lines are equally distant from the centre and those which are equally distant from the centre are equal to one another*

Let $ABDC$ be a circle and let $AB$, $CD$ be equal straight lines in it,

I say that $AB$, $CD$ are equally distant from the centre

For let the centre of the circle $ABDC$ be taken [III 1] and let it be $E$, from $E$ let $EF$, $EG$ be drawn perpendicular to $AB$ $CD$ and let $AF$ $EC$ be joined

Then since a straight line $EF$ through the centre cuts a straight line $AB$ not through the centre at right angles it also bisects it [III 3]

Therefore $AF$ is equal to $FB$,

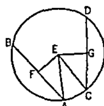
therefore $AB$ is double of $AF$

For the same reason

$CD$ is also double of $CG$,

and $AB$ is equal to $CD$

therefore $AF$ is also equal to $CG$




And since $AE$ is equal to $EC$,

the square on $AE$ is also equal to the square on $EC$

But the squares on $AF$ $EF$ are equal to the square on $AE$ for the angle at $F$ is right,
and the squares on $EG$ $GC$ are equal to the square on $EC$, for the angle at $G$ is right, [I 47]

therefore the squares on $AF$ $FE$ are equal to the squares on $CG$ $GE$,
of which the square on $AF$ is equal to the square on $CG$ for $AF$ is equal to $CG$,
therefore the square on $FE$ which remains is equal to the square on $EG$
therefore $EF$ is equal to $EG$

But in a circle straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal [III Def 4],

therefore $AB$ $CD$ are equally distant from the centre

Next let the straight lines $AB$ $CD$ be equally distant from the centre that is let $EF$ be equal to $EG$

I say that $AB$ is also equal to $CD$

For with the same construction we can prove similarly, that $AB$ is double of $AF$ and $CD$ of $CG$

And since $AE$ is equal to $CE$

the square on $AE$ is equal to the square on $CE$

But the squares on $EF$ $FA$ are equal to the square on $AE$ and the squares on $EG$ $GC$ equal to the square on $CE$ [I 47]

Therefore the squares on $EF$ $FA$ are equal to the squares on $EG$ $GC$,
of which the square on $EF$ is equal to the square on $EG$ for $EF$ is equal to $EG$,

therefore the square on $AF$ which remains is equal to the square on $CG$,
therefore $AF$ is equal to $CG$
And $AB$ is double of $AF$, and $CD$ double of $CG$
therefore $AB$ is equal to $CD$
Therefore etc
Q E D

## Proposition 15

*Of straight lines in a circle the diameter is greatest and of the rest the nearer to the centre is always greater than the more remote*

Let $ABCD$ be a circle, let $AD$ be its diameter and $E$ the centre, and let $BC$ be nearer to the diameter $AD$, and $FG$ more remote

I say that $AD$ is greatest and $BC$ greater than $FG$

For from the centre $E$ let $EH$, $EK$ be drawn perpendicular to $BC$ $FG$

Then since $BC$ is nearer to the centre and $FG$ more remote $EK$ is greater than $EH$ [III Def 5]

Let $EL$ be made equal to $EH$ through $L$ let $LM$ be drawn at right angles to $EK$ and carried through to $N$, and let $ME$ $EN$, $FE$, $EG$ be joined

Then since $EH$ is equal to $EL$
$BC$ is also equal to $MN$ [III 14]

Again since $AE$ is equal to $EM$ and $ED$ to $EN$,
$AD$ is equal to $ME$, $EN$

But $ME$, $EN$ are greater than $MN$, [I 20]
and $MN$ is equal to $BC$,
*therefore $AD$ is greater than $BC$*

And since the two sides $ME$ $EN$ are equal to the two sides $FE$ $EG$
and the angle $MEN$ greater than the angle $FEG$
therefore the base $MN$ is greater than the base $FG$ [I 24]

But $MN$ was proved equal to $BC$

Therefore the diameter $AD$ is greatest and $BC$ greater than $FG$

Therefore etc
Q E D

## Proposition 16

*The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle and into the space between the straight line and the circumference another straight line cannot be interposed further the angle of the semicircle is greater and the remaining angle less, than any acute rectilineal angle*

Let $ABC$ be a circle about $D$ as centre and $AB$ as diameter,

I say that the straight line drawn from $A$ at right angles to $AB$ from its extremity will fall outside the circle

For suppose it does not but if possible let it fall within as $CA$ and let $DC$ be joined

Since $DA$ is equal to $DC$
the angle $DAC$ is also equal to the angle $ACD$ [I 5]

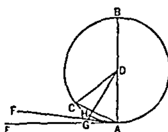
But the angle $DAC$ is right
therefore the angle $ACD$ is also right

thus, in the triangle $ACD$ the two angles $DAC$ $ACD$ are equal to two right angles which is impossible [I 17]

Therefore the straight line drawn from the point $A$ at right angles to $BA$ will not fall within the circle

Similarly we can prove that neither will it fall on the circumference,

therefore it will fall outside




Let it fall as $AE$,

I say next that into the space between the straight line $AE$ and the circumference $CHA$ another straight line cannot be interposed

For if possible let another straight line be so interposed, as $FA$ and let $DG$ be drawn from the point $D$ perpendicular to $FA$

Then, since the angle $AGD$ is right,

and the angle $DAG$ is less than a right angle,

$AD$ is greater than $DG$ [I 19]

But $DA$ is equal to $DH$,

therefore $DH$ is greater than $DG$, the less than the greater which is impossible

Therefore another straight line cannot be interposed into the space between the straight line and the circumference

I say further that the angle of the semicircle contained by the straight line $BA$ and the circumference $CHA$ is greater than any acute rectilineal angle and the remaining angle contained by the circumference $CHA$ and the straight line $AE$ is less than any acute rectilineal angle

For if there is any rectilineal angle greater than the angle contained by the straight line $BA$ and the circumference $CHA$ and any rectilineal angle less than the angle contained by the circumference $CHA$ and the straight line $AE$, then into the space between the circumference and the straight line $AE$ a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line $BA$ and the circumference $CHA$, and another angle contained by straight lines which is less than the angle contained by the circumference $CHA$ and the straight line $AE$

But such a straight line cannot be interposed

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line $BA$ and the circumference $CHA$ nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference $CHA$ and the straight line $AE$ —

PORISM From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle Q E D

## PROPOSITION 17

*From a given point to draw a straight line touching a given circle*

Let $A$ be the given point and $BCD$ the given circle

thus it is required to draw from the point $A$ a straight line touching the circle $BCD$

For let the centre $F$ of the circle be taken, [III 1]

let $AE$ be joined and with centre $E$ and distance $EA$ let the circle $AFG$ be described,

from $D$ let $DF$ be drawn at right angles to $EA$,

and let $EF$ $AB$ be joined

I say that $AB$ has been drawn from the point $A$ touching the circle $BCD$

For, since $E$ is the centre of the circles $BCD$ $AFG$

$EA$ is equal to $EF$, and $ED$ to $EB$,

therefore the two sides $AE$, $EB$ are equal to the two sides $FE$, $ED$,

and they contain a common angle, the angle at $E$,

therefore the base $DF$ is equal to the base $AB$,

and the triangle $DEF$ is equal to the triangle $BEA$,

and the remaining angles to the remaining angles [I 4]

therefore the angle $EDF$ is equal to the angle $EBA$

But the angle $EDF$ is right,

therefore the angle $EBA$ is also right

Now $EB$ is a radius

and the straight line drawn at right angles to the diameter of a circle from its extremity, touches the circle, [III 16 Por]

therefore $AB$ touches the circle $BCD$

Therefore from the given point $A$ the straight line $AB$ has been drawn touching the circle $BCD$ Q E F

## Proposition 18

*If a straight line touch a circle, and a straight line be joined from the centre to the point of contact the straight line so joined will be perpendicular to the tangent*

For let a straight line $DE$ touch the circle $ABC$ at the point $C$, let the centre $F$ of the circle $ABC$ be taken, and let $FC$ be joined from $F$ to $C$,

I say that $FC$ is perpendicular to $DE$

For, if not let $FG$ be drawn from $F$ perpendicular to $DE$

Then since the angle $FGC$ is right

the angle $FCG$ is acute [I 17]

and the greater angle is subtended by the greater side, [I 19]

*therefore $FC$ is greater than $FG$*

But $FC$ is equal to $FB$,

therefore $FB$ is also greater than $FG$

the less than the greater which is impossible

Therefore $FG$ is not perpendicular to $DE$

Similarly we can prove that neither is any other straight line except $FC$,

therefore $FC$ is perpendicular to $DE$

Therefore etc Q E D

## Proposition 19

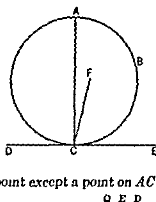
*If a straight line touch a circle and from the point of contact a straight line be drawn at right angles to the tangent the centre of the circle will be on the straight line so drawn*

For let a straight line $DE$ touch the circle $ABC$ at the point $C$ and from $C$ let $CA$ be drawn at right angles to $DE$

I say that the centre of the circle is on $AC$

For suppose it is not but if possible let $F$ be the centre

and let $CF$ be joined




Since a straight line $DE$ touches the circle $ABC$,
and $FC$ has been joined from the centre to the point of contact,

$FC$ is perpendicular to $DE$, [III 18]

therefore the angle $FCE$ is right

But the angle $ACE$ is also right,
therefore the angle $FCE$ is equal to the angle $ACE$,

the less to the greater which is impossible

Therefore $F$ is not the centre of the circle $ABC$

Similarly we can prove that neither is any other point except a point on $AC$

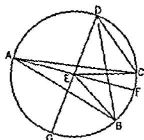
Therefore etc Q E D

## Proposition 20

*In a circle the angle at the centre is double of the angle at the circumference when the angles have the same circumference as base*

Let $ABC$ be a circle let the angle $BEC$ be an angle at its centre, and the angle $BAC$ an angle at the circumference and let them have the same circumference $BC$ as base

I say that the angle $BEC$ is double of the angle $BAC$




For let $AE$ be joined and drawn through to $F$

Then since $EA$ is equal to $EB$,
the angle $EAB$ is also equal to the angle $EBA$, [I 5]
therefore the angles $EAB$, $EBA$ are double of the angle $EAB$

But the angle $BEF$ is equal to the angles $EAB$ $EBA$ [I 32]
therefore the angle $BEF$ is also double of the angle $EAB$

For the same reason

the angle $FEC$ is also double of the angle $EAC$

Therefore the whole angle $BEC$ is double of the whole angle $BAC$

Again let another straight line be inflected and let there be another angle $BDC$, let $DE$ be joined and produced to $G$

Similarly then we can prove that the angle $GEC$ is double of the angle $EDC$,

of which the angle $GEB$ is double of the angle $EDB$

therefore the angle $BEC$ which remains is double of the angle $BDC$

Therefore etc Q E D

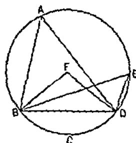
## Proposition 21

*In a circle the angles in the same segment are equal to one another*

Let $ABCD$ be a circle and let the angles $BAD$ $BED$ be angles in the same segment $BAED$

I say that the angles $BAD$ $BED$ are equal to one another

For let the centre of the circle $ABCD$ be taken and let it be $F$, let $BF$, $FD$ be joined




Now, since the angle $BFD$ is at the centre, and the angle $BAD$ at the circumference and they have the same circumference $BCD$ as base therefore the angle $BFD$ is double of the angle $BAD$ [III 20]

For the same reason the angle $BFD$ is also double of the angle $BFD$ therefore the angle $BAD$ is equal to the angle $BED$

Therefore etc Q E D

## Proposition 22

*The opposite angles of quadrilaterals in circles are equal to two right angles*

Let $ABCD$ be a circle and let $ABCD$ be a quadrilateral in it, I say that the opposite angles are equal to two right angles

Let $AC$ $BD$ be joined

Then since in any triangle the three angles are equal to two right angles [I 32] the three angles $CAB$ $ABC$, $BCA$ of the triangle $ABC$ are equal to two right angles

But the angle $CAB$ is equal to the angle $BDC$ for they are in the same segment $BADC$ [III 21] and the angle $ACB$ is equal to the angle $ADB$ for they are in the same segment $ADCB$

therefore the whole angle $ADC$ is equal to the angles $BAC$ $ACB$

Let the angle $ABC$ be added to each

therefore the angles $ABC$, $BAC$ $ACB$ are equal to the angles $ABC$ $ADC$

But the angles $ABC$ $BAC$, $ACB$ are equal to two right angles,

therefore the angles $ABC$ $ADC$ are also equal to two right angles

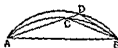
Similarly we can prove that the angles $BAD$, $DCB$ are also equal to two right angles

Therefore etc Q E D

## Proposition 23

*On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side*

For if possible on the same straight line $AB$ let two similar and unequal segments of circles $ACB$ $ADB$ be constructed on the same side




let $ACD$ be drawn through and let $CB$ $DB$ be joined

Then since the segment $ACB$ is similar to the segment $ADB$, and similar segments of circles are those which admit equal angles [III Def 11] the angle $ACB$ is equal to the angle $ADB$, the exterior to the interior which is impossible [I 16]

Therefore etc Q E D

PROPOSITION 24

*Similar segments of circles on equal straight lines are equal to one another*

For let $AEB$, $CFD$ be similar segments of circles on equal straight lines $AB$, $CD$,

I say that the segment $AEB$ is equal to the segment $CFD$

For, if the segment $AEB$ be applied to $CFD$ and if the point $A$ be placed on $C$ and the straight line $AB$ on $CD$,

the point $B$ will also coincide with the point $D$, because $AB$ is equal to $CD$,




and $AB$ coinciding with $CD$,

the segment $AEB$ will also coincide with $CFD$

For if the straight line $AB$ coincide with $CD$ but the segment $AEB$ do not coincide with $CFD$

it will either fall within it or outside it

or it will fall awry as $CGD$ and a circle cuts a circle at more points than two which is impossible [III 10]

Therefore if the straight line $AB$ be applied to $CD$ the segment $AEB$ will not fail to coincide with $CFD$ also,

therefore it will coincide with it and will be equal to it

Therefore etc Q E D

PROPOSITION 25

*Given a segment of a circle to describe the complete circle of which it is a segment*

Let $ABC$ be the given segment of a circle,

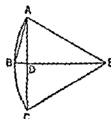
thus it is required to describe the complete circle belonging to the segment $ABC$ that is of which it is a segment

For let $AC$ be bisected at $D$, let $DB$ be drawn from the point $D$ at right angles to $AC$ and let $AB$ be joined

the angle $ABD$ is then greater than equal to or less than the angle $BAD$

First let it be greater

and on the straight line $BA$ and at the point $A$ on it, let the angle $BAE$ be constructed equal to the angle $ABD$, let $DB$ be drawn through to $E$ and let $EC$ be joined




Then since the angle $ABE$ is equal to the angle $BAE$

the straight line $EB$ is also equal to $EA$ [I 6]

And since $AD$ is equal to $DC$ and $DE$ is common

the two sides $AD$, $DE$ are equal to the two sides $CD$, $DE$ respectively

and the angle $ADE$ is equal to the angle $CDE$ for each is right,

therefore the base $AE$ is equal to the base $CE$

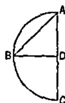
But $AE$ was proved equal to $BE$

therefore $BE$ is also equal to $CE$,

therefore the three straight lines $AE$ $EB$, $EC$ are equal to one another

Therefore the circle drawn with centre $E$ and distance one of the straight lines $AE$, $EB$ $EC$ will also pass through the remaining points and will have been completed [III 9]

Therefore, given a segment of a circle, the complete circle has been described




And it is manifest that the segment $ABC$ is less than a semi circle because the centre $E$ happens to be outside it

Similarly, even if the angle $ABD$ be equal to the angle $BAD$,

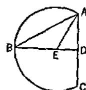
$AD$ being equal to each of the two $BD$ $DC$

the three straight lines $DA$, $DB$, $DC$ will be equal to one another,

$D$ will be the centre of the completed circle

and $ABC$ will clearly be a semicircle

But, if the angle $ABD$ be less than the angle $BAD$, and if we construct on the straight line $BA$ and at the point $A$ on it an angle equal to the angle $ABD$ the centre will fall on $DB$ within the segment $ABC$, and the segment $ABC$ will clearly be greater than a semicircle



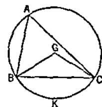

Therefore, given a segment of a circle, the complete circle has been described Q E F

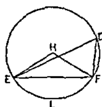
PROPOSITION 26

*In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences*

Let $ABC$, $DEF$ be equal circles, and in them let there be equal angles, namely at the centres the angles $BGC$, $EHF$ and at the circumferences the angles $BAC$, $EDF$

I say that the circumference $BKC$ is equal to the circumference $ELF$




For let $BC$ $EF$ be joined

Now, since the circles $ABC$, $DEF$ are equal

the radii are equal

Thus the two straight lines $BG$, $GC$ are equal to the two straight lines $EH$, $HF$,

and the angle at $G$ is equal to the angle at $H$

therefore the base $BC$ is equal to the base $EF$ [I 4]

And since the angle at $A$ is equal to the angle at $D$

the segment $BAC$ is similar to the segment $EDF$ [III Def 11]

and they are upon equal straight lines

But similar segments of circles on equal straight lines are equal to one another, [III 24]

therefore the segment $BAC$ is equal to $EDF$

But the whole circle $ABC$ is also equal to the whole circle $DEF$,

therefore the circumference $BKC$ which remains is equal to the circumference $ELF$

Therefore etc Q E D

## Proposition 27

*In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences*

For in equal circles *ABC*, *DEF*, on equal circumferences *BC EF*, let the angles *BGC*, *EHF* stand at the centres *G H*, and the angles *BAC*, *EDF* at the circumferences,

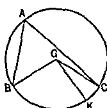
I say that the angle *BGC* is equal to the angle *EHF*,

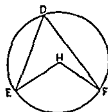
and the angle *BAC* is equal to the angle *EDF*

For, if the angle *BGC* is unequal to the angle *EHF*,

one of them is greater

Let the angle *BGC* be greater and on the straight line *BG*, and at the point *G* on it let the angle *BGK* be constructed equal to the angle *EHF* [I 23]




Now equal angles stand on equal circumferences, when they are at the centres [III 26]

therefore the circumference *BK* is equal to the circumference *EF*

But *EF* is equal to *BC*,

therefore *BK* is also equal to *BC* the less to the greater which is impossible

Therefore the angle *BGC* is not unequal to the angle *EHF*,

therefore it is equal to it

And the angle at *A* is half of the angle *BGC*

and the angle at *D* half of the angle *EHF*, [III 20]

therefore the angle at *A* is also equal to the angle at *D*

Therefore etc Q E D

## Proposition 28

*In equal circles equal straight lines cut off equal circumferences the greater equal to the greater and the less to the less*

Let *ABC*, *DEF* be equal circles and in the circles let *AB*, *DE* be equal straight lines cutting off *ACB*, *DFE* as greater circumferences and *AGB*, *DHE* as lesser

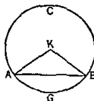
I say that the greater circumference *ACB* is equal to the greater circumference *DFE* and the less circumference *AGB* to *DHE*

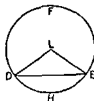
For let the centres *K L* of the circles be taken and let *AK KB DL LE* be joined

Now since the circles are equal

the radii are also equal

therefore the two sides *AK KB* are equal to the two sides *DL LE*,




and the base *AB* is equal to the base *DE*

therefore the angle *AKB* is equal to the angle *DLE* [I 8]

But equal angles stand on equal circumferences when they are at the centres [III 26]

therefore the circumference *AGB* is equal to *DHE*

And the whole circle *ABC* is also equal to the whole circle *DEF*,

therefore the circumference $ACB$ which remains is also equal to the circumference $DFE$ which remains

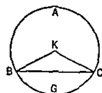
Therefore etc Q E D

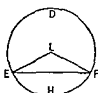
## Proposition 29

*In equal circles equal circumferences are subtended by equal straight lines*

Let $ABC$ $DEF$ be equal circles and in them let equal circumferences $BGC$ $EHF$ be cut off and let the straight lines $BC$, $EF$ be joined

I say that $BC$ is equal to $EF$






For let the centres of the circles be taken and let them be $K$ $L$ let $BK$ $KC$ $EL$ $LF$ be joined

Now, since the circumference $BGC$ is equal to the circumference $EHF$,

the angle $BKC$ is also equal to the angle $ELF$ [III 27]

And since the circles $ABC$ $DEF$ are equal

the radii are also equal

therefore the two sides $BK$, $KC$ are equal to the two sides $EL$ $LF$ and they contain equal angles

therefore the base $BC$ is equal to the base $EF$ [I 4]

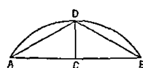
Therefore etc Q E D

## Proposition 30

*To bisect a given circumference*

Let $ADB$ be the given circumference,

thus it is required to bisect the circumference $ADB$




Let $AB$ be joined and bisected at $C$ from the point $C$ let $CD$ be drawn at right angles to the straight line $AB$ and let $AD$ $DB$ be joined

Then since $AC$ is equal to $CB$ and $CD$ is common

the two sides $AC$ $CD$ are equal to the two sides $BC$ $CD$

and the angle $ACD$ is equal to the angle $BCD$ for each is right

therefore the base $AD$ is equal to the base $DB$ [I 4]

But equal straight lines cut off equal circumferences the greater equal to the greater, and the less to the less [III 28]

and each of the circumferences $AD$ $DB$ is less than a semicircle

therefore the circumference $AD$ is equal to the circumference $DB$

Therefore the given circumference has been bisected at the point $D$

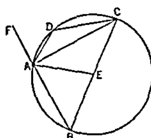
Q E F

## Proposition 31

*In a circle the angle in the semicircle is right that in a greater segment less than a right angle and that in a less segment greater than a right angle and further the angle of the greater segment is greater than a right angle and the angle of the less segment less than a right angle*

Let $ABCD$ be a circle let $BC$ be its diameter and $E$ its centre and $AC$, $AD$ $DC$ be joined

I say that the angle $BAC$ in the semicircle $BAC$ is right
the angle $ABC$ in the segment $ABC$ greater than the semicircle is less than a right angle,
and the angle $ADC$ in the segment $ADC$ less than the semicircle is greater than a right angle




Let $AE$ be joined, and let $BA$ be carried through to $F$

Then since $BE$ is equal to $EA$
the angle $ABE$ is also equal to the angle $BAE$ [I 5]

Again since $CE$ is equal to $EA$,
the angle $ACE$ is also equal to the angle $CAE$ [I 5]

Therefore the whole angle $BAC$ is equal to the two angles $ABC$, $ACB$

But the angle $FAC$ exterior to the triangle $ABC$ is also equal to the two angles $ABC$, $ACB$, [I 32]

therefore the angle $BAC$ is also equal to the angle $FAC$,

therefore each is right, [I Def 10]

therefore the angle $BAC$ in the semicircle $BAC$ is right

Next since in the triangle $ABC$ the two angles $ABC$, $BAC$ are less than two right angles [I 17]

and the angle $BAC$ is a right angle,

the angle $ABC$ is less than a right angle,

and it is the angle in the segment $ABC$ greater than the semicircle

Next since $ABCD$ is a quadrilateral in a circle
and the opposite angles of quadrilaterals in circles are equal to two right angles [III 22]

while the angle $ABC$ is less than a right angle

therefore the angle $ADC$ which remains is greater than a right angle

and it is the angle in the segment $ADC$ less than the semicircle

I say further that the angle of the greater segment namely that contained by the circumference $ABC$ and the straight line $AC$, is greater than a right angle
and the angle of the less segment namely that contained by the circumference $ADC$ and the straight line $AC$ is less than a right angle

This is at once manifest

For since the angle contained by the straight lines $BA$, $AC$ is right
the angle contained by the circumference $ABC$ and the straight line $AC$ is greater than a right angle

Again since the angle contained by the straight lines $AC$ $AF$ is right
the angle contained by the straight line $CA$ and the circumference $ADC$ is less than a right angle

Therefore etc Q E D

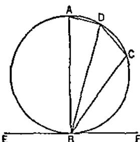
## Proposition 32

*If a straight line touch a circle and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle*

For let a straight line $EF$ touch the circle $ABCD$ at the point $B$, and from

the point $B$ let there be drawn across, in the circle $ABCD$ a straight line $BD$ cutting it,

I say that the angles which $BD$ makes with the tangent $EF$ will be equal to the angles in the alternate segments of the circle that is that the angle $FBD$ is equal to the angle constructed in the segment $BAD$ and the angle $EBD$ is equal to the angle constructed in the segment $DCB$




For let $BA$ be drawn from $B$ at right angles to $EF$,

let a point $C$ be taken at random on the circumference $BD$,

and let $AD$ $DC$, $CB$ be joined

Then, since a straight line $EF$ touches the circle $ABCD$ at $B$,

and $BA$ has been drawn from the point of contact at right angles to the tangent the centre of the circle $ABCD$ is on $BA$ [III 19]

Therefore $BA$ is a diameter of the circle $ABCD$

therefore the angle $ADB$, being an angle in a semicircle is right [III 31]

*Therefore the remaining angles $BAD$ $ABD$ are equal to one right angle* [I 32]

But the angle $ABF$ is also right,

therefore the angle $ABF$ is equal to the angles $BAD$ $ABD$

Let the angle $ABD$ be subtracted from each

therefore the angle $DBF$ which remains is equal to the angle $BAD$ in the alternate segment of the circle

Next since $ABCD$ is a quadrilateral in a circle,

its opposite angles are equal to two right angles [III 22]

But the angles $DBF$ $DBE$ are also equal to two right angles,

therefore the angles $DBF$ $DBE$ are equal to the angles $BAD$ $BCD$,

of which the angle $BAD$ was proved equal to the angle $DBF$,

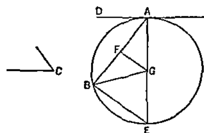
therefore the angle $DBE$ which remains is equal to the angle $DCB$ in the alternate segment $DCB$ of the circle

Therefore etc Q E D

## Proposition 33

*On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle*

Let $AB$ be the given straight line and the angle at $C$ the given rectilineal angle,




thus it is required to describe on the given straight line $AB$ a segment of a circle admitting an angle equal to the angle at $C$

The angle at $C$ is then acute or right or obtuse

First let it be acute and as in the first figure on the straight line $AB$ and at the point $A$ let the angle $BAD$ be constructed equal to the angle at $C$

therefore the angle $BAD$ is also acute

Let $AE$ be drawn at right angles to $DA$, let $AB$ be bisected at $F$, let $FG$ be drawn from the point $F$ at right angles to $AB$, and let $GB$ be joined

Then, since $AF$ is equal to $FB$

and $FG$ is common

the two sides $AF$ $FG$ are equal to the two sides $BF$ $FG$,

and the angle $AFG$ is equal to the angle $BFG$,

therefore the base $AG$ is equal to the base $BG$ [I 4]

Therefore the circle described with centre $G$ and distance $GA$ will pass through $B$ also

Let it be drawn, and let it be $ABE$,

let $EB$ be joined

Now since $AD$ is drawn from $A$ the extremity of the diameter $AE$, at right angles to $AE$

therefore $AD$ touches the circle $ABE$ [III 16 Por]

Since then a straight line $AD$ touches the circle $ABE$

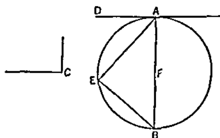
and from the point of contact at $A$ a straight line $AB$ is drawn across in the circle $ABE$

the angle $DAB$ is equal to the angle $AEB$ in the alternate segment of the circle [III 32]

But the angle $DAB$ is equal to the angle at $C$,

therefore the angle at $C$ is also equal to the angle $AEB$

Therefore on the given straight line $AB$ the segment $AEB$ of a circle has been described admitting the angle $AEB$ equal to the given angle the angle at $C$




Next let the angle at $C$ be right and let it be again required to describe on $AB$ a segment of a circle admitting an angle equal to the right angle at $C$

Let the angle $BAD$ be constructed equal to the right angle at $C$ as is the case in the second figure

let $AB$ be bisected at $F$ and with centre $F$ and distance either $FA$ or $FB$ let the circle $AEB$ be described

Therefore the straight line $AD$ touches the circle $ABE$ because the angle at $A$ is right [III 16 Por]

And the angle $BAD$ is equal to the angle in the segment $AEB$ for the latter too is itself a right angle being an angle in a semicircle [III 31]

But the angle $BAD$ is also equal to the angle at $C$

Therefore the angle $AEB$ is also equal to the angle at $C$

Therefore again the segment $AEB$ of a circle has been described on $AB$ admitting an angle equal to the angle at $C$

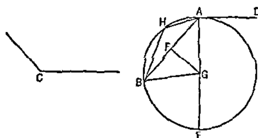
Next let the angle at $C$ be obtuse

and on the straight line $AB$ and at the point $A$ let the angle $BAD$ be constructed equal to it as is the case in the third figure

let $AE$ be drawn at right angles to $AD$ let $AB$ be again bisected at $F$ let $FG$ be drawn at right angles to $AB$ and let $GB$ be joined

Then, since $AF$ is again equal to $FB$,

and $FG$ is common,




the two sides $AF$, $FG$ are equal to the two sides $BF$ $FG$,

and the angle $AFG$ is equal to the angle $BFG$,

therefore the base $AG$ is equal to the base $BG$ [I 4]

Therefore the circle described with centre $G$ and distance $GA$ will pass through $B$ also, let it so pass, as $AEB$

Now, since $AD$ is drawn at right angles to the diameter $AE$ from its extremity,

$AD$ touches the circle $AEB$ [III 16, Por]

And $AB$ has been drawn across from the point of contact at $A$, therefore the angle $BAD$ is equal to the angle constructed in the alternate segment $AHB$ of the circle [III 32]

But the angle $BAD$ is equal to the angle at $C$

Therefore the angle in the segment $AHB$ is also equal to the angle at $C$

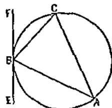
Therefore on the given straight line $AB$ the segment $AHB$ of a circle has been described admitting an angle equal to the angle at $C$ Q E F

PROPOSITION 34

*From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle*

Let $ABC$ be the given circle and the angle at $D$ the given rectilineal angle, thus it is required to cut off from the circle $ABC$ a segment admitting an angle equal to the given rectilineal angle, the angle at $D$

Let $EF$ be drawn touching $ABC$ at the point $B$ and on the straight line $FB$ and at the point $B$ on it let the angle $FBC$ be constructed equal to the angle at $D$ [I 23]




Then since a straight line $EF$ touches the circle $ABC$,

and $BC$ has been drawn across from the point of contact at $B$

the angle $FBC$ is equal to the angle constructed in the alternate segment $BAC$ [III 32]

But the angle $FBC$ is equal to the angle at $D$,

therefore the angle in the segment $BAC$ is equal to the angle at $D$

Therefore from the given circle $ABC$ the segment $BAC$ has been cut off admitting an angle equal to the given rectilineal angle the angle at $D$ Q E F

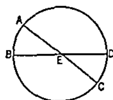
PROPOSITION 35

*If in a circle two straight lines cut one another the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other*

For in the circle $ABCD$ let the two straight lines $AC$ $BD$ cut one another at the point $E$

I say that the rectangle contained by $AE$, $EC$ is equal to the rectangle contained by $DE$, $FB$

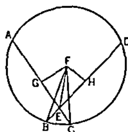
If now $AC$, $BD$ are through the centre so that $E$ is the centre of the circle $ABCD$




it is manifest that, $AE$ $EC$ $DE$ $EB$ being equal the rectangle contained by $AE$, $EC$ is also equal to the rectangle contained by $DE$, $EB$

Next let $AC$ $DB$ not be through the centre, let the centre of $ABCD$ be taken, and let it be $F$ from $F$ let $FG$, $FH$ be drawn perpendicular to the straight lines $AC$, $DB$, and let $FB$ $FC$, $FE$ be joined

Then since a straight line $GF$ through the centre cuts a straight line $AC$ not through the centre at right angles




it also bisects it, [III 3]

therefore $AG$ is equal to $GC$

Since then, the straight line $AC$ has been cut into equal parts at $G$ and into unequal parts at $E$ the rectangle contained by $AE$ $EC$ together with the square on $EG$ is equal to the square on $GC$, [II 5]

Let the square on $GF$ be added

therefore the rectangle $AE$, $EC$ together with the squares on $GF$, $GF$ is equal to the squares on $CG$ $GF$

But the square on $FE$ is equal to the squares on $EG$ $GF$, and the square on $FC$ is equal to the squares on $CG$ $GF$, [I 47]

therefore the rectangle $AE$, $EC$ together with the square on $FE$ is equal to the square on $FC$

And $FC$ is equal to $FB$

therefore the rectangle $AE$ $EC$ together with the square on $EF$ is equal to the square on $FB$

For the same reason also

the rectangle $DE$ $EB$ together with the square on $FE$ is equal to the square on $FB$

But the rectangle $AE$ $EC$ together with the square on $FE$ was also proved equal to the square on $FB$,

therefore the rectangle $AE$ $EC$ together with the square on $FE$ is equal to the rectangle $DE$ $EB$ together with the square on $FE$

Let the square on $FE$ be subtracted from each,

therefore the rectangle contained by $AE$ $EC$ which remains is equal to the rectangle contained by $DE$ $EB$

Therefore etc Q E D

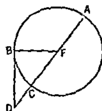
PROPOSITION 36

*If a point be taken outside a circle and from it there fall on the circle two straight lines and if one of them cut the circle and the other touch it the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent*

For let a point $D$ be taken outside the circle $ABC$ and from $D$ let the two

straight lines $DCA$, $DB$ fall on the circle $ABC$, let $DCA$ cut the circle $ABC$ and let $BD$ touch it

I say that the rectangle contained by $AD$ $DC$ is equal to the square on $DB$




Then $DCA$ is either through the centre or not through *the centre*

First let it be through the centre and let $F$ be the centre of the circle $ABC$ let $FB$ be joined,

therefore the angle $FBD$ is right [III 18]

And, since $AC$ has been bisected at $F$ and $CD$ is added to it,

the rectangle $AD$, $DC$ together with the square on $FC$ is equal to the square on $FD$ [II 6]

But $FC$ is equal to $FB$,

therefore the rectangle $AD$, $DC$ together with the square on $FB$ is equal to the square on $FD$

And the squares on $FB$, $BD$ are equal to the square on $FD$ [I 47]

therefore the rectangle $AD$, $DC$ together with the square on $FB$ is equal to the squares on $FB$, $BD$

Let the square on $FB$ be subtracted from each

therefore the rectangle $AD$ $DC$ which remains is equal to the square on the tangent $DB$

Again let $DCA$ *not be through the centre of the circle $ABC$*,

let the centre $E$ be taken, and from $E$ let $EF$ be drawn perpendicular to $AC$,

let $EB$, $EC$, $ED$ be joined

Then the angle $EBD$ is right [III 18]

And since a straight line $EF$ through the centre cuts a straight line $AC$ not through the centre at right angles,

it also bisects it [III 3]

therefore $AF$ is equal to $FC$

Now, since the straight line $AC$ has been bisected at the point $F$ and $CD$ is added to it

the rectangle contained by $AD$ $DC$ together with the square on $FC$ is equal to the square on $FD$ [II 6]

Let the square on $FE$ be added to each

therefore the rectangle $AD$, $DC$ together with the squares on $CF$ $FE$ is equal to the squares on $FD$ $FE$

But the square on $EC$ is equal to the squares on $CF$ $FE$ for the angle $EFC$ is right [I 47]

and the square on $ED$ is equal to the squares on $DF$ $FE$

therefore the rectangle $AD$, $DC$ together with the square on $EC$ is equal to the square on $ED$

*And $EC$ is equal to $EB$*

therefore the rectangle $AD$ $DC$ together with the square on $EB$ is equal to the square on $ED$

But the squares on $EB$ $BD$ are equal to the square on $ED$ for the angle $EBD$ is right [I

therefore the rectangle $AD$, $DC$ together with the square on $EB$ is equal to the squares on $EB$, $BD$

Let the square on $EB$ be subtracted from each,

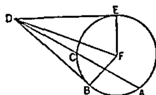
therefore the rectangle $AD$ $DC$ which remains is equal to the square on $DB$

Therefore etc Q E D

## Proposition 37

*If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it and if further the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle the straight line which falls on it will touch the circle*

For let a point $D$ be taken outside the circle $ABC$, from $D$ let the two straight lines $DCA$ $DB$ fall on the circle $ABC$, let $DCA$ cut the circle and $DB$ fall on it and let the rectangle $AD$ $DC$ be equal to the square on $DB$




I say that $DB$ touches the circle $ABC$

For let $DF$ be drawn touching $ABC$, let the centre of the circle $ABC$ be taken, and let it be $F$ let $FE$ $FB$ $FD$ be joined

Thus the angle $FED$ is right [III 18]

Now since $DE$ touches the circle $ABC$ and $DCA$ cuts it the rectangle $AD$ $DC$ is equal to the square on $DE$ [III 36]

But the rectangle $AD$ $DC$ was also equal to the square on $DB$

therefore the square on $DE$ is equal to the square on $DB$

therefore $DE$ is equal to $DB$

And $FE$ is equal to $FB$,

therefore the two sides $DE$, $EF$ are equal to the two sides $DB$, $BF$,

and $FD$ is the common base of the triangles,

therefore the angle $DEF$ is equal to the angle $DBF$ [I 8]

But the angle $DEF$ is right

therefore the angle $DBF$ is also right

And $FB$ produced is a diameter

and the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle [III 16, Por]

therefore $DB$ touches the circle

Similarly this can be proved to be the case even if the centre be on $AC$

Therefore etc Q E D

# BOOK FOUR

## *DEFINITIONS*

1 A rectilineal figure is said to be *inscribed in a rectilineal figure* when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed

2 Similarly a figure is said to be *circumscribed about a figure* when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed

3 A rectilineal figure is said to be *inscribed in a circle* when each angle of the inscribed figure lies on the circumference of the circle

4 A rectilineal figure is said to be *circumscribed about a circle,* when each side of the circumscribed figure touches the circumference of the circle

5 Similarly a circle is said to be *inscribed in a figure* when the circumference of the circle touches each side of the figure in which it is inscribed

6 A circle is said to be *circumscribed about a figure* when the circumference of the circle passes through each angle of the figure about which it is circumscribed

7 A straight line is said to be *fitted into a circle* when its extremities are on the circumference of the circle

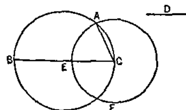
## BOOK IV PROPOSITIONS

### Proposition 1

*Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle*

Let $ABC$ be the given circle, and $D$ the given straight line not greater than the diameter of the circle,

thus it is required to fit into the circle $ABC$ a straight line equal to the straight line $D$




Let a diameter $BC$ of the circle $ABC$ be drawn

Then if $BC$ is equal to $D$ that which was enjoined will have been done for $BC$ has been fitted into the circle $ABC$ equal to the straight line $D$

But if $BC$ is greater than $D$

let $CE$ be made equal to $D$, and with centre $C$ and distance $CE$ let the circle $EAF$ be described

let $CA$ be joined

Then since the point $C$ is the centre of the circle $EAF$,

$CA$ is equal to $CE$

But $CE$ is equal to $D$,

therefore $D$ is also equal to $CA$

Therefore into the given circle $ABC$ there has been fitted $CA$ equal to the given straight line $D$ Q E F

PROPOSITION 2

*In a given circle to inscribe a triangle equiangular with a given triangle*

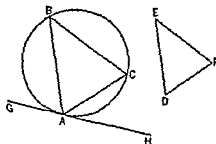
Let $ABC$ be the given circle, and $DEF$ the given triangle,

thus it is required to inscribe in the circle $ABC$ a triangle equiangular with the triangle $DEF$

Let $GH$ be drawn touching the circle $ABC$ at $A$ [III 16 Por], on the straight line $AH$, and at the point $A$ on it let the angle $HAC$ be constructed equal to the angle $DEF$,

and on the straight line $AG$ and at the point $A$ on it, let the angle $GAB$ be constructed equal to the angle $DFE$, [I 23]

let $BC$ be joined




Then since a straight line $AH$ touches the circle $ABC$,

and from the point of contact at $A$ the straight line $AC$ is drawn across in the circle,

therefore the angle $HAC$ is equal to the angle $ABC$ in the alternate segment of the circle [III 32]

But the angle $HAC$ is equal to the angle $DEF$,

therefore the angle $ABC$ is also equal to the angle $DEF$

For the same reason

the angle $ACB$ is also equal to the angle $DFE$,

therefore the remaining angle $BAC$ is also equal to the remaining angle $EDF$ [I 32]

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle Q E F

PROPOSITION 3

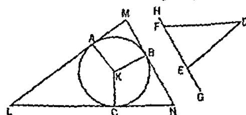
*About a given circle to circumscribe a triangle equiangular with a given triangle*

Let $ABC$ be the given circle and $DEF$ the given triangle,

thus it is required to circumscribe about the circle $ABC$ a triangle equiangular with the triangle $DEF$

Let $EF$ be produced in both directions to the points $G$, $H$

let the centre $K$ of the circle $ABC$ be taken [III 1] and let the straight line $KB$ be drawn across at random,




on the straight line $KB$ and at the point $K$ on it, let the angle $BKA$ be constructed equal to the angle $DEG$,

and the angle $BKC$ equal to the angle $DFH$, [I 23]

and through the points *A*, *B*, *C* let *LAM*, *MBN*, *NCL* be drawn touching the circle *ABC* [III 16, Por]

Now since *LM*, *MN*, *NL* touch the circle *ABC* at the points *A*, *B*, *C*, and *KA*, *KB*, *KC* have been joined from the centre *K* to the points *A*, *B*, *C*, therefore the angles at the points *A*, *B*, *C* are right [III 18]

And since the four angles of the quadrilateral *AMBK* are equal to four right angles, inasmuch as *AMBK* is in fact divisible into two triangles, and the angles *KAM*, *KBM* are right,

therefore the remaining angles *AKB*, *AMB* are equal to two right angles

But the angles *DEG*, *DEF* are also equal to two right angles [I 13]

therefore the angles *AKB*, *AMB* are equal to the angles *DEG*, *DEF*, of which the angle *AKB* is equal to the angle *DEG*,

therefore the angle *AMB* which remains is equal to the angle *DEF* which remains

Similarly it can be proved that the angle *LNB* is also equal to the angle *DFE*, therefore the remaining angle *MLN* is equal to the angle *EDF* [I 32]

Therefore the triangle *LMN* is equiangular with the triangle *DEF*, and it has been circumscribed about the circle *ABC*

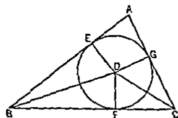
Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle Q E F

## Proposition 4

*In a given triangle to inscribe a circle*

Let *ABC* be the given triangle

thus it is required to inscribe a circle in the triangle *ABC*




Let the angles *ABC*, *ACB* be bisected by the straight lines *BD*, *CD* [I 9] and let these meet one another at the point *D*, from *D* let *DE*, *DF*, *DG* be drawn perpendicular to the straight lines *AB*, *BC*, *CA*

Now since the angle *ABD* is equal to the angle *CBD*

and the right angle *BED* is also equal to the right angle *BFD*,

*EBD*, *FBD* are two triangles having two angles equal to two angles and one side equal to one side namely that subtending one of the equal angles which is *BD* common to the triangles

therefore they will also have the remaining sides equal to the remaining sides [I 26]

therefore *DE* is equal to *DF*

For the same reason

*DG* is also equal to *DF*

Therefore the three straight lines *DE*, *DF*, *DG* are equal to one another, therefore the circle described with centre *D* and distance one of the straight lines *DE*, *DF*, *DG* will pass also through the remaining points and will touch the straight lines *AB*, *BC*, *CA* because the angles at the points *E*, *F*, *G* are right

For, if it cuts them the straight line drawn at right angles to the d

the circle from its extremity will be found to fall within the circle which was proved absurd, [III 16]
therefore the circle described with centre $D$ and distance one of the straight lines $DE$ $DF$, $DG$ will not cut the straight lines $AB$, $BC$ $CA$,
therefore it will touch them and will be the circle inscribed in the triangle $ABC$ [IV Def 5]

Let it be inscribed as $FGF$

Therefore in the given triangle $ABC$ the circle $EFG$ has been inscribed

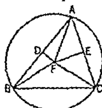
Q E F

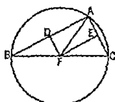
PROPOSITION 5

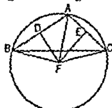
*About a given triangle to circumscribe a circle*

Let $ABC$ be the given triangle

thus it is required to circumscribe a circle about the given triangle $ABC$








Let the straight lines $AB$ $AC$ be bisected at the points $D$, $E$ [I 10], and from the points $D$ $E$ let $DF$ $EF$ be drawn at right angles to $AB$, $AC$,
they will then meet within the triangle $ABC$, or on the straight line $BC$, or outside $BC$

First let them meet within at $F$, and let $FB$, $FC$ $FA$ be joined

Then, since $AD$ is equal to $DB$,

and $DF$ is common and at right angles,

therefore the base $AF$ is equal to the base $FB$ [I 4]

Similarly we can prove that

$CF$ is also equal to $AF$,

so that $FB$ is also equal to $FC$

therefore the three straight lines $FA$ $FB$ $FC$ are equal to one another

Therefore the circle described with centre $F$ and distance one of the straight lines $FA$ $FB$ $FC$ will pass also through the remaining points and the circle will have been circumscribed about the triangle $ABC$

Let it be circumscribed as $ABC$

Next let $DF$ $EF$ meet on the straight line $BC$ at $F$, as is the case in the second figure, and let $AF$ be joined

Then similarly we shall prove that the point $F$ is the centre of the circle circumscribed about the triangle $ABC$

Again let $DF$, $EF$ meet outside the triangle $ABC$ at $F$, as is the case in the third figure and let $AF$ $BF$ $CF$ be joined

Then again since $AD$ is equal to $DB$,

and $DF$ is common and at right angles,

therefore the base $AF$ is equal to the base $BF$ [I 4]

Similarly we can prove that

$CF$ is also equal to $AF$

so that $BF$ is also equal to $FC$,

therefore the circle described with centre $F$ and distance one of the straight lines $FA$, $FB$, $FC$ will pass also through the remaining points and will have been circumscribed about the triangle $ABC$

Therefore about the given triangle a circle has been circumscribed

Q E F

And it is manifest that, when the centre of the circle falls within the triangle the angle $BAC$, being in a segment greater than the semicircle is less than a right angle,

when the centre falls on the straight line $BC$, the angle $BAC$, being in a semi circle is right,

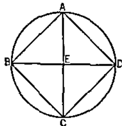
and when the centre of the circle falls outside the triangle, the angle $BAC$, being in a segment less than the semicircle, is greater than a right angle [III 31]

## PROPOSITION 6

*In a given circle to inscribe a square*

Let $ABCD$ be the given circle,

thus it is required to inscribe a square in the circle $ABCD$




Let two diameters $AC$, $BD$ of the circle $ABCD$ be drawn at right angles to one another, and let $AB$ $BC$ $CD$ $DA$ be joined

Then since $BE$ is equal to $ED$ for $E$ is the centre and $EA$ is common and at right angles

therefore the base $AB$ is equal to the base $AD$ [I 4]

For the same reason

each of the straight lines $BC$ $CD$ is also equal to each of the straight lines $AB$, $AD$,

therefore the quadrilateral $ABCD$ is equilateral

I say next that it is also right-angled

For, since the straight line $BD$ is a diameter of the circle $ABCD$

therefore $BAD$ is a semicircle

therefore the angle $BAD$ is right [III 31]

For the same reason

each of the angles $ABC$, $BCD$, $CDA$ is also right,

therefore the quadrilateral $ABCD$ is right angled

But it was also proved equilateral

therefore it is a square [I Def 22]

and it has been inscribed in the circle $ABCD$

Therefore in the given circle the square $ABCD$ has been inscribed Q E F

## PROPOSITION 7

*About a given circle to circumscribe a square*

Let $ABCD$ be the given circle

thus it is required to circumscribe a square about the circle $ABCD$

Let two diameters $AC$ $BD$ of the circle $ABCD$ be drawn at right angles to one another and through the points $A$ $B$ $C$ $D$ let $FG$, $GH$ $HK$, $KF$ be drawn touching the circle $ABCD$ [III 16 Por]

Then since $FG$ touches the circle $ABCD$

and $EA$ has been joined from the centre $E$ to the point of contact at $A$

therefore the angles at $A$ are right [III 18]

For the same reason

the angles at the points $B$ $C$ $D$ are also right

Now, since the angle $AEB$ is right

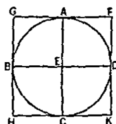
and the angle $EBG$ is also right

therefore $GH$ is parallel to $AC$ [I 28]

*For the same reason*

$AC$ is also parallel to $FK$,

so that $GH$ is also parallel to $FK$ [I 30]




Similarly we can prove that

each of the straight lines $GF$ $HK$ is parallel to $BED$

Therefore $GK$, $GC$ $AK$, $FB$, $BK$ are parallelograms,

therefore $GF$ is equal to $HK$, and $GH$ to $FK$ [I 34]

And since $AC$ is equal to $BD$

and $AC$ is also equal to each of the straight lines $GH$ $FK$,

while $BD$ is equal to each of the straight lines $GF$, $HK$, [I 34]

therefore the quadrilateral $FGHK$ is equilateral

I say next that it is also right-angled

For, since $GBEA$ is a parallelogram

and the angle $AEB$ is right,

*therefore the angle $AGB$ is also right* [I 34]

Similarly we can prove that

the angles at $H$, $K$, $F$ are also right

Therefore $FGHK$ is right angled

But it was also proved equilateral,

therefore it is a square

and it has been circumscribed about the circle $ABCD$

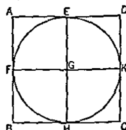
Therefore about the given circle a square has been circumscribed Q E F

## PROPOSITION 8

*In a given square to inscribe a circle*

Let $ABCD$ be the given square

thus it is required to inscribe a circle in the given square $ABCD$




Let the straight lines $AD$, $AB$ be bisected at the points $E$, $F$ respectively [I 10]
through $E$ let $EH$ be drawn parallel to either $AB$ or $CD$, and through $F$ let $FK$ be drawn parallel to either $AD$ or $BC$, [I 31]
therefore each of the figures $AK$ $KB$ $AH$ $HD$ $AG$ $GC$ $BG$ $GD$ is a parallelogram and their opposite sides are evidently equal [I 34]

Now since $AD$ is equal to $AB$

and $AE$ is half of $AD$, and $AF$ half of $AB$,

therefore $AE$ is equal to $AF$,

so that the opposite sides are also equal

*therefore $FG$ is equal to $GE$*

Similarly we can prove that each of the straight lines $GH$ $GK$ is equal to each of the straight lines $FG$ $GE$

therefore the four straight lines $GE$ $GF$ $GH$ $GK$ are equal to one another

Therefore the circle described with centre $G$ and distance one of the straight lines $GE$, $GF$, $GH$ $GK$ will pass also through the remaining points

And it will touch the straight lines $AB$, $BC$ $CD$ $DA$, because the angles at $E$, $F$, $H$, $K$ are right

For, if the circle cuts $AB$, $BC$, $CD$ $DA$, the straight line drawn at right angles to the diameter of the circle from its extremity will fall within the circle which was proved absurd, [III 16]
therefore the circle described with centre $G$ and distance one of the straight lines $GE$, $GF$ $GH$, $GK$ will not cut the straight lines $AB$ $BC$ $CD$ $DA$

Therefore it will touch them and will have been inscribed in the square $ABCD$

Therefore in the given square a circle has been inscribed Q E F

PROPOSITION 9

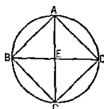
*About a given square to circumscribe a circle*

Let $ABCD$ be the given square,

thus it is required to circumscribe a circle about the square $ABCD$

For let $AC$, $BD$ be joined and let them cut one another at $E$

Then, since $DA$ is equal to $AB$ and $AC$ is common




therefore the two sides $DA$ $AC$ are equal to the two sides $BA$, $AC$,

and the base $DC$ is equal to the base $BC$,

therefore the angle $DAC$ is equal to the angle $BAC$ [I 8]

Therefore the angle $DAB$ is bisected by $AC$

Similarly we can prove that each of the angles $ABC$, $BCD$ $CDA$ is bisected by the straight lines $AC$, $DB$

Now, since the angle $DAB$ is equal to the angle $ABC$

and the angle $EAB$ is half the angle $DAB$,

and the angle $EBA$ half the angle $ABC$,

therefore the angle $EAB$ is also equal to the angle $EBA$,

so that the side $EA$ is also equal to $EB$ [I 6]

Similarly we can prove that each of the straight lines $EA$, $EB$ is equal to each of the straight lines $EC$ $ED$

Therefore the four straight lines $EA$, $EB$ $EC$, $ED$ are equal to one another

Therefore the circle described with centre $E$ and distance one of the straight lines $EA$ $EB$ $EC$ $ED$ will pass also through the remaining points,

and it will have been circumscribed about the square $ABCD$

Let it be circumscribed as $ABCD$

Therefore about the given square a circle has been circumscribed Q E F

PROPOSITION 10

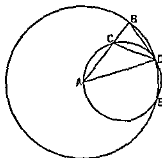
*To construct an isosceles triangle having each of the angles at the base double of the remaining one*

Let any straight line $AB$ be set out and let it be cut at the point $C$ so that the rectangle contained by $AB$ $BC$ is equal to the square on $CA$ [II 11]

with centre $A$ and distance $AB$ let the circle $BDE$ be described

and let there be fitted in the circle $BDE$ the straight line $BD$ equal to the str line $AC$ which is not greater than the diameter of the circle $BDE$

Let $AD$, $DC$ be joined and let the circle $ACD$ be circumscribed about the triangle $ACD$ [IV 5]




Then, since the rectangle $AB$, $BC$ is equal to the square on $AC$,

and $AC$ is equal to $BD$

therefore the rectangle $AB$, $BC$ is equal to the square on $BD$

And since a point $B$ has been taken outside the circle $ACD$ and from $B$ the two straight lines $BA$, $BD$ have fallen on the circle $ACD$, and one of them cuts it while the other falls on it and the rectangle $AB$, $BC$ is equal to the square on $BD$,

therefore $BD$ touches the circle $ACD$ [III 37]

Since, then $BD$ touches it and $DC$ is drawn across from the point of contact at $D$

therefore the angle $BDC$ is equal to the angle $DAC$ in the alternate segment of the circle [III 32]

Since, then the angle $BDC$ is equal to the angle $DAC$,

let the angle $CDA$ be added to each,

therefore the whole angle $BDA$ is equal to the two angles $CDA$, $DAC$

But the exterior angle $BCD$ is equal to the angles $CDA$ $DAC$ [I 32]

therefore the angle $BDA$ is also equal to the angle $BCD$

But the angle $BDA$ is equal to the angle $CBD$ since the side $AD$ is also equal to $AB$ [I 5]

so that the angle $DBA$ is also equal to the angle $BCD$

Therefore the three angles $BDA$ $DBA$ $BCD$ are equal to one another

And since the angle $DBC$ is equal to the angle $BCD$

the side $BD$ is also equal to the side $DC$ [I 6]

But $BD$ is by hypothesis equal to $CA$

therefore $CA$ is also equal to $CD$

so that the angle $CDA$ is also equal to the angle $DAC$, [I 5]

therefore the angles $CDA$ $DAC$ are double of the angle $DAC$

But the angle $BCD$ is equal to the angles $CDA$ $DAC$

therefore the angle $BCD$ is also double of the angle $CAD$

But the angle $BCD$ is equal to each of the angles $BDA$, $DBA$,

therefore each of the angles $BDA$ $DBA$ is also double of the angle $DAB$

Therefore the isosceles triangle $ABD$ has been constructed having each of the angles at the base $DB$ double of the remaining one Q E F

## Proposition 11

*In a given circle to inscribe an equilateral and equiangular pentagon*

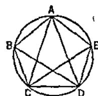
Let $ABCDE$ be the given circle

thus it is required to inscribe in the circle $ABCDE$ an equilateral and equiangular pentagon

Let the isosceles triangle $FGH$ be set out having each of the angles at $G$ $H$ double of the angle at $F$ [IV 10]

let there be inscribed in the circle $ABCDE$ the triangle $ACD$ equiangular with

the triangle $FGH$, so that the angle $CAD$ is equal to the angle at $F$ and the angles at $G$, $H$ respectively equal to the angles $ACD$ $CDA$, [IV 2]







therefore each of the angles $ACD$, $CDA$ is also double of the angle $CAD$

Now let the angles $ACD$, $CDA$ be bisected respectively by the straight lines $CE$, $DB$ [I 9] and let $AB$, $BC$, $DE$ $EA$ be joined

Then since each of the angles $ACD$ $CDA$ is double of the angle $CAD$ and they have been bisected by the straight lines $CE$ $DB$,

therefore the five angles $DAC$, $ACE$ $ECD$ $CDB$, $BDA$ are equal to one another

But equal angles stand on equal circumferences [III 26]
therefore the five circumferences $AB$, $BC$ $CD$ $DE$, $EA$ are equal to one another

But equal circumferences are subtended by equal straight lines [III 29]
therefore the five straight lines $AB$ $BC$, $CD$ $DE$, $EA$ are equal to one another
therefore the pentagon $ABCDE$ is equilateral

I say next that it is also equiangular

For since the circumference $AB$ is equal to the circumference $DE$, let $BCD$ be added to each

therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$

And the angle $AED$ stands on the circumference $ABCD$ and the angle $BAE$ on the circumference $EDCB$,
therefore the angle $BAE$ is also equal to the angle $AED$ [III 27]

For the same reason
each of the angles $ABC$ $BCD$ $CDE$ is also equal to each of the angles $BAE$ $AED$
therefore the pentagon $ABCDE$ is equiangular

But it was also proved equilateral
therefore in the given circle an equilateral and equiangular pentagon has been inscribed

Q E F

## Proposition 12

*About a given circle to circumscribe an equilateral and equiangular pentagon*

Let $ABCDE$ be the given circle
thus it is required to circumscribe an equilateral and equiangular pentagon about the circle $ABCDE$

Let $A$, $B$, $C$ $D$ $E$ be conceived to be the angular points of the inscribed pentagon so that the circumferences $AB$ $BC$ $CD$ $DE$ $EA$ are equal [IV 11]
through $A$ $B$ $C$ $D$ $E$ let $GH$, $HK$, $KL$ $LM$ $MG$ be drawn touching the circle, [III 16 Por]
let the centre $F$ of the circle $ABCDE$ be taken [III 1] and let $FB$ $FK$ $FC$ $FL$ $FD$ be joined

Then since the straight line $KL$ touches the circle $ABCDE$ at $C$
and $FC$ has been joined from the centre $F$ to the point of contact at $C$
therefore $FC$ is perpendicular to $KL$, [III 18]

therefore each of the angles at $C$ is right

For the same reason

the angles at the points $B$, $D$ are also right

And, since the angle $FCK$ is right

therefore the square on $FK$ is equal to the squares on $FC$, $CK$

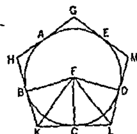
For the same reason [I 47]

the square on $FK$ is also equal to the squares on $FB$, $BK$,

so that the squares on $FC$, $CK$ are equal to the squares on $FB$ $BK$,

of which the square on $FC$ is equal to the square on $FB$

therefore the square on $CK$ which remains is equal to the square on $BK$




Therefore $BK$ is equal to $CK$

And, since $FB$ is equal to $FC$

and $FK$ common

the two sides $BF$, $FK$ are equal to the two sides $CF$ $FK$, and the base $BK$ equal to the base $CK$,

therefore the angle $BFK$ is equal to the angle $KFC$, [I 8]

and the angle $BKF$ to the angle $FKC$

Therefore the angle $BFC$ is double of the angle $KFC$,

and the angle $BKC$ of the angle $FKC$

For the same reason

the angle $CFD$ is also double of the angle $CFL$,

and the angle $DLC$ of the angle $FLC$

Now since the circumference $BC$ is equal to $CD$

the angle $BFC$ is also equal to the angle $CFD$ [III 27]

And the angle $BFC$ is double of the angle $KFC$, and the angle $DFC$ of the angle $LFC$

therefore the angle $KFC$ is also equal to the angle $LFC$

But the angle $FCK$ is also equal to the angle $FCL$,

therefore $FKC$ $FLC$ are two triangles having two angles equal to two angles and one side equal to one side, namely $FC$ which is common to them, therefore they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle, [I 26]

therefore the straight line $KC$ is equal to $CL$,

and the angle $FKC$ to the angle $FLC$,

And, since $KC$ is equal to $CL$

therefore $KL$ is double of $KC$

For the same reason it can be proved that

$HK$ is also double of $BK$

And $BK$ is equal to $KC$,

therefore $HK$ is also equal to $KL$

Similarly each of the straight lines $HG$ $GM$ $ML$ can also be proved equal to each of the straight lines $HK$ $KL$

therefore the pentagon $GHKLM$ is equilateral

I say next that it is also equiangular

For, since the angle $FKC$ is equal to the angle $FLC$,

and the angle $HKL$ was proved double of the angle $FKC$,

and the angle $KLM$ double of the angle $FLC$

therefore the angle $HKL$ is also equal to the angle $KLM$

Similarly each of the angles $KHG$, $HGM$, $GML$ can also be proved equal to each of the angles $HKL$, $KLM$,

therefore the five angles $GHK$, $HKL$, $KLM$, $LMG$, $MGH$ are equal to one another

Therefore the pentagon $GHKLM$ is equiangular

And it was also proved equilateral, and it has been circumscribed about the circle $ABCDE$ Q. E. F.

PROPOSITION 13

*In a given pentagon, which is equilateral and equiangular, to inscribe a circle*

Let $ABCDE$ be the given equilateral and equiangular pentagon,

thus it is required to inscribe a circle in the pentagon $ABCDE$

For let the angles $BCD$, $CDE$ be bisected by the straight lines $CF$, $DF$ respectively, and from the point $F$ at which the straight lines $CF$, $DF$ meet one another, let the straight lines $FB$, $FA$, $FE$ be joined

Then since $BC$ is equal to $CD$ and $CF$ common,

the two sides $BC$, $CF$ are equal to the two sides $DC$, $CF$,

and the angle $BCF$ is equal to the angle $DCF$,

therefore the base $BF$ is equal to the base $DF$

and the triangle $BCF$ is equal to the triangle $DCF$, and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]

Therefore the angle $CBF$ is equal to the angle $CDF$

And since the angle $CDE$ is double of the angle $CDF$,

and the angle $CDE$ is equal to the angle $ABC$,

while the angle $CDF$ is equal to the angle $CBF$

therefore the angle $CBA$ is also double of the angle $CBF$,

therefore the angle $ABF$ is equal to the angle $FBC$

therefore the angle $ABC$ has been bisected by the straight line $BF$

Similarly it can be proved that

the angles $BAE$, $AED$ have also been bisected by the straight lines $FA$, $FE$ respectively

Now let $FG$, $FH$, $FK$, $FL$, $FM$ be drawn from the point $F$ perpendicular to the straight lines $AB$, $BC$, $CD$, $DE$, $EA$

Then, since the angle $HCF$ is equal to the angle $KCF$

and the right angle $FHC$ is also equal to the angle $FKC$

$FHC$, $FKC$ are two triangles having two angles equal to two angles and one side equal to one side, namely $FC$ which is common to them and subtends one of the equal angles

therefore they will also have the remaining sides equal to the remaining sides [I 26]

therefore the perpendicular $FH$ is equal to the perpendicular $FK$

Similarly it can be proved that

each of the straight lines *FL*, *FM*, *FG* is also equal to each of the straight lines *FH*, *FK*,

therefore the five straight lines *FG*, *FH*, *FK*, *FL*, *FM* are equal to one another

Therefore the circle described with centre *F* and distance one of the straight lines *FG*, *FH*, *FK*, *FL*, *FM* will pass also through the remaining points,
and it will touch the straight lines *AB*, *BC*, *CD*, *DE*, *EA*, because the angles at the points *G*, *H*, *K*, *L*, *M* are right

For if it does not touch them, but cuts them,
it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle which was proved absurd [III 16]

Therefore the circle described with centre *F* and distance one of the straight lines *FG*, *FH*, *FK*, *FL*, *FM* will not cut the straight lines *AB*, *BC*, *CD*, *DE*, *EA*,
therefore it will touch them

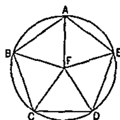
Let it be described as *GHKLM*

Therefore in the given pentagon, which is equilateral and equiangular, a circle has been inscribed Q E F

PROPOSITION 14

*About a given pentagon, which is equilateral and equiangular, to circumscribe a circle*

Let *ABCDE* be the given pentagon, which is equilateral and equiangular, thus it is required to circumscribe a circle about the pentagon *ABCDE*




Let the angles *BCD*, *CDE* be bisected by the straight lines *CF*, *DF* respectively, and from the point *F* at which the straight lines meet let the straight lines *FB*, *FA*, *FE* be joined to the points *B*, *A*, *E*

Then in manner similar to the preceding it can be proved that the angles *CBA*, *BAE*, *AED* have also been bisected by the straight lines *FB*, *FA*, *FE* respectively

Now, since the angle *BCD* is equal to the angle *CDE*,
and the angle *FCD* is half of the angle *BCD*
and the angle *CDF* half of the angle *CDE*,
therefore the angle *FCD* is also equal to the angle *CDF*,
so that the side *FC* is also equal to the side *FD* [I 6]

Similarly it can be proved that
each of the straight lines *FB*, *FA*, *FE* is also equal to each of the straight lines *FC*, *FD*,
therefore the five straight lines *FA*, *FB*, *FC*, *FD*, *FE* are equal to one another

Therefore the circle described with centre *F* and distance one of the straight lines *FA*, *FB*, *FC*, *FD*, *FE* will pass also through the remaining points, and will have been circumscribed

Let it be circumscribed, and let it be *ABCDE*

Therefore about the given pentagon, which is equilateral and equiangular, a circle has been circumscribed Q E F

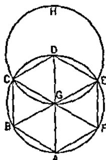
PROPOSITION 15

*In a given circle to inscribe an equilateral and equiangular hexagon*

Let $ABCDEF$ be the given circle,

thus it is required to inscribe an equilateral and equiangular hexagon in the circle $ABCDEF$

Let the diameter $AD$ of the circle $ABCDEF$ be drawn,




let the centre $G$ of the circle be taken and with centre $D$ and distance $DG$ let the circle $EGCH$ be described

let $EG$ $CG$ be joined and carried through to the points $B$, $F$,

and let $AB$ $BC$, $CD$ $DE$, $EF$, $FA$ be joined

I say that the hexagon $ABCDEF$ is equilateral and equiangular

For since the point $G$ is the centre of the circle $ABCDEF$,

$GE$ is equal to $GD$

Again, since the point $D$ is the centre of the circle $GCH$,

$DE$ is equal to $DG$

But $GE$ was proved equal to $GD$,

therefore $GE$ is also equal to $ED$,

therefore the triangle $EGD$ is equilateral

and therefore its three angles $EGD$, $GDE$ $DEG$ are equal to one another inasmuch as in isosceles triangles the angles at the base are equal to one another [I 5]

And the three angles of the triangle are equal to two right angles, [I 32]

therefore the angle $EGD$ is one third of two right angles

Similarly the angle $DGC$ can also be proved to be one-third of two right angles

And since the straight line $CG$ standing on $EB$ makes the adjacent angles $EGC$, $CGB$ equal to two right angles

therefore the remaining angle $CGB$ is also one-third of two right angles

Therefore the angles $EGD$ $DGC$ $CGB$ are equal to one another

so that the angles vertical to them the angles $BGA$ $AGF$ $FGE$ are equal [I 15]

Therefore the six angles $EGD$ $DGC$ $CGB$, $BGA$ $AGF$, $FGE$ are equal to one another

But equal angles stand on equal circumferences [III 26]

therefore the six circumferences $AB$ $BC$ $CD$ $DE$ $EF$ $FA$ are equal to one another

And equal circumferences are subtended by equal straight lines [III 29]

therefore the six straight lines are equal to one another

therefore the hexagon $ABCDEF$ is equilateral

I say next that it is also equiangular

For, since the circumference $FA$ is equal to the circumference $ED$,

let the circumference $ABCD$ be added to each

therefore the whole $FABCD$ is equal to the whole $EDCBA$

and the angle $FED$ stands on the circumference $FABCD$

and the angle $AFE$ on the circumference $EDCBA$,

therefore the angle $AFE$ is equal to the angle $DEF$ [III 27]

Similarly it can be proved that the remaining angles of the hexagon $ABCDEF$ are also severally equal to each of the angles $AFE$, $FED$,

therefore the hexagon $ABCDEF$ is equiangular

But it was also proved equilateral

and it has been inscribed in the circle $ABCDEF$

Therefore in the given circle an equilateral and equiangular hexagon has been inscribed Q E F

PORISM From this it is manifest that the side of the hexagon is equal to the radius of the circle

And in like manner as in the case of the pentagon if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it

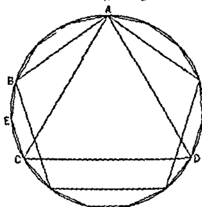
Q E F

## PROPOSITION 16

*In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular*

Let $ABCD$ be the given circle,

thus it is required to inscribe in the circle $ABCD$ a fifteen angled figure which shall be both equilateral and equiangular




In the circle $ABCD$ let there be inscribed a side $AC$ of the equilateral triangle inscribed in it and a side $AB$ of an equilateral pentagon

therefore of the equal segments of which there are fifteen in the circle $ABCD$ there will be five in the circumference $ABC$ which is one-third of the circle and there will be three in the circumference $AB$ which is one-fifth of the circle

therefore in the remainder $BC$ there will be two of the equal segments

Let $BC$ be bisected at $E$, [III 30]

therefore each of the circumferences $BE$ $EC$ is a fifteenth of the circle $ABCD$

If therefore we join $BE$ $EC$ and fit into the circle $ABCD$ straight lines equal to them and in contiguity a fifteen angled figure which is both equilateral and equiangular will have been inscribed in it Q E F

And, in like manner as in the case of the pentagon if through the points of division on the circle we draw tangents to the circle there will be circumscribed about the circle a fifteen angled figure which is equilateral and equiangular

And further by proofs similar to those in the case of the pentagon we can both inscribe a circle in the given fifteen angled figure and circumscribe one about it Q E F

# BOOK FIVE

## DEFINITIONS

1 A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater

2 The greater is a *multiple* of the less when it is measured by the less

3 A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind

4 Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another

5 Magnitudes are said to *be in the same ratio*, the first to the second and the third to the fourth when if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth the former equimultiples alike exceed are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order

6 Let magnitudes which have the same ratio be called *proportional*

7 When of the equimultiples the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to *have a greater ratio* to the second than the third has to the fourth

8 A proportion in three terms is the least possible

9 When three magnitudes are proportional the first is said to have to the third the *duplicate ratio* of that which it has to the second

10 When four magnitudes are <continuously> proportional, the first is said to have to the fourth the *triplicate ratio* of that which it has to the second, and so on continually whatever be the proportion

11 The term *corresponding magnitudes* is used of antecedents in relation to antecedents and of consequents in relation to consequents

12 *Alternate ratio* means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent

13 *Inverse ratio* means taking the consequent as antecedent in relation to the antecedent as consequent

14 *Composition of a ratio* means taking the antecedent together with the consequent as one in relation to the consequent by itself

15 *Separation of a ratio* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself

16 *Conversion of a ratio* means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent

17 A ratio *ex aequali* arises when there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes,

Or in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms

18 A *perturbed proportion* arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes so is antecedent to consequent among the second magnitudes while as the consequent is to a third among the first magnitudes so is a third to the antecedent among the second magnitudes

## BOOK V PROPOSITIONS

### PROPOSITION 1

*If there be any number of magnitudes whatever which are respectively equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one that multiple also will all be of all*

Let any number of magnitudes whatever $AB$ $CD$ be respectively equimultiples of any magnitudes $E$ $F$ equal in multitude,

I say that whatever multiple $AB$ is of $E$ that multiple will $AB$, $CD$ also be of $E$ $F$




For since $AB$ is the same multiple of $E$ that $CD$ is of $F$, as many magnitudes as there are in $AB$ equal to $E$ so many also are there in $CD$ equal to $F$

Let $AB$ be divided into the magnitudes $AG$, $GB$ equal to $E$,

and $CD$ into $CH$ $HD$ equal to $F$

then the multitude of the magnitudes $AG$ $GB$ will be equal to the multitude of the magnitudes $CH$ $HD$

Now since $AG$ is equal to $E$, and $CH$ to $F$

therefore $AG$ is equal to $E$, and $AG$, $CH$ to $E$ $F$

For the same reason

$GB$ is equal to $F$ and $GB$ $HD$ to $E$ $F$

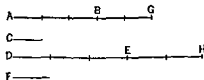
therefore as many magnitudes as there are in $AB$ equal to $E$ so many also are there in $AB$ $CD$ equal to $E$ $F$

therefore whatever multiple $AB$ is of $E$, that multiple will $AB$ $CD$ also be of $E$ $F$

Therefore etc Q E D

### PROPOSITION 2

*If a first magnitude be the same multiple of a second that a third is of a fourth and a fifth also be the same multiple of the second that a sixth is of the fourth the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth*




Let a first magnitude $AB$ be the same multiple of a second $C$, that a third $DE$ is of a fourth $F$, and let a fifth $BG$ also be the same multiple of the second $C$ that a sixth $EH$ is of the fourth $F$

I say that the sum of the first and fifth $AG$ will be the same multiple of the

second, $C$, that the sum of the third and sixth, $DH$ is of the fourth $F$

For since $AB$ is the same multiple of $C$ that $DE$ is of $F$ therefore, as many magnitudes as there are in $AB$ equal to $C$, so many also are there in $DE$ equal to $F$

For the same reason also

as many as there are in $BG$ equal to $C$ so many are there also in $EH$ equal to $F$, therefore as many as there are in the whole $AG$ equal to $C$, so many also are there in the whole $DH$ equal to $F$

Therefore whatever multiple $AG$ is of $C$, that multiple also is $DH$ of $F$

Therefore the sum of the first and fifth $AG$ is the same multiple of the second $C$ that the sum of the third and sixth, $DH$ is of the fourth, $F$

Therefore etc Q E D

## Proposition 3

*If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third then also* ex aequali *the magnitudes taken will be equimultiples respectively the one of the second and the other of the fourth*

Let a first magnitude $A$ be the same multiple of a second $B$ that a third $C$ is of a fourth $D$, and let equimultiples $EF$, $GH$ be taken of $A$, $C$,

I say that $EF$ is the same multiple of $B$ that $GH$ is of $D$

For, since $EF$ is the same multiple of $A$ that $GH$ is of $C$, therefore as many magnitudes as there are in $EF$ equal to $A$ so many also are there in $GH$ equal to $C$

Let $EF$ be divided into the magnitudes $EK$, $KF$ equal to $A$ and $GH$ into the magnitudes $GL$, $LH$ equal to $C$

then the multitude of the magnitudes $EK$, $KF$ will be equal to the multitude of the magnitudes $GL$ $LH$

A
B
E K F
C
D
G L H

And since $A$ is the same multiple of $B$ that $C$ is of $D$, while $EK$ is equal to $A$ and $GL$ to $C$, therefore $EK$ is the same multiple of $B$ that $GL$ is of $D$

For the same reason $KF$ is the same multiple of $B$ that $LH$ is of $D$

Since then a first magnitude $EK$ is the same multiple of a second $B$ that a third $GL$ is of a fourth $D$

and a fifth $KF$ is also the same multiple of the second $B$ that a sixth $LH$ is of the fourth $D$

therefore the sum of the first and fifth $EF$ is also the same multiple of the second $B$ that the sum of the third and sixth $GH$ is of the fourth $D$ [v 2]

Therefore etc Q E D

## Proposition 4

*If a first magnitude have to a second the same ratio as a third to a fourth any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively taken in corresponding order*

For let a first magnitude $A$ have to a second $B$ the same ratio as a third $C$ to a fourth $D$, and let equimultiples $E$ $F$ be taken of $A$, $C$, and $G$, $H$ other, chance, equimultiples of $B$, $D$,

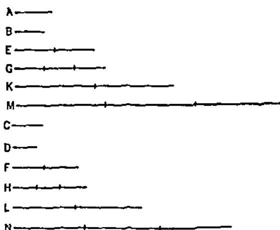
I say that as $E$ is to $G$, so is $F$ to $H$

For let equimultiples $K$, $L$ be taken of $E$, $F$, and other, chance equimultiples $M$ $N$ of $G$, $H$

Since $E$ is the same multiple of $A$ that $F$ is of $C$, and equimultiples $K$, $L$ of $E$, $F$ have been taken, therefore $K$ is the same multiple of $A$ that $L$ is of $C$ [v 3]

For the same reason $M$ is the same multiple of $B$ that $N$ is of $D$

And since as $A$ is to $B$, so is $C$ to $D$




and of $A$ $C$ equimultiples $K$ $L$ have been taken,
and of $B$ $D$ other chance equimultiples $M$ $N$
therefore if $K$ is in excess of $M$, $L$ also is in excess of $N$,
if it is equal equal, and if less less [v Def 5]
And $K$ $L$ are equimultiples of $E$, $F$,
and $M$ $N$ other chance equimultiples of $G$, $H$,
therefore as $E$ is to $G$ so is $F$ to $H$ [v Def 5]

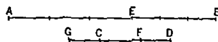
Therefore etc Q E D

## Proposition 5

*If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted the remainder will also be the same multiple of the remainder that the whole is of the whole*

For let the magnitude $AB$ be the same multiple of the magnitude $CD$ that the part $AE$ subtracted is of the part $CF$ subtracted,

I say that the remainder $EB$ is also the same multiple of the remainder $FD$ that the whole $AB$ is of the whole $CD$




For whatever multiple $AE$ is of $CF$ let $EB$ be made that multiple of $CG$

Then, since $AE$ is the same multiple of $CF$ that $EB$ is of $GC$
therefore $AE$ is the same multiple of $CF$ that $AB$ is of $GF$ [v 1]

But by the assumption $AE$ is the same multiple of $CF$ that $AB$ is of $CD$

Therefore $AB$ is the same multiple of each of the magnitudes $GF$ $CD$,
therefore $GF$ is equal to $CD$

Let $CF$ be subtracted from each,

therefore the remainder $GC$ is equal to the remainder $FD$

And, since $AE$ is the same multiple of $CF$ that $EB$ is of $GC$,

and $GC$ is equal to $DF$

therefore $AE$ is the same multiple of $CF$ that $EB$ is of $FD$

But, by hypothesis

$AE$ is the same multiple of $CF$ that $AB$ is of $CD$,

therefore $EB$ is the same multiple of $FD$ that $AB$ is of $CD$

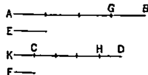
That is the remainder $EB$ will be the same multiple of the remainder $FD$ that the whole $AB$ is of the whole $CD$

Therefore etc Q E D

## Proposition 6

*If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them*

For let two magnitudes $AB$, $CD$ be equimultiples of two magnitudes $E$ $F$, and let $AG$, $CH$ subtracted from them be equimultiples of the same two $E$ $F$,




I say that the remainders also $GB$ $HD$, are either equal to $E$ $F$ or equimultiples of them

For, first let $GB$ be equal to $E$,

I say that $HD$ is also equal to $F$

For let $CK$ be made equal to $F$

Since $AG$ is the same multiple of $E$ that $CH$ is of $F$,

while $GB$ is equal to $E$ and $KC$ to $F$,

therefore $AB$ is the same multiple of $E$ that $KH$ is of $F$ [v 2]

But by hypothesis $AB$ is the same multiple of $E$ that $CD$ is of $F$,

therefore $KH$ is the same multiple of $F$ that $CD$ is of $F$

Since then each of the magnitudes $KH$, $CD$ is the same multiple of $F$,

therefore $KH$ is equal to $CD$

Let $CH$ be subtracted from each,

therefore the remainder $KC$ is equal to the remainder $HD$

But $F$ is equal to $KC$,

therefore $HD$ is also equal to $F$

Hence if $GB$ is equal to $E$ $HD$ is also equal to $F$

Similarly we can prove that, even if $GB$ be a multiple of $E$, $HD$ is also the same multiple of $F$

Therefore etc Q E D

## Proposition 7

*Equal magnitudes have to the same the same ratio as also has the same to equal magnitudes*

Let $A$, $B$ be equal magnitudes and $C$ any other chance, magnitude,

I say that each of the magnitudes $A$, $B$ has the same ratio to $C$, and $C$ has the same ratio to each of the magnitudes $A$ $B$

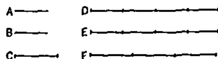
For let equimultiples $D$ $E$ of $A$ $B$ be taken, and of $C$ another, chance multiple $F$

Then since $D$ is the same multiple of $A$ that $E$ is of $B$, while $A$ is equal to $B$,

therefore $D$ is equal to $E$

But $F$ is another, chance, magnitude

If therefore $D$ is in excess of $F$, $E$ is also in excess of $F$, if equal to it, equal, and if less less




And $D$, $E$ are equimultiples of $A$, $B$, while $F$ is another, chance multiple of $C$, therefore as $A$ is to $C$, so is $B$ to $C$ [v Def 5]

I say next that $C$ also has the same ratio to each of the magnitudes $A$, $B$

For, with the same construction, we can prove similarly that $D$ is equal to $E$, and $F$ is some other magnitude

If therefore $F$ is in excess of $D$ it is also in excess of $E$ if equal equal, and if less, less

And $F$ is a multiple of $C$, while $D$ $E$ are other chance equimultiples of $A$ $B$, therefore as $C$ is to $A$, so is $C$ to $B$ [v Def 5]

Therefore etc

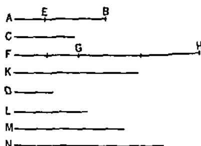
Porism From this it is manifest that if any magnitudes are proportional they will also be proportional inversely Q E D

## Proposition 8

*Of unequal magnitudes the greater has to the same a greater ratio than the less has and the same has to the less a greater ratio than it has to the greater*

Let $AB$, $C$ be unequal magnitudes, and let $AB$ be greater let $D$ be another chance magnitude

I say that $AB$ has to $D$ a greater ratio than $C$ has to $D$ and $D$ has to $C$ a greater ratio than it has to $AB$




For since $AB$ is greater than $C$ let $BE$ be made equal to $C$

then the less of the magnitudes $AE$ $EB$ if multiplied will sometime be greater than $D$ [v Def 4]

First let $AE$ be less than $EB$ let $AE$ be multiplied and let $FG$ be a multiple of it which is greater than $D$,

then whatever multiple $FG$ is of $AE$ let $GH$ be made the same multiple of $EB$ and $K$ of $C$

and let $L$ be taken double of $D$ $M$ triple of it and successive multiples increasing by one until what is taken is a multiple of $D$ and the first that is greater than $K$ Let it be taken and let it be $N$ which is quadruple of $D$ and the first multiple of it that is greater than $K$

Then since $K$ is less than $N$ first

therefore $K$ is not less than $M$

And since $FG$ is the same multiple of $AE$ that $GH$ is of $EB$

therefore $FG$ is the same multiple of $AE$ that $FH$ is of $AB$ [v 1]

But $FG$ is the same multiple of $AE$ that $K$ is of $C$

therefore $FH$ is the same multiple of $AB$ that $K$ is of $C$,

therefore $FH$ $K$ are equimultiples of $AB$ $C$

Again since $GH$ is the same multiple of $EB$ that $K$ is of $C$

and $EB$ is equal to $C$

*therefore GH is equal to K*

But $K$ is not less than $M$

therefore neither is $GH$ less than $M$

And $FG$ is greater than $D$,

therefore the whole $FH$ is greater than $D$ $M$ together

But $D$ $M$ together are equal to $N$ inasmuch as $M$ is triple of $D$, and $M$, $D$ together are quadruple of $D$ while $N$ is also quadruple of $D$, whence $M$ $D$ together are equal to $N$

But $FH$ is greater than $M$ $D$,

therefore $FH$ is in excess of $N$

while $K$ is not in excess of $N$

And $FH$, $K$ are equimultiples of $AB$, $C$, while $N$ is another chance multiple of $D$

therefore $AB$ has to $D$ a greater ratio than $C$ has to $D$ [v Def 7]

I say next that $D$ also has to $C$ a greater ratio than $D$ has to $AB$

For with the same construction we can prove similarly that $N$ is in excess of $K$ while $N$ is not in excess of $FH$

And $N$ is a multiple of $D$,

while $FH$ $K$ are other, chance equimultiples of $AB$ $C$,

therefore $D$ has to $C$ a greater ratio than $D$ has to $AB$ [v Def 7]

Again let $AE$ be greater than $EB$

Then the less $EB$, if multiplied will sometime be greater than $D$ [v Def 4]

Let it be multiplied and let $GH$ be a multiple of $EB$ and greater than $D$ and whatever multiple $GH$ is of $EB$ let $FG$ be made the same multiple of $AE$ and $K$ of $C$

Then we can prove similarly that $FH$ $K$ are equimultiples of $AB$ $C$ and similarly let $N$ be taken a multiple of $D$ but the first that is greater than $FG$,

so that $FG$ is again not less than $M$

But $GH$ is greater than $D$

therefore the whole $FH$ is in excess of $D$ $M$ that is of $N$

*Now K is not in excess of N inasmuch as FG also which is greater than GH* that is than $K$ is not in excess of $N$

And in the same manner by following the above argument we complete the demonstration

*Therefore etc* Q E D

PROPOSITION 9

*Magnitudes which have the same ratio to the same are equal to one another and magnitudes to which the same has the same ratio are equal*

For let each of the magnitudes $A$ $B$ have the same ratio to $C$

I say that $A$ is equal to $B$




For otherwise each of the magnitudes $A$ $B$ would not have had the same ratio to $C$ but it has [v 8]

therefore $A$ is equal to $B$

Again, let $C$ have the same ratio to each of the magnitudes $A$, $B$,

I say that $A$ is equal to $B$

For otherwise, $C$ would not have had the same ratio to each of the magnitudes $A$, $B$, [v 8]

but it has,

therefore $A$ is equal to $B$

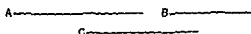
Therefore etc Q E D

PROPOSITION 10

*Of magnitudes which have a ratio to the same, that which has a greater ratio is greater and that to which the same has a greater ratio is less*

For let $A$ have to $C$ a greater ratio than $B$ has to $C$,

I say that $A$ is greater than $B$




For if not $A$ is either equal to $B$ or less

Now $A$ is not equal to $B$,

for in that case each of the magnitudes $A$, $B$ would have had the same ratio to $C$, [v 7]

but they have not,

therefore $A$ is not equal to $B$

Nor again is $A$ less than $B$

for in that case $A$ would have had to $C$ a less ratio than $B$ has to $C$, [v 8]

but it has not

therefore $A$ is not less than $B$

But it was proved not to be equal either

therefore $A$ is greater than $B$

Again, let $C$ have to $B$ a greater ratio than $C$ has to $A$,

I say that $B$ is less than $A$

For, if not it is either equal or greater

Now $B$ is not equal to $A$

for in that case $C$ would have had the same ratio to each of the magnitudes $A$, $B$, [v 7]

but it has not

therefore $A$ is not equal to $B$

Nor again is $B$ greater than $A$

for in that case $C$ would have had to $B$ a less ratio than it has to $A$, [v 8]

but it has not

therefore $B$ is not greater than $A$

But it was proved that it is not equal either

therefore $B$ is less than $A$

Therefore etc Q E D

PROPOSITION 11

*Ratios which are the same with the same ratio are also the same with one another*

For, as $A$ is to $B$ so let $C$ be to $D$

and as $C$ is to $D$ so let $E$ be to $F$,

I say that as $A$ is to $B$, so is $E$ to $F$

For of $A$, $C$, $E$ let equimultiples $G$, $H$, $K$ be taken and of $B$, $D$, $F$ other, chance, equimultiples $L$, $M$ $N$

Then since, as $A$ is to $B$, so is $C$ to $D$,

and of $A$, $C$ equimultiples $G$, $H$ have been taken,

and of $B$ $D$ other, chance equimultiples $L$, $M$

therefore, if $G$ is in excess of $L$, $H$ is also in excess of $M$,

if equal equal,

and if less less

Again, since as $C$ is to $D$, so is $E$ to $F$,

and of $C$ $E$ equimultiples $H$, $K$ have been taken

and of $D$ $F$ other chance equimultiples $M$ $N$

therefore if $H$ is in excess of $M$, $K$ is also in excess of $N$,

if equal equal

and if less less

But we saw that if $H$ was in excess of $M$, $G$ was also in excess of $L$, if equal equal, and if less less

so that, in addition, if $G$ is in excess of $L$ $K$ is also in excess of $N$,

if equal equal

and if less less

And $G$, $K$ are equimultiples of $A$ $E$,

while $L$ $N$ are other chance equimultiples of $B$, $F$,

therefore as $A$ is to $B$, so is $E$ to $F$

Therefore etc Q E D

## Proposition 12

*If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents*

Let any number of magnitudes $A$ $B$ $C$ $D$, $E$, $F$ be proportional, so that, as $A$ is to $B$ so is $C$ to $D$ and $E$ to $F$,

I say that as $A$ is to $B$ so are $A$, $C$ $E$ to $B$, $D$ $F$

For of $A$ $C$ $E$ let equimultiples $G$ $H$, $K$ be taken and of $B$ $D$, $F$ other chance equimultiples $L$, $M$ $N$

Then since as $A$ is to $B$, so is $C$ to $D$, and $E$ to $F$, and of $A$, $C$ $E$ equimultiples $G$ $H$, $K$ have been taken

and of $B$ $D$ $F$ other chance equimultiples $L$ $M$ $N$,

therefore if $G$ is in excess of $L$ $H$ is also in excess of $M$ and $K$ of $N$,

if equal equal,

and if less less,
so that in addition
if $G$ is in excess of $L$ then $G$ $H$, $K$ are in excess of $L$, $M$ $N$,
if equal, equal
and if less less

Now $G$ and $G$ $H$ $K$ are equimultiples of $A$ and $A$ $C$, $E$ since, if any number of magnitudes whatever are respectively equimultiples of any magnitudes equal in multitude whatever multiple one of the magnitudes is of one that multiple also will all be of all [v 1]

For the same reason
$L$ and $L$ $M$, $N$ are also equimultiples of $B$ and $B$ $D$ $F$,
therefore as $A$ is to $B$, so are $A$, $C$, $E$ to $B$, $D$, $F$ [v Def 5]

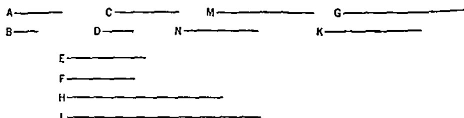
Therefore etc Q E D

## Proposition 13

*If a first magnitude have to a second the same ratio as a third to a fourth and the third have to the fourth a greater ratio than a fifth has to a sixth the first will also have to the second a greater ratio than the fifth to the sixth*

For let a first magnitude $A$ have to a second $B$ the same ratio as a third $C$ has to a fourth $D$
and let the third $C$ have to the fourth $D$ a greater ratio than a fifth $E$ has to a sixth $F$,

I say that the first $A$ will also have to the second $B$ a greater ratio than the fifth $E$ to the sixth $F$




*For since there are some equimultiples of $C$ $E$,*
and of $D$ $F$ other chance, equimultiples such that the multiple of $C$ is in excess of the multiple of $D$
while the multiple of $E$ is not in excess of the multiple of $F$, [v Def 7
let them be taken
and let $G$ $H$ be equimultiples of $C$, $E$,
and $K$, $L$ other chance equimultiples of $D$ $F$
so that $G$ is in excess of $K$ but $H$ is not in excess of $L$
and whatever multiple $G$ is of $C$ let $M$ be also that multiple of $A$,
and, whatever multiple $K$ is of $D$ let $N$ be also that multiple of $B$

Now, since as $A$ is to $B$ so is $C$ to $D$,
and of $A$, $C$ equimultiples $M$ $G$ have been taken
and of $B$ $D$ other chance equimultiples $N$ $K$
therefore if $M$ is in excess of $N$ $G$ is also in excess of $K$,
if equal equal

and if less less [v Def 5]

But $G$ is in excess of $K$

therefore $M$ is also in excess of $N$

But $H$ is not in excess of $L$

and $M$, $H$ are equimultiples of $A$, $E$,

and $N$ $L$ other chance equimultiples of $B$, $F$,

therefore $A$ has to $B$ a greater ratio than $E$ has to $F$ [v Def 7]

Therefore etc Q E D

PROPOSITION 14

*If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third the second will also be greater than the fourth if equal equal and if less less*

For let a first magnitude $A$ have the same ratio to a second $B$ as a third $C$ has to a fourth $D$, and let $A$ be greater than $C$,

I say that $B$ is also greater than $D$

For, since $A$ is greater than $C$ and $B$ is another, chance magnitude therefore $A$ has to $B$ a greater ratio than $C$ has to $B$ [v 8]

But as $A$ is to $B$ so is $C$ to $D$,

therefore $C$ has also to $D$ a greater ratio than $C$ has to $B$ [v 13]

But that to which the same has a greater ratio is less, [v 10]

therefore $D$ is less than $B$

so that $B$ is greater than $D$

Similarly we can prove that if $A$ be equal to $C$ $B$ will also be equal to $D$

and if $A$ be less than $C$, $B$ will also be less than $D$

Therefore etc Q E D

PROPOSITION 15

*Parts have the same ratio as the same multiples of them taken in corresponding order*

For let $AB$ be the same multiple of $C$ that $DE$ is of $F$

I say that as $C$ is to $F$ so is $AB$ to $DE$

For since $AB$ is the same multiple of $C$ that $DE$ is of $F$ as many magnitudes as there are in $AB$ equal to $C$ so many are there also in $DE$ equal to $F$

Let $AB$ be divided into the magnitudes $AG$ $GH$ $HB$ equal to $C$

and $DE$ into the magnitudes $DK$ $KL$ $LE$ equal to $F$

then the multitude of the magnitudes $AG$ $GH$ $HB$ will be equal to the multitude of the magnitudes $DK$ $KL$ $LE$

And since $AG$ $GH$ $HB$ are equal to one another

and $DK$ $KL$ $LE$ are also equal to one another

therefore as $AG$ is to $DK$ so is $GH$ to $KL$ and $HB$ to $LE$ [v 7]

Therefore as one of the antecedents is to one of the consequents so will all the antecedents be to all the consequents [v 12]

therefore as $AG$ is to $DK$ so is $AB$ to $DE$

But $AG$ is equal to $C$ and $DK$ to $F$,

therefore, as $C$ is to $F$, so is $AB$ to $DE$

Therefore etc Q E D

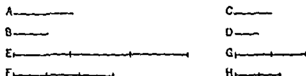
PROPOSITION 16

*If four magnitudes be proportional, they will also be proportional alternately*

Let $A$, $B$, $C$, $D$ be four proportional magnitudes

so that as $A$ is to $B$, so is $C$ to $D$,

I say that they will also be so alternately, that is, as $A$ is to $C$, so is $B$ to $D$




For of $A$ $B$ let equimultiples $E$, $F$ be taken

and of $C$, $D$ other, chance, equimultiples $G$ $H$

Then, since $E$ is the same multiple of $A$ that $F$ is of $B$,

and parts have the same ratio as the same multiples of them [v 15]

therefore as $A$ is to $B$, so is $E$ to $F$

But as $A$ is to $B$ so is $C$ to $D$,

therefore also as $C$ is to $D$ so is $E$ to $F$ [v 11]

Again, since $G$, $H$ are equimultiples of $C$, $D$,

therefore, as $C$ is to $D$, so is $G$ to $H$ [v 15]

But as $C$ is to $D$ so is $E$ to $F$,

therefore also as $E$ is to $F$ so is $G$ to $H$ [v 11]

But if four magnitudes be proportional and the first be greater than the third,

the second will also be greater than the fourth,

if equal equal

and if less less [v 14]

Therefore if $E$ is in excess of $G$, $F$ is also in excess of $H$,

if equal equal,

and if less less

Now $E$ $F$ are equimultiples of $A$ $B$

and $G$ $H$ other, chance equimultiples of $C$ $D$,

therefore as $A$ is to $C$ so is $B$ to $D$ [v Def 5]

Therefore etc Q E D

PROPOSITION 17

*If magnitudes be proportional* componendo *they will also be proportional* separando

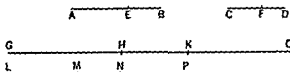
Let $AB$, $BE$ $CD$, $DF$ be magnitudes proportional *componendo* so that as $AB$ is to $BE$ so is $CD$ to $DF$,

I say that they will also be proportional *separando*, that is as $AE$ is to $EB$, so is $CF$ to $DF$

For of $AE$ $EB$ $CF$ $FD$ let equimultiples $GH$ $HK$ $LM$ $MN$ be taken,

and of $EB$ $FD$ other chance equimultiples $KO$ $NP$

Then, since $GH$ is the same multiple of $AE$ that $HK$ is of $EB$,
therefore $GH$ is the same multiple of $AE$ that $GK$ is of $AB$ [v 1]
But $GH$ is the same multiple of $AE$ that $LM$ is of $CF$
therefore $GK$ is the same multiple of $AB$ that $LM$ is of $CF$




Again, since $LM$ is the same multiple of $CF$ that $MN$ is of $FD$
therefore $LM$ is the same multiple of $CF$ that $LN$ is of $CD$ [v 1]
But $LM$ was the same multiple of $CF$ that $GK$ is of $AB$
therefore $GK$ is the same multiple of $AB$ that $LN$ is of $CD$
Therefore $GK$ $LN$ are equimultiples of $AB$, $CD$
Again, since $HK$ is the same multiple of $EB$ that $MN$ is of $FD$,
and $KO$ is also the same multiple of $EB$ that $NP$ is of $FD$,
therefore the sum $HO$ is also the same multiple of $EB$ that $MP$ is of $FD$ [v 2]
And, since as $AB$ is to $BE$ so is $CD$ to $DF$
and of $AB$, $CD$ equimultiples $GK$, $LN$ have been taken
and of $EB$, $FD$ equimultiples $HO$ $MP$
therefore, if $GK$ is in excess of $HO$, $LN$ is also in excess of $MP$,
if equal, equal
and if less, less
Let $GH$ be in excess of $KO$,
then if $HK$ be added to each,
$GK$ is also in excess of $HO$
But we saw that if $GK$ was in excess of $HO$ $LN$ was also in excess of $MP$,
therefore $LN$ is also in excess of $MP$
and if $MN$ be subtracted from each,
$LM$ is also in excess of $NP$
so that if $GH$ is in excess of $KO$ $LM$ is also in excess of $NP$
Similarly we can prove that
if $GH$ be equal to $KO$ $LM$ will also be equal to $NP$
and if less less
And $GH$ $LM$ are equimultiples of $AE$ $CF$
while $KO$ $NP$ are other chance equimultiples of $EB$, $FD$,
therefore, as $AE$ is to $EB$ so is $CF$ to $FD$
Therefore etc Q E D

PROPOSITION 18

*If magnitudes be proportional* separando *they will also be proportional* componendo

Let $AE$ $EB$ $CF$ $FD$ be magnitudes proportional *separando*, so that as $AE$ is to $EB$ so is $CF$ to $FD$

I say that they will also be proportional *componendo*, that is as $AB$ is to $BE$ so is $CD$ to $FD$

For, if $CD$ be not to $DF$ as $AB$ to $BE$,
then, as $AB$ is to $BE$, so will $CD$ be either to some magnitude less than $DF$ or to a greater

First, let it be in that ratio to a less magnitude $DG$

Then, since as $AB$ is to $BE$, so is $CD$ to $DG$,

they are magnitudes proportional *componendo*,
so that they will also be proportional *separando* [v 17]

Therefore as $AE$ is to $EB$ so is $CG$ to $GD$

But also by hypothesis

as $AE$ is to $EB$, so is $CF$ to $FD$

Therefore also as $CG$ is to $GD$, so is $CF$ to $FD$ [v 11]

But the first $CG$ is greater than the third $CF$,

therefore the second $GD$ is also greater than the fourth $FD$ [v 14]

But it is also less which is impossible

Therefore as $AB$ is to $BE$ so is not $CD$ to a less magnitude than $FD$

Similarly we can prove that neither is it in that ratio to a greater,

it is therefore in that ratio to $FD$ itself

Therefore etc Q E D

PROPOSITION 19

*If as a whole is to a whole so is a part subtracted to a part subtracted the remainder will also be to the remainder as whole to whole*

For as the whole $AB$ is to the whole $CD$, so let the part $AE$ subtracted be to the part $CF$ subtracted,

I say that the remainder $EB$ will also be to the remainder $FD$ as the whole $AB$ to the whole $CD$

For since as $AB$ is to $CD$, so is $AE$ to $CF$,

alternately also as $BA$ is to $AE$, so is $DC$ to $CF$ [v 16]

And since the magnitudes are proportional *componendo* they will also be proportional *separando*, [v 17]

that is as $BE$ is to $EA$ so is $DF$ to $CF$

and alternately

as $BE$ is to $DF$ so is $EA$ to $FC$ [v 16]

But as $AE$ is to $CF$ so by hypothesis is the whole $AB$ to the whole $CD$

Therefore also the remainder $EB$ will be to the remainder $FD$ as the whole $AB$ is to the whole $CD$ [v 11]

Therefore etc

[PORISM From this it is manifest that if magnitudes be proportional *componendo* they will also be proportional *convertendo* ] Q E D

PROPOSITION 20

*If there be three magnitudes and others equal to them in multitude which taken two and two are in the same ratio and if ex aequali the first be greater than the third the fourth will also be greater than the sixth if equal equal and if less less*

Let there be three magnitudes $A$ $B$ $C$ and others $D$ $E$, $F$ equal to them in multitude which taken two and two are in the same ratio so that,

as $A$ is to $B$ so is $D$ to $E$

as $B$ is to $C$ so is $E$ to $F$
and let $A$ be greater than $C$ *ex aequali*,

I say that $D$ will also be greater than $F$, if $A$ is equal to $C$, equal, and if less, less

For since $A$ is greater than $C$, and $B$ is some other magnitude and the greater has to the same a greater ratio than the less has [V 8] therefore $A$ has to $B$ a greater ratio than $C$ has to $B$

But as $A$ is to $B$, so is $D$ to $E$,
and as $C$ is to $B$ inversely so is $F$ to $E$,
therefore $D$ has also to $E$ a greater ratio than $F$ has to $E$ [V 13]

But of magnitudes which have a ratio to the same, that which has a greater ratio is greater, [V 10]
therefore $D$ is greater than $F$

Similarly we can prove that, if $A$ be equal to $C$, $D$ will also be equal to $F$, and if less less

Therefore etc Q E D

## Proposition 21

*If there be three magnitudes and others equal to them in multitude which taken two and two together are in the same ratio and the proportion of them be perturbed then if* ex aequali *the first magnitude is greater than the third the fourth will also be greater than the sixth if equal equal and if less less*

Let there be three magnitudes $A$, $B$, $C$ and others $D$, $E$ $F$ equal to them in multitude which taken two and two are in the same ratio and let the proportion of them be perturbed so that,

as $A$ is to $B$ so is $E$ to $F$
and as $B$ is to $C$ so is $D$ to $E$,
and let $A$ be greater than $C$ *ex aequali*,

I say that $D$ will also be greater than $F$ if $A$ is equal to $C$ equal, and if less less

For since $A$ is greater than $C$ and $B$ is some other magnitude, therefore $A$ has to $B$ a greater ratio than $C$ has to $B$ [V 8]

But as $A$ is to $B$ so is $E$ to $F$
and as $C$ is to $B$ inversely so is $E$ to $D$
Therefore also $E$ has to $F$ a greater ratio than $E$ has to $D$ [V 13]

But that to which the same has a greater ratio is less [V 10]
therefore $F$ is less than $D$
therefore $D$ is greater than $F$

Similarly we can prove that if $A$ be equal to $C$ $D$ will also be equal to $F$ and if less less

Therefore etc Q E D

PROPOSITION 22

*If there be any number of magnitudes whatever, and others equal to them in multitude which taken two and two together are in the same ratio they will also be in the same ratio* ex aequali

Let there be any number of magnitudes A B, C, and others D E, F equal to them in multitude which taken two and two together are in the same ratio, so that,

as A is to B so is D to E

and, as B is to C so is E to F,

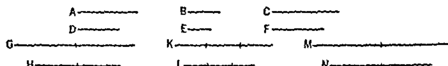
I say that they will also be in the same ratio *ex aequali*,

<that is as A is to C so is D to F>

For of A, D let equimultiples G H be taken

and of B, E other chance, equimultiples K L,

and, further of C, F other, chance, equimultiples M, N




Then, since as A is to B so is D to E,

and of A D equimultiples G, H have been taken,

and of B, E other chance equimultiples K, L,

therefore as G is to K, so is H to L [v 4]

For the same reason also

as K is to M so is L to N

Since then there are three magnitudes G K, M and others H, L N equal to them in multitude, which taken two and two together are in the same ratio, therefore *ex aequali* if G is in excess of M, H is also in excess of N

if equal, equal and if less less [v 20]

And G H are equimultiples of A D

and M N other chance equimultiples of C, F

Therefore as A is to C so is D to F [v Def 5]

Therefore etc Q E D

PROPOSITION 23

*If there be three magnitudes and others equal to them in multitude which taken two and two together are in the same ratio and the proportion of them be perturbed they will also be in the same ratio* ex aequali

Let there be three magnitudes A B C, and others equal to them in multitude, which taken two and two together are in the same proportion, namely D, E, F and let the proportion of them be perturbed so that,

as A is to B so is E to F

and, as B is to C so is D to E

I say that as A is to C so is D to F

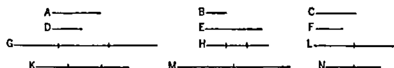
Of A, B D let equimultiples G H K be taken

and of C E F other chance equimultiples L M, N

Then since G H are equimultiples of A B

and parts have the same ratio as the same multiples of them, [v 15]
therefore, as $A$ is to $B$, so is $G$ to $H$

For the same reason also




as $E$ is to $F$, so is $M$ to $N$

And as $A$ is to $B$, so is $E$ to $F$,
therefore also, as $G$ is to $H$, so is $M$ to $N$ [v 11]

Next, since as $B$ is to $C$, so is $D$ to $E$,
alternately, also, as $B$ is to $D$, so is $C$ to $E$ [v 16]

And since $H$ $K$ are equimultiples of $B$, $D$
and parts have the same ratio as their equimultiples
therefore as $B$ is to $D$, so is $H$ to $K$ [v 15]

But as $B$ is to $D$ so is $C$ to $E$,
therefore also as $H$ is to $K$, so is $C$ to $E$ [v 11]

Again, since $L$, $M$ are equimultiples of $C$, $E$,
therefore as $C$ is to $E$, so is $L$ to $M$ [v 15]

But, as $C$ is to $E$ so is $H$ to $K$,
therefore also as $H$ is to $K$, so is $L$ to $M$ [v 11]
and alternately as $H$ is to $L$ so is $K$ to $M$ [v 16]

But it was also proved that,
as $G$ is to $H$, so is $M$ to $N$

Since then there are three magnitudes $G$ $H$ $L$ and others equal to them in multitude $K$, $M$, $N$ which taken two and two together are in the same ratio,
and the proportion of them is perturbed,
therefore, *ex aequali*, if $G$ is in excess of $L$, $K$ is also in excess of $N$,
if equal equal, and if less, less [v 21]

And $G$, $K$ are equimultiples of $A$ $D$
and $L$ $N$ of $C$, $F$

Therefore as $A$ is to $C$, so is $D$ to $F$

Therefore etc Q E D

### Proposition 24

*If a first magnitude have to a second the same ratio as a third has to a fourth and also a fifth have to the second the same ratio as a sixth to the fourth the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth*

Let a first magnitude $AB$ have to a second $C$ the same ratio as a third $DE$ has to a fourth $F$ and let also a fifth $BG$ have to the second $C$ the same ratio as a sixth $EH$ has to the fourth $F$,

I say that the first and fifth added together, $AG$ will have to the second $C$ the same ratio as the third and sixth $DH$ has to the fourth $F$

For since as $BG$ is to $C$, so is $EH$ to $F$

inversely as $C$ is to $BG$, so is $F$ to $EH$

Since then, as $AB$ is to $C$, so is $DE$ to $F$,

and as $C$ is to $BG$ so is $F$ to $EH$,

therefore *ex aequali* as $AB$ is to $BG$, so is $DE$ to $EH$ [v 22]

And, since the magnitudes are proportional *separando*, they will also be proportional *componendo*, [v 18]

therefore, as $AG$ is to $GB$, so is $DH$ to $HE$

But also, as $BG$ is to $C$, so is $EH$ to $F$,

therefore, *ex aequali* as $AG$ is to $C$, so is $DH$ to $F$ [v 22]

Therefore etc Q E D

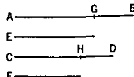
## Proposition 25

*If four magnitudes be proportional the greatest and the least are greater than the remaining two*

Let the four magnitudes $AB$, $CD$ $E$, $F$ be proportional so that as $AB$ is to $CD$, so is $E$ to $F$ and let $AB$ be the greatest of them and $F$ the least

I say that $AB$, $F$ are greater than $CD$, $E$

For let $AG$ be made equal to $E$, and $CH$ equal to $F$




Since as $AB$ is to $CD$, so is $E$ to $F$,

and $E$ is equal to $AG$ and $F$ to $CH$,

therefore as $AB$ is to $CD$ so is $AG$ to $CH$

And since as the whole $AB$ is to the whole $CD$ so is the part $AG$ subtracted to the part $CH$ subtracted

the remainder $GB$ will also be to the remainder $HD$ as the whole $AB$ is to the whole $CD$ [v 19]

But $AB$ is greater than $CD$

therefore $GB$ is also greater than $HD$

And since $AG$ is equal to $E$ and $CH$ to $F$,

therefore $AG$ $F$ are equal to $CH$, $E$

And if $GB$ $HD$ being unequal and $GB$ greater $AG$, $F$ be added to $GB$ and $CH$ $E$ be added to $HD$,

it follows that $AB$, $F$ are greater than $CD$, $E$

Therefore etc Q E D

# BOOK SIX

## DEFINITIONS

1 *Similar rectilineal figures* are such as have their angles severally equal and the sides about the equal angles proportional

2 A straight line is said to have been *cut in extreme and mean ratio* when, as the whole line is to the greater segment so is the greater to the less

3 The *height* of any figure is the perpendicular drawn from the vertex to the base

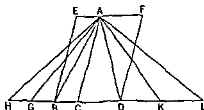
## BOOK VI PROPOSITIONS

### Proposition 1

*Triangles and parallelograms which are under the same height are to one another as their bases*

Let $ABC$ $ACD$ be triangles and $EC$ $CF$ parallelograms under the same height,

I say that as the base $BC$ is to the base $CD$ so is the triangle $ABC$ to the triangle $ACD$ and the parallelogram $EC$ to the parallelogram $CF$




For let $BD$ be produced in both directions to the points $H$, $L$ and let [any number of straight lines] $BG$, $GH$ be made equal to the base $BC$, and any number of straight lines $DK$ $KL$ equal to the base $CD$,

let $AG$ $AH$ $AK$, $AL$ be joined

Then since $CB$, $BG$, $GH$ are equal to one another

the triangles $ABC$ $AGB$ $AHG$ are also equal to one another [I 38]

Therefore whatever multiple the base $HC$ is of the base $BC$ that multiple also is the triangle $AHC$ of the triangle $ABC$

For the same reason

whatever multiple the base $LC$ is of the base $CD$ that multiple also is the triangle $ALC$ of the triangle $ACD$

and if the base $HC$ is equal to the base $CL$ the triangle $AHC$ is also equal to the triangle $ACL$

if the base $HC$ is in excess of the base $CL$ the triangle $AHC$ is also in excess of the triangle $ACL$

and if less less

Thus there being four magnitudes two bases $BC$ $CD$ and the triangles $ABC$ $ACD$

equimultiples have been taken of the base $BC$ and the triangle $ABC$, namely the base $HC$ and the triangle $AHC$
and of the base $CD$ and the triangle $ADC$ other chance equimultiples namely the base $LC$ and the triangle $ALC$,

and it has been proved that

if the base $HC$ is in excess of the base $CL$, the triangle $AHC$ is also in excess of *the triangle* $ALC$,

if equal equal, and, if less less

Therefore as the base $BC$ is to the base $CD$, so is the triangle $ABC$ to the triangle $ACD$ [v Def 5]

Next, since the parallelogram $EC$ is double of the triangle $ABC$, [i 41]

*and the parallelogram* $FC$ *is double of the triangle* $ACD$

while parts have the same ratio as the same multiples of them [v 15]

therefore as the triangle $ABC$ is to the triangle $ACD$ so is the parallelogram $EC$ to the parallelogram $FC$

Since then it was proved that as the base $BC$ is to $CD$ so is the triangle $ABC$ to the triangle $ACD$
and as the triangle $ABC$ is to the triangle $ACD$ so is the parallelogram $EC$ to the parallelogram $CF$
therefore also as the base $BC$ is to the base $CD$ so is the parallelogram $EC$ to the parallelogram $FC$ [v 11]

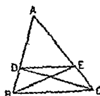
Therefore etc Q E D

## Proposition 2

*If a straight line be drawn parallel to one of the sides of a triangle it will cut the sides of the triangle proportionally and if the sides of the triangle be cut proportionally the line joining the points of section will be parallel to the remaining side of the triangle*

For let $DE$ be drawn parallel to $BC$ one of the sides of the triangle $ABC$,

I say that as $BD$ is to $DA$ so is $CE$ to $EA$




For let $BE$ $CD$ be joined

Therefore the triangle $BDE$ is equal to the triangle $CDE$
for they are on the same base $DE$ and in the same parallels $DE$ $BC$ [i 38]

And the triangle $ADE$ is another area

But equals have the same ratio to the same [v 7]

therefore as the triangle $BDE$ is to the triangle $ADE$ so is the triangle $CDE$ to the triangle $ADE$

But as the triangle $BDE$ is to $ADE$ so is $BD$ to $DA$
for being under the same height the perpendicular drawn from $E$ to $AB$ they are to one another as their bases [vi 1]

For the same reason also

as the triangle $CDE$ is to $ADE$ so is $CE$ to $EA$

Therefore also as $BD$ is to $DA$ so is $CE$ to $EA$ [v 11]

Again let the sides $AB$ $AC$ of the triangle $ABC$ be cut proportionally so that as $BD$ is to $DA$ so is $CE$ to $EA$ and let $DE$ be joined

I say that $DE$ is parallel to $BC$

For, with the same construction

since as $BD$ is to $DA$ so is $CE$ to $EA$

but, as $BD$ is to $DA$, so is the triangle $BDE$ to the triangle $ADE$

and as $CE$ is to $EA$, so is the triangle $CDE$ to the triangle $ADE$, [VI 1]

therefore also,

as the triangle $BDE$ is to the triangle $ADE$, so is the triangle $CDE$ to the triangle $ADE$ [V 11]

Therefore each of the triangles $BDE$ $CDE$ has the same ratio to $ADE$

Therefore the triangle $BDE$ is equal to the triangle $CDE$, [V 9]

and they are on the same base $DE$

But equal triangles which are on the same base are also in the same parallels [I 39]

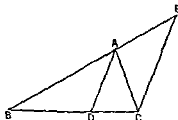
Therefore $DE$ is parallel to $BC$

Therefore etc Q E D

## Proposition 3

*If an angle of a triangle be bisected and the straight line cutting the angle cut the base also the segments of the base will have the same ratio as the remaining sides of the triangle and, if the segments of the base have the same ratio as the remaining sides of the triangle the straight line joined from the vertex to the point of section will bisect the angle of the triangle*

Let $ABC$ be a triangle and let the angle $BAC$ be bisected by the straight line $AD$,




I say that as $BD$ is to $CD$ so is $BA$ to $AC$

For let $CE$ be drawn through $C$ parallel to $DA$, and let $BA$ be carried through and meet it at $E$

Then since the straight line $AC$ falls upon the parallels $AD$ $EC$

the angle $ACE$ is equal to the angle $CAD$ [I 29]

But the angle $CAD$ is by hypothesis equal to the angle $BAD$,

therefore the angle $BAD$ is also equal to the angle $ACE$

Again since the straight line $BAE$ falls upon the parallels $AD$ $EC$,

the exterior angle $BAD$ is equal to the interior angle $AEC$ [I 29]

But the angle $ACE$ was also proved equal to the angle $BAD$

therefore the angle $ACE$ is also equal to the angle $AEC$,

so that the side $AE$ is also equal to the side $AC$ [I 6]

And since $AD$ has been drawn parallel to $EC$ one of the sides of the triangle $BCE$

therefore proportionally as $BD$ is to $DC$, so is $BA$ to $AE$

But $AE$ is equal to $AC$, [VI 2]

therefore as $BD$ is to $DC$ so is $BA$ to $AC$

Again let $BA$ be to $AC$ as $BD$ to $DC$, and let $AD$ be joined,

I say that the angle $BAC$ has been bisected by the straight line $AD$

For with the same construction,

since as $BD$ is to $DC$, so is $BA$ to $AC$

and also as $BD$ is to $DC$ so is $BA$ to $AE$ for $AD$ has been drawn parallel to

*EC*, one of the sides of the triangle *BCE* [VI 2]

therefore also, as *BA* is to *AC*, so is *BA* to *AE* [V 11]

Therefore *AC* is equal to *AE* [V 9]

so that the angle *AEC* is also equal to the angle *ACE* [I 5]

But the angle *AEC* is equal to the exterior angle *BAD* [I 29]

and the angle *ACE* is equal to the alternate angle *CAD*, [id]

therefore the angle *BAD* is also equal to the angle *CAD*

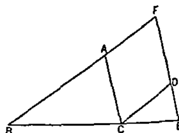
Therefore the angle *BAC* has been bisected by the straight line *AD*

Therefore etc Q E D

PROPOSITION 4

*In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles*

Let *ABC*, *DCE* be equiangular triangles having the angle *ABC* equal to the angle *DCE* the angle *BAC* to the angle *CDE* and further the angle *ACB* to the angle *CED*,




I say that in the triangles *ABC*, *DCE* the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles

For let *BC* be placed in a straight line with *CE*

Then since the angles *ABC* *ACB* are less than two right angles [I 17]

and the angle *ACB* is equal to the angle *DEC*,

therefore the angles *ABC* *DEC* are less than two right angles

therefore *BA*, *ED* when produced, will meet [I Post 5]

Let them be produced and meet at *F*

Now since the angle *DCE* is equal to the angle *ABC*,

*BF* is parallel to *CD* [I 28]

Again since the angle *ACB* is equal to the angle *DEC*

*AC* is parallel to *FE* [I 28]

Therefore *FACD* is a parallelogram,

therefore *FA* is equal to *DC*, and *AC* to *FD* [I 34]

And since *AC* has been drawn parallel to *FE*, one side of the triangle *FBE*,

therefore as *BA* is to *AF* so is *BC* to *CE* [VI 2]

But *AF* is equal to *CD*

therefore as *BA* is to *CD*, so is *BC* to *CE*,

and alternately as *AB* is to *BC*, so is *DC* to *CE* [V 16]

Again since *CD* is parallel to *BF*,

therefore as *BC* is to *CE* so is *FD* to *DE* [VI 2]

But *FD* is equal to *AC*,

therefore as *BC* is to *CE* so is *AC* to *DE*,

and alternately as *BC* is to *CA*, so is *CE* to *ED* [V 16]

Since then it was proved that

as *AB* is to *BC* so is *DC* to *CE*

and, as *BC* is to *CA* so is *CE* to *ED*,

therefore *ex aequali* as *BA* is to *AC*, so is *CD* to *DE* [V 22]

Therefore etc Q E D

PROPOSITION 5

*If two triangles have their sides proportional the triangles will be equiangular and will have those angles equal which the corresponding sides subtend*

Let $ABC$, $DEF$ be two triangles having their sides proportional, so that

as $AB$ is to $BC$, so is $DE$ to $EF$,
as $BC$ is to $CA$, so is $EF$ to $FD$,
and further, as $BA$ is to $AC$, so is $ED$ to $DF$,

I say that the triangle $ABC$ is equiangular with the triangle $DEF$, and they will have those angles equal which the corresponding sides subtend namely the angle $ABC$ to the angle $DEF$ the angle $BCA$ to the angle $EFD$ and further the angle $BAC$ to the angle $EDF$

For on the straight line $EF$, and at the points $E$, $F$ on it, let there be constructed the angle $FEG$ equal to the angle $ABC$, and the angle $EFG$ equal to the angle $ACB$, [I 23]

therefore the remaining angle at $A$ is equal to the remaining angle at $G$ [I 32]

Therefore the triangle $ABC$ is equiangular with the triangle $GEF$




Therefore in the triangles $ABC$ $GEF$ the sides about the equal angles are proportional and those are corresponding sides which subtend the equal angles [VI 4]

therefore as $AB$ is to $BC$, so is $GE$ to $EF$

But as $AB$ is to $BC$, so by hypothesis is $DE$ to $EF$,

therefore as $DE$ is to $EF$, so is $GE$ to $EF$ [V 11]

Therefore each of the straight lines $DE$ $GE$ has the same ratio to $EF$

therefore $DE$ is equal to $GE$ [V 9]

For the same reason

$DF$ is also equal to $GF$

Since then $DE$ is equal to $EG$

and $EF$ is common,

the two sides $DE$, $EF$ are equal to the two sides $GE$ $EF$,
and the base $DF$ is equal to the base $FG$
therefore the angle $DEF$ is equal to the angle $GEF$, [I 8]
and the triangle $DEF$ is equal to the triangle $GEF$,

and the remaining angles are equal to the remaining angles namely those which the equal sides subtend [I 4]

Therefore the angle $DFE$ is also equal to the angle $GFE$,
and the angle $EDF$ to the angle $EGF$

And, since the angle $FED$ is equal to the angle $GEF$
while the angle $GEF$ is equal to the angle $ABC$
therefore the angle $ABC$ is also equal to the angle $DEF$

For the same reason

the angle $ACB$ is also equal to the angle $DFE$
and further the angle at $A$ to the angle at $D$
therefore the triangle $ABC$ is equiangular with the triangle $DEF$

Therefore etc Q E D

PROPOSITION 6

*If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend*

Let *ABC* *DEF* be two triangles having one angle *BAC* equal to one angle *EDF* and the sides about the equal angles proportional so that,

as *BA* is to *AC*, so is *ED* to *DF*,

I say that the triangle *ABC* is equiangular with the triangle *DEF*, and will have the angle *ABC* equal to the angle *DEF*, and the angle *ACB* to the angle *DFE*

For on the straight line *DF* and at the points *D*, *F* on it, let there be constructed the angle *FDG* equal to either of the angles *BAC*, *EDF*, and the angle *DFG* equal to the angle *ACB*, [I 23]
therefore the remaining angle at *B* is equal to the remaining angle at *G* [I 32]

Therefore the triangle *ABC* is equiangular with the triangle *DGF*

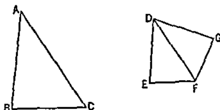
Therefore proportionally as *BA* is to *AC*, so is *GD* to *DF* [VI 4]

But by hypothesis as *BA* is to *AC* so also is *ED* to *DF*,
therefore also as *ED* is to *DF* so is *GD* to *DF* [V 11]

Therefore *ED* is equal to *DG*, [V 9]

and *DF* is common

therefore the two sides *ED*, *DF* are equal to the two sides *GD* *DF*, and the angle *EDF* is equal to the angle *GDF*,




therefore the base *EF* is equal to the base *GF*,

and the triangle *DEF* is equal to the triangle *DGF*

and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]

Therefore the angle *DFG* is equal to the angle *DFE*

and the angle *DGF* to the angle *DEF*

But the angle *DFG* is equal to the angle *ACB*,

therefore the angle *ACB* is also equal to the angle *DFE*

And by hypothesis the angle *BAC* is also equal to the angle *EDF*, therefore the remaining angle at *B* is also equal to the remaining angle at *E*, [I 32]

therefore the triangle *ABC* is equiangular with the triangle *DEF*

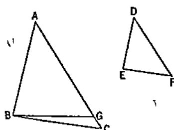
Therefore etc Q E D

PROPOSITION 7

*If two triangles have one angle equal to one angle the sides about other angles proportional and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional*

Let *ABC*, *DEF* be two triangles having one angle equal to one angle the angle *BAC* to the angle *EDF*, the sides about other angles *ABC* *DEF* proportional so that as *AB* is to *BC* so is *DE* to *EF*, and first each of the remaining angles at *C* *F* less than a right angle,

I say that the triangle $ABC$ is equiangular with the triangle $DEF$, the angle $ABC$ will be equal to the angle $DEF$, and the remaining angle namely the angle at $C$, equal to the remaining angle the angle at $F$




For if the angle $ABC$ is unequal to the angle $DEF$, one of them is greater

Let the angle $ABC$ be greater, and on the straight line $AB$, and at the point $B$ on it, let the angle $ABG$ be constructed equal to the angle $DEF$ [I 23]

Then, since the angle $A$ is equal to $D$,
and the angle $ABG$ to the angle $DEF$,
therefore the remaining angle $AGB$ is equal to the remaining angle $DFE$ [I 32]

Therefore the triangle $ABG$ is equiangular with the triangle $DEF$

Therefore, as $AB$ is to $BG$, so is $DE$ to $EF$ [VI 4]

But as $DE$ is to $EF$ so by hypothesis is $AB$ to $BC$,
therefore $AB$ has the same ratio to each of the straight lines $BC$, $BG$, [V 11]
therefore $BC$ is equal to $BG$ [V 9]
so that the angle at $C$ is also equal to the angle $BGC$ [I 5]

But, by hypothesis the angle at $C$ is less than a right angle
therefore the angle $BGC$ is also less than a right angle,
so that the angle $AGB$ adjacent to it is greater than a right angle [I 13]

And it was proved equal to the angle at $F$,
therefore the angle at $F$ is also greater than a right angle

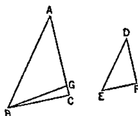
But it is by hypothesis less than a right angle which is absurd

Therefore the angle $ABC$ is not unequal to the angle $DEF$,
therefore it is equal to it

But the angle at $A$ is also equal to the angle at $D$
therefore the remaining angle at $C$ is equal to the remaining angle at $F$ [I 32]

Therefore the triangle $ABC$ is equiangular with the triangle $DEF$

But again, let each of the angles at $C$, $F$ be supposed not less than a right angle,




I say again that in this case too the triangle $ABC$ is equiangular with the triangle $DEF$

For, with the same construction, we can prove similarly that
$BC$ is equal to $BG$
so that the angle at $C$ is also equal to the angle $BGC$ [I 5]

But the angle at $C$ is not less than a right angle,
therefore neither is the angle $BGC$ less than a right angle

Thus in the triangle $BGC$ the two angles are not less than two right angles which is impossible [I 17]

Therefore once more the angle $ABC$ is not unequal to the angle $DEF$,
therefore it is equal to it

But the angle at $A$ is also equal to the angle at $D$
therefore the remaining angle at $C$ is equal to the remaining angle at $F$ [I 32]

Therefore the triangle $ABC$ is equiangular with the triangle $DEF$

Therefore etc Q E D

### Proposition 8

*If in a right angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another*

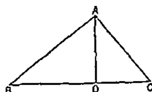
Let $ABC$ be a right-angled triangle having the angle $BAC$ right, and let $AD$ be drawn from $A$ perpendicular to $BC$,

I say that each of the triangles $ABD$, $ADC$ is similar to the whole $ABC$ and further, they are similar to one another

For, since the angle $BAC$ is equal to the angle $ADB$, for each is right and the angle at $B$ is common to the two triangles $ABC$ and $ABD$

therefore the remaining angle $ACB$ is equal to the remaining angle $BAD$, [I 32]

therefore the triangle $ABC$ is equiangular with the triangle $ABD$




Therefore as $BC$ which subtends the right angle in the triangle $ABC$ is to $BA$ which subtends the right angle in the triangle $ABD$, so is $AB$ itself which subtends the angle at $C$ in the triangle $ABC$ to $BD$ which subtends the equal angle $BAD$ in the triangle $ABD$ and so also is $AC$ to $AD$ which subtends the angle at $B$ common to the two triangles [VI 4]

Therefore the triangle $ABC$ is both equiangular to the triangle $ABD$ and has the sides about the equal angles proportional

Therefore the triangle $ABC$ is similar to the triangle $ABD$ [VI Def 1]

Similarly we can prove that

the triangle $ABC$ is also similar to the triangle $ADC$

therefore each of the triangles $ABD$, $ADC$ is similar to the whole $ABC$

I say next that the triangles $ABD$, $ADC$ are also similar to one another

For since the right angle $BDA$ is equal to the right angle $ADC$,

and moreover the angle $BAD$ was also proved equal to the angle at $C$

therefore the remaining angle at $B$ is also equal to the remaining angle $DAC$, [I 32]

therefore the triangle $ABD$ is equiangular with the triangle $ADC$

Therefore as $BD$ which subtends the angle $BAD$ in the triangle $ABD$ is to $DA$ which subtends the angle at $C$ in the triangle $ADC$ equal to the angle $BAD$, so is $AD$ itself which subtends the angle at $B$ in the triangle $ABD$ to $DC$ which subtends the angle $DAC$ in the triangle $ADC$ equal to the angle at $B$, and so also is $BA$ to $AC$ these sides subtending the right angles, [VI 4]

therefore the triangle $ABD$ is similar to the triangle $ADC$ [VI Def 1]

Therefore etc

PORISM From this it is clear that if in a right angled triangle a perpendicular be drawn from the right angle to the base the straight line so drawn is a mean proportional between the segments of the base Q E D

### Proposition 9

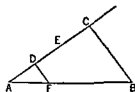
*From a given straight line to cut off a prescribed part*

Let $AB$ be the given straight line,

thus it is required to cut off from $AB$ a prescribed part

Let the third part be that prescribed

Let a straight line $AC$ be drawn through from $A$ containing with $AB$ any angle,




let a point $D$ be taken at random on $AC$, and let $DE$, $EC$ be made equal to $AD$ [I 3]

Let $BC$ be joined, and through $D$ let $DF$ be drawn parallel to it [I 31]

Then since $FD$ has been drawn parallel to $BC$ one of the sides of the triangle $ABC$,

therefore, proportionally, as $CD$ is to $DA$, so is $BF$ to $FA$ [VI 2]

But $CD$ is double of $DA$

therefore $BF$ is also double of $FA$,

therefore $BA$ is triple of $AF$

Therefore from the given straight line $AB$ the prescribed third part $AF$ has been cut off

Q E F

## Proposition 10

*To cut a given uncut straight line similarly to a given cut straight line*

Let $AB$ be the given uncut straight line and $AC$ the straight line cut at the points $D$ $E$, and let them be so placed as to contain any angle,

let $CB$ be joined and through $D$, $E$ let $DF$ $EG$ be drawn parallel to $BC$, and through $D$ let $DHK$ be drawn parallel to $AB$ [I 31]

Therefore each of the figures $FH$, $HB$ is a parallelogram,

therefore $DH$ is equal to $FG$ and $HK$ to $GB$ [I 34]

Now since the straight line $HE$ has been drawn parallel to $KC$, one of the sides of the triangle $DKC$

therefore, proportionally, as $CE$ is to $ED$, so is $KH$ to $HD$ [VI 2]

But $KH$ is equal to $BG$ and $HD$ to $GF$,

therefore, as $CE$ is to $ED$ so is $BG$ to $GF$

Again since $FD$ has been drawn parallel to $GE$, one of the sides of the triangle $AGE$

therefore proportionally, as $ED$ is to $DA$ so is $GF$ to $FA$ [VI 2]

But it was also proved that

as $CE$ is to $ED$, so is $BG$ to $GF$,

therefore as $CE$ is to $ED$ so is $BG$ to $GF$

and as $ED$ is to $DA$ so is $GF$ to $FA$

Therefore the given uncut straight line $AB$ has been cut similarly to the given cut straight line $AC$

Q E F

## Proposition 11

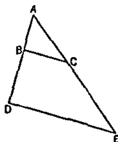
*To two given straight lines to find a third proportional*

Let $BA$ $AC$ be the two given straight lines and let them be placed so as to contain any angle

thus it is required to find a third proportional to $BA$, $AC$

For let them be produced to the points $D$, $E$, and let $BD$ be made equal to $AC$, [I 3]

let $BC$ be joined, and through $D$ let $DE$ be drawn parallel to it [I 31]




Since then, $BC$ has been drawn parallel to $DE$, one of the sides of the triangle $ADE$, proportionally as $AB$ is to $BD$, so is $AC$ to $CE$ [VI 2]

But $BD$ is equal to $AC$,

therefore as $AB$ is to $AC$, so is $AC$ to $CE$

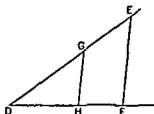
Therefore to two given straight lines $AB$, $AC$ a third proportional to them $CE$, has been found Q E F

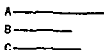
PROPOSITION 12

*To three given straight lines to find a fourth proportional*

Let $A$ $B$, $C$ be the three given straight lines,

thus it is required to find a fourth proportional to $A$, $B$, $C$




Let two straight lines $DE$, $DF$ be set out containing any angle $EDF$,

let $DG$ be made equal to $A$ $GE$ equal to $B$, and further $DH$ equal to $C$,

let $GH$ be joined, and let $EF$ be drawn through $E$ parallel to it [I 31]

Since then, $GH$ has been drawn parallel to $EF$, one of the sides of the triangle $DEF$,

therefore as $DG$ is to $GE$ so is $DH$ to $HF$ [VI 2]

But $DG$ is equal to $A$, $GE$ to $B$ and $DH$ to $C$,

therefore as $A$ is to $B$ so is $C$ to $HF$

Therefore to the three given straight lines $A$, $B$ $C$ a fourth proportional $HF$ has been found Q E F

PROPOSITION 13

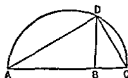
*To two given straight lines to find a mean proportional*

Let $AB$, $BC$ be the two given straight lines,

thus it is required to find a mean proportional to $AB$, $BC$

Let them be placed in a straight line, and let the semicircle $ADC$ be described on $AC$

let $BD$ be drawn from the point $B$ at right angles to the straight line $AC$ and let $AD$ $DC$ be joined




Since the angle $ADC$ is an angle in a semicircle it is right [III 31]

And, since in the right-angled triangle $ADC$ $DB$ has been drawn from the right angle perpendicular to the base,

therefore $DB$ is a mean proportional between the segments of the base, $AB$ $BC$ [VI 8 Por]

Therefore to the two given straight lines $AB$, $BC$ a mean proportional $DB$ has been found Q E F

PROPOSITION 14

*In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal*

Let $AB$, $BC$ be equal and equiangular parallelograms having the angles at $B$ equal and let $DB$ $BE$ be placed in a straight line

therefore $FB$, $BG$ are also in a straight line [I 14]

I say that in $AB$ $BC$ the sides about the equal angles are reciprocally proportional that is to say, that, as $DB$ is to $BE$, so is $GB$ to $BF$

For let the parallelogram $FE$ be completed

Since then, the parallelogram $AB$ is equal to the parallelogram $BC$

and $FE$ is another area

therefore as $AB$ is to $FE$, so is $BC$ to $FE$ [V 7]

But, as $AB$ is to $FE$ so is $DB$ to $BE$ [VI 1]

and as $BC$ is to $FE$ so is $GB$ to $BF$ [id]

therefore also as $DB$ is to $BE$ so is $GB$ to $BF$ [V 11]

Therefore in the parallelograms $AB$ $BC$ the sides about the equal angles are reciprocally proportional

Next let $GB$ be to $BF$ as $DB$ to $BE$

I say that the parallelogram $AB$ is equal to the parallelogram $BC$

For since as $DB$ is to $BE$ so is $GB$ to $BF$

while as $DB$ is to $BE$ so is the parallelogram $AB$ to the parallelogram $FE$ [VI 1]

and as $GB$ is to $BF$ so is the parallelogram $BC$ to the parallelogram $FE$, [VI 1]

therefore also as $AB$ is to $FE$ so is $BC$ to $FE$ [V 11]

therefore the parallelogram $AB$ is equal to the parallelogram $BC$ [V 9]

Therefore etc Q E D

PROPOSITION 15

*In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional and those triangles which have one angle equal to one angle and in which the sides about the equal angles are reciprocally proportional are equal*

Let $ABC$ $ADE$ be equal triangles having one angle equal to one angle namely the angle $BAC$ to the angle $DAE$

I say that in the triangles $ABC$, $ADE$ the sides about the equal angles are reciprocally proportional that is to say that

as $CA$ is to $AD$ so is $EA$ to $AB$

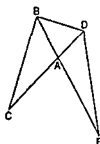
For let them be placed so that $CA$ is in a straight line with $AD$

therefore $EA$ is also in a straight line with $AB$

Let $BD$ be joined

Since, then the triangle $ABC$ is equal to the triangle $ADE$, and $BAD$ is an other area

therefore as the triangle $CAB$ is to the triangle $BAD$ so is the triangle $EAD$ to the triangle $BAD$ [v 7]




But, as $CAB$ is to $BAD$, so is $CA$ to $AD$ [vi 1]

and, as $EAD$ is to $BAD$ so is $EA$ to $AB$ [id]

Therefore also as $CA$ is to $AD$, so is $EA$ to $AB$ [v 11]

Therefore in the triangles $ABC$, $ADE$ the sides about the equal angles are reciprocally proportional

Next, let the sides of the triangles $ABC$ $ADE$ be reciprocally proportional that is to say, let $EA$ be to $AB$ as $CA$ to $AD$,

I say that the triangle $ABC$ is equal to the triangle $ADE$

For if $BD$ be again joined

since as $CA$ is to $AD$, so is $EA$ to $AB$,

while, as $CA$ is to $AD$, so is the triangle $ABC$ to the triangle $BAD$,

and as $EA$ is to $AB$, so is the triangle $EAD$ to the triangle $BAD$, [vi 1]

therefore as the triangle $ABC$ is to the triangle $BAD$, so is the triangle $EAD$ to the triangle $BAD$ [v 11]

Therefore each of the triangles $ABC$, $EAD$ has the same ratio to $BAD$

Therefore the triangle $ABC$ is equal to the triangle $EAD$ [v 9]

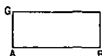
Therefore etc Q E D

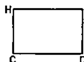
## Proposition 16

*If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means and if the rectangle contained by the extremes be equal to the rectangle contained by the means the four straight lines will be proportional*

Let the four straight lines $AB$ $CD$, $E$ $F$ be proportional so that, as $AB$ is to $CD$ so is $E$ to $F$

I say that the rectangle contained by $AB$ $F$ is equal to the rectangle contained by $CD$, $E$




Let $AG$ $CH$ be drawn from the points $A$ $C$ at right angles to the straight lines $AB$, $CD$ and let $AG$ be made equal to $F$ and $CH$ equal to $E$

Let the parallelograms $BG$ $DH$ be completed

Then since, as $AB$ is to $CD$ so is $E$ to $F$

while $E$ is equal to $CH$ and $F$ to $AG$

therefore as $AB$ is to $CD$, so is $CH$ to $AG$

Therefore in the parallelograms $BG$, $DH$ the sides about the equal angles are reciprocally proportional

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal [vi 14]

therefore the parallelogram *BG* is equal to the parallelogram *DH*

And *BG* is the rectangle *AB*, *F* for *AG* is equal to *F*

and *DH* is the rectangle *CD* *E*, for *E* is equal to *CH*,

therefore the rectangle contained by *AB* *F* is equal to the rectangle contained by *CD*, *E*

Next let the rectangle contained by *AB*, *F* be equal to the rectangle contained by *CD*, *E*,

I say that the four straight lines will be proportional so that, as *AB* is to *CD* so is *E* to *F*

For, with the same construction,

since the rectangle *AB*, *F* is equal to the rectangle *CD* *E*,

and the rectangle *AB*, *F* is *BG*, for *AG* is equal to *F*

and the rectangle *CD*, *E* is *DH*, for *CH* is equal to *E*

therefore *BG* is equal to *DH*

And they are equiangular

But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional [VI 14]

Therefore, as *AB* is to *CD* so is *CH* to *AG*

But *CH* is equal to *E*, and *AG* to *F*,

therefore, as *AB* is to *CD*, so is *E* to *F*

Therefore etc Q E D

## Proposition 17

*If three straight lines be proportional the rectangle contained by the extremes is equal to the square on the mean and, if the rectangle contained by the extremes be equal to the square on the mean the three straight lines will be proportional*

Let the three straight lines *A* *B*, *C* be proportional, so that as *A* is to *B*, so is *B* to *C*,

I say that the rectangle contained by *A* *C* is equal to the square on *B*

A———————
B—————— D——————
C————

Let *D* be made equal to *B*

Then since as *A* is to *B*, so is *B* to *C*

and *B* is equal to *D*

therefore as *A* is to *B* so is *D* to *C*

But if four straight lines be proportional the rectangle contained by the extremes is equal to the rectangle contained by the means [VI 16]

Therefore the rectangle *A*, *C* is equal to the rectangle *B* *D*

But the rectangle *B* *D* is the square on *B* for *B* is equal to *D*,

therefore the rectangle contained by *A*, *C* is equal to the square on *B*

Next let the rectangle *A*, *C* be equal to the square on *B* I say that as *A* is to *B*, so is *B* to *C*

For with the same construction

since the rectangle *A* *C* is equal to the square on *B*

while the square on *B* is the rectangle *B* *D* for *B* is equal to *D*

therefore the rectangle *A* *C* is equal to the rectangle *B* *D*

But if the rectangle contained by the extremes be equal to that contained by the means the four straight lines are proportional [VI 16]

Therefore as *A* is to *B* so is *D* to *C*

But $B$ is equal to $D$,

therefore as $A$ is to $B$, so is $B$ to $C$

Therefore etc Q E D

## Proposition 18

*On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure*

Let $AB$ be the given straight line and $CE$ the given rectilineal figure thus it is required to describe on the straight line $AB$ a rectilineal figure similar and similarly situated to the rectilineal figure $CE$

Let $DF$ be joined and on the straight line $AB$, and at the points $A$, $B$ on it, let the angle $GAB$ be constructed equal to the angle at $C$, and the angle $ABG$ equal to the angle $CDF$ [I 23]

Therefore the remaining angle $CFD$ is equal to the angle $AGB$, [I 32]

therefore the triangle $FCD$ is equiangular with the triangle $GAB$

Therefore proportionally, as $FD$ is to $GB$, so is $FC$ to $GA$ and $CD$ to $AB$

Again, on the straight line $BG$, and at the points $B$, $G$ on it, let the angle $BGH$ be constructed equal to the angle $DFE$ and the angle $GBH$ equal to the angle $FDE$ [I 23]

Therefore the remaining angle at $E$ is equal to the remaining angle at $H$, [I 32]

therefore the triangle $FDE$ is equiangular with the triangle $GBH$, therefore, proportionally, as $FD$ is to $GB$, so is $FE$ to $GH$, and $ED$ to $HB$ [VI 4]

But it was also proved that as $FD$ is to $GB$ so is $FC$ to $GA$, and $CD$ to $AB$, therefore also, as $FC$ is to $AG$ so is $CD$ to $AB$, and $FE$ to $GH$, and further $ED$ to $HB$

And, since the angle $CFD$ is equal to the angle $AGB$,

and the angle $DFE$ to the angle $BGH$,

therefore the whole angle $CFE$ is equal to the whole angle $AGH$

For the same reason

the angle $CDE$ is also equal to the angle $ABH$

And the angle at $C$ is also equal to the angle at $A$

and the angle at $E$ to the angle at $H$

Therefore $AH$ is equiangular with $CE$

and they have the sides about their equal angles proportional

therefore the rectilineal figure $AH$ is similar to the rectilineal figure $CE$ [VI Def 1]

Therefore on the given straight line $AB$ the rectilineal figure $AH$ has been described similar and similarly situated to the given rectilineal figure $CE$

Q E F

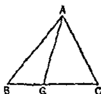
## Proposition 19

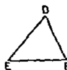
*Similar triangles are to one another in the duplicate ratio of the corresponding sides*

Let $ABC$, $DEF$ be similar triangles having the angle at $B$ equal to the angle

at $E$, and such that as $AB$ is to $BC$, so is $DE$ to $EF$ so that $BC$ corresponds to $EF$ [v Def 11]

I say that the triangle $ABC$ has to the triangle $DEF$ a ratio duplicate of that which $BC$ has to $EF$




For let a third proportional $BG$ be taken to $BC$, $EF$, so that as $BC$ is to $EF$, so is $EF$ to $BG$, [vi 11]

and let $AG$ be joined

Since then as $AB$ is to $BC$, so is $DE$ to $EF$

therefore alternately, as $AB$ is to $DE$ so is $BC$ to $EF$ [v 16]

But, as $BC$ is to $EF$, so is $EF$ to $BG$,

therefore also as $AB$ is to $DE$, so is $EF$ to $BG$ [v 11]

Therefore in the triangles $ABG$, $DEF$ the sides about the equal angles are reciprocally proportional

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional are equal, [vi 15]

therefore the triangle $ABG$ is equal to the triangle $DEF$

Now since, as $BC$ is to $EF$ so is $EF$ to $BG$,

and if three straight lines be proportional the first has to the third a ratio duplicate of that which it has to the second [v Def 9]

therefore $BC$ has to $BG$ a ratio duplicate of that which $CB$ has to $EF$

But, as $CB$ is to $BG$ so is the triangle $ABC$ to the triangle $ABG$, [vi 1]

therefore the triangle $ABC$ also has to the triangle $ABG$ a ratio duplicate of that which $BC$ has to $EF$

But the triangle $ABG$ is equal to the triangle $DEF$

therefore the triangle $ABC$ also has to the triangle $DEF$ a ratio duplicate of that which $BC$ has to $EF$

Therefore etc

PORISM From this it is manifest that if three straight lines be proportional then, as the first is to the third so is the figure described on the first to that which is similar and similarly described on the second Q E D

PROPOSITION 20

*Similar polygons are divided into similar triangles and into triangles equal in multitude and in the same ratio as the wholes and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side*

Let $ABCDE$ $FGHKL$ be similar polygons and let $AB$ correspond to $FG$

I say that the polygons $ABCDE$ $FGHKL$ are divided into similar triangles and into triangles equal in multitude and in the same ratio as the wholes and the polygon $ABCDE$ has to the polygon $FGHKL$ a ratio duplicate of that which $AB$ has to $FG$

Let $BE$ $EC$, $GL$ $LH$ be joined

Now, since the polygon $ABCDE$ is similar to the polygon $FGHKL$,

the angle $BAE$ is equal to the angle $GFL$

and as $BA$ is to $AE$ so is $GF$ to $FL$ [vi Def 1]

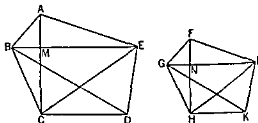
Since then $ABE$ $FGL$ are two triangles having one angle equal to one angle

and the sides about the equal angles proportional,

therefore the triangle $ABE$ is equiangular with the triangle $FGL$, [VI 6]

so that it is also similar, [VI 4 and Def 1]

therefore the angle $ABE$ is equal to the angle $FGL$




But the whole angle $ABC$ is also equal to the whole angle $FGH$ because of the similarity of the polygons,

therefore the remaining angle $EBC$ is equal to the angle $LGH$

And, since because of the similarity of the triangles $ABE$, $FGL$

as $EB$ is to $BA$, so is $LG$ to $GF$,

and moreover also because of the similarity of the polygons

as $AB$ is to $BC$ so is $FG$ to $GH$,

therefore, *ex aequali* as $EB$ is to $BC$, so is $LG$ to $GH$, [V 22]

that is the sides about the equal angles $EBC$, $LGH$ are proportional,

therefore the triangle $EBC$ is equiangular with the triangle $LGH$, [VI 6]

so that the triangle $EBC$ is also similar to the triangle $LGH$ [VI 4 and Def 1]

For the same reason

the triangle $ECD$ is also similar to the triangle $LHK$

Therefore the similar polygons $ABCDE$ $FGHKL$ have been divided into similar triangles and into triangles equal in multitude

I say that they are also in the same ratio as the wholes that is in such manner that the triangles are proportional and $ABE$, $EBC$, $ECD$ are antecedents while $FGL$ $LGH$ $LHK$ are their consequents and that the polygon $ABCDE$ has to the polygon $FGHKL$ a ratio duplicate of that which the corresponding side has to the corresponding side that is $AB$ to $FG$

For let $AC$ $FH$ be joined

Then since because of the similarity of the polygons

the angle $ABC$ is equal to the angle $FGH$,

and as $AB$ is to $BC$ so is $FG$ to $GH$

the triangle $ABC$ is equiangular with the triangle $FGH$ [VI 6]

therefore the angle $BAC$ is equal to the angle $GFH$

and the angle $BCA$ to the angle $GHF$

And since the angle $BAM$ is equal to the angle $GFN$,

and the angle $ABM$ is also equal to the angle $FGN$,

therefore the remaining angle $AMB$ is also equal to the remaining angle $FNG$, [I 32]

therefore the triangle $ABM$ is equiangular with the triangle $FGN$

Similarly we can prove that

the triangle $BMC$ is also equiangular with the triangle $GNH$

Therefore proportionally as $AM$ is to $MB$ so is $FN$ to $NG$

and as $BM$ is to $MC$ so is $GN$ to $NH$,

so that in addition *ex aequali*

as $AM$ is to $MC$, so is $FN$ to $NH$

But, as $AM$ is to $MC$ so is the triangle $ABM$ to $MBC$, and $AME$ to $EMC$, for they are to one another as their bases [VI 1]

Therefore also as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

*therefore, as the triangle AMB is to BMC, so is ABE to CBE*

But as $AMB$ is to $BMC$, so is $AM$ to $MC$,

therefore also as $AM$ is to $MC$, so is the triangle $ABE$ to the triangle $EBC$

For the same reason also

as $FN$ is to $NH$, so is the triangle $FGL$ to the triangle $GLH$

And as $AM$ is to $MC$, so is $FN$ to $NH$,

therefore also, as the triangle $ABE$ is to the triangle $BEC$, so is the triangle $FGL$ to the triangle $GLH$,

and, alternately, as the triangle $ABE$ is to the triangle $FGL$ so is the triangle $BEC$ to the triangle $GLH$

Similarly we can prove if $BD$, $GK$ be joined, that as the triangle $BEC$ is to the triangle $LGH$, so also is the triangle $ECD$ to the triangle $LHK$

And since *as the triangle ABE is to the triangle FGL, so is EBC to LGH,* and further $ECD$ to $LHK$,

therefore also as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [v 12]

therefore as the triangle $ABE$ is to the triangle $FGL$

so is the polygon $ABCDE$ to the polygon $FGHKL$

But the triangle $ABE$ has to the triangle $FGL$ a ratio duplicate of that which the corresponding side $AB$ has to the corresponding side $FG$, for similar triangles are in the duplicate ratio of the corresponding sides [vi 19]

Therefore the polygon $ABCDE$ also has to the polygon $FGHKL$ a ratio duplicate of that which the corresponding side $AB$ has to the corresponding side $FG$

Therefore etc

Porism Similarly also it can be proved in the case of quadrilaterals that they are in the duplicate ratio of the corresponding sides And it was also proved in the case of triangles therefore also generally similar rectilineal figures are to one another in the duplicate ratio of the corresponding sides

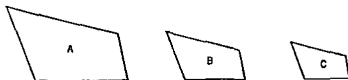
Q E D

## Proposition 21

*Figures which are similar to the same rectilineal figure are also similar to one another*

For let each of the rectilineal figures $A$, $B$ be similar to $C$, I say that $A$ is also similar to $B$

For since $A$ is similar to $C$,




it is equiangular with it and has the sides about the equal angles proportional [vi Def 1]

Again since $B$ is similar to $C$

it is equiangular with it and has the sides about the equal angles proportional

Therefore each of the figures $A$, $B$ is equiangular with $C$ and with $C$ has the sides about the equal angles proportional,

therefore $A$ is similar to $B$ Q E D

PROPOSITION 22

*If four straight lines be proportional the rectilineal figures similar and similarly described upon them will also be proportional and if the rectilineal figures similar and similarly described upon them be proportional the straight lines will themselves also be proportional*

Let the four straight lines $AB$ $CD$, $EF$, $GH$ be proportional,

so that as $AB$ is to $CD$, so is $EF$ to $GH$,

and let there be described on $AB$, $CD$ the similar and similarly situated rectilineal figures $KAB$, $LCD$

and on $EF$, $GH$ the similar and similarly situated rectilineal figures $MF$, $NH$,

I say that as $KAB$ is to $LCD$, so is $MF$ to $NH$

For let there be taken a third proportional $O$ to $AB$, $CD$, and a third proportional $P$ to $EF$ $GH$ [VI 11]

Then since as $AB$ is to $CD$, so is $EF$ to $GH$

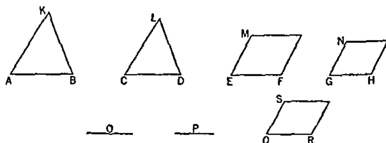
and, as $CD$ is to $O$ so is $GH$ to $P$,

therefore *ex aequali* as $AB$ is to $O$ so is $EF$ to $P$ [V 22]

But as $AB$ is to $O$ so is $KAB$ to $LCD$, [VI 19 Por]

and as $EF$ is to $P$, so is $MF$ to $NH$,

therefore also as $KAB$ is to $LCD$ so is $MF$ to $NH$ [V 11]




Next let $MF$ be to $NH$ as $KAB$ is to $LCD$,

I say also that as $AB$ is to $CD$ so is $EF$ to $GH$

For if $EF$ is not to $GH$ as $AB$ to $CD$

let $EF$ be to $QR$ as $AB$ to $CD$ [VI 12]

and on $QR$ let the rectilineal figure $SR$ be described similar and similarly situated to either of the two $MF$ $NH$ [VI 18]

Since then as $AB$ is to $CD$ so is $EF$ to $QR$

and there have been described on $AB$ $CD$ the similar and similarly situated figures $KAB$ $LCD$

and on $EF$ $QR$ the similar and similarly situated figures $MF$ $SR$,

therefore as $KAB$ is to $LCD$ so is $MF$ to $SR$

But also by hypothesis

as $KAB$ is to $LCD$ so is $MF$ to $NH$

therefore also as $MF$ is to $SR$ so is $MF$ to $NH$ [V 11]

Therefore $MF$ has the same ratio to each of the figures $NH$, $SR$,

therefore $NH$ is equal to $SR$ [v 9]

But it is also similar and similarly situated to it

therefore $GH$ is equal to $QR$

And, since, as $AB$ is to $CD$ so is $EF$ to $QR$,

while $QR$ is equal to $GH$

therefore as $AB$ is to $CD$, so is $EF$ to $GH$

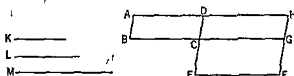
Therefore etc Q E D

## Proposition 23

*Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides*

Let $AC$ $CF$ be equiangular parallelograms having the angle $BCD$ equal to the angle $ECG$,

I say that the parallelogram $AC$ has to the parallelogram $CF$ the ratio compounded of the ratios of the sides




For let them be placed so that $BC$ is in a straight line with $CG$,

therefore $DC$ is also in a straight line with $CE$

Let the parallelogram $DG$ be completed,

let a straight line $K$ be set out and let it be contrived that

as $BC$ is to $CG$, so is $K$ to $L$

and, as $DC$ is to $CE$ so is $L$ to $M$ [vi 12]

Then the ratios of $K$ to $L$ and of $L$ to $M$ are the same as the ratios of the sides namely of $BC$ to $CG$ and of $DC$ to $CE$

But the ratio of $K$ to $M$ is compounded of the ratio of $K$ to $L$ and of that of $L$ to $M$,

so that $K$ has also to $M$ the ratio compounded of the ratios of the sides

Now since, as $BC$ is to $CG$, so is the parallelogram $AC$ to the parallelogram $CH$, [vi 1]

while as $BC$ is to $CG$ so is $K$ to $L$,

therefore also as $K$ is to $L$ so is $AC$ to $CH$ [v 11]

Again since as $DC$ is to $CE$, so is the parallelogram $CH$ to $CF$ [vi 1]

while as $DC$ is to $CE$ so is $L$ to $M$,

therefore also, as $L$ is to $M$ so is the parallelogram $CH$ to the parallelogram $CF$ [v 11]

Since then it was proved that, as $K$ is to $L$ so is the parallelogram $AC$ to the parallelogram $CH$

and as $L$ is to $M$ so is the parallelogram $CH$ to the parallelogram $CF$

therefore, *ex aequali* as $K$ is to $M$ so is $AC$ to the parallelogram $CF$

But $K$ has to $M$ the ratio compounded of the ratios of the sides

therefore $AC$ also has to $CF$ the ratio compounded of the ratios of the sides

Therefore etc Q E D

PROPOSITION 24

*In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another*

Let $ABCD$ be a parallelogram and $AC$ its diameter, and let $EG$, $HK$ be parallelograms about $AC$,

I say that each of the parallelograms $EG$ $HK$ is similar both to the whole $ABCD$ and to the other

For, since $EF$ has been drawn parallel to $BC$ one of the sides of the triangle $ABC$,

proportionally, as $BE$ is to $EA$, so is $CF$ to $FA$ [vi 2]

Again since $FG$ has been drawn parallel to $CD$, one of the sides of the triangle $ACD$,

proportionally as $CF$ is to $FA$, so is $DG$ to $GA$ [vi 2]

But it was proved that

as $CF$ is to $FA$, so also is $BE$ to $EA$,

therefore also as $BE$ is to $EA$ so is $DG$ to $GA$,

and therefore *componendo*

as $BA$ is to $AE$, so is $DA$ to $AG$, [v 18]

and alternately,

as $BA$ is to $AD$, so is $EA$ to $AG$ [v 16]

Therefore in the parallelograms $ABCD$, $EG$, the sides about the common angle $BAD$ are proportional

And, since $GF$ is parallel to $DC$,

the angle $AFG$ is equal to the angle $DCA$,

and the angle $DAC$ is common to the two triangles $ADC$, $AGF$,

therefore the triangle $ADC$ is equiangular with the triangle $AGF$

For the same reason

the triangle $ACB$ is also equiangular with the triangle $AFE$

and the whole parallelogram $ABCD$ is equiangular with the parallelogram $EG$

Therefore proportionally

as $AD$ is to $DC$, so is $AG$ to $GF$

as $DC$ is to $CA$ so is $GF$ to $FA$,

as $AC$ is to $CB$, so is $AF$ to $FE$

and further as $CB$ is to $BA$, so is $FE$ to $EA$

And since it was proved that

as $DC$ is to $CA$ so is $GF$ to $FA$,

and, as $AC$ is to $CB$, so is $AF$ to $FE$,

therefore *ex aequali* as $DC$ is to $CB$ so is $GF$ to $FE$ [v 22]

Therefore in the parallelograms $ABCD$ $EG$ the sides about the equal angles are proportional,

therefore the parallelogram $ABCD$ is similar to the parallelogram $EG$ [vi Def 1]

For the same reason

the parallelogram $ABCD$ is also similar to the parallelogram $KH$

therefore each of the parallelograms $EG$ $HK$ is similar to $ABCD$

But figures similar to the same rectilineal figure are also similar to one another [vi 21]

therefore the parallelogram $EG$ is also similar to the parallelogram $HK$

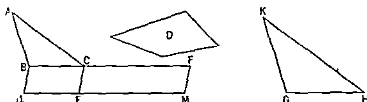
Therefore etc Q E D

PROPOSITION 25

*To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure*

Let $ABC$ be the given rectilineal figure to which the figure to be constructed must be similar and $D$ that to which it must be equal

thus it is required to construct one and the same figure similar to $ABC$ and equal to $D$




Let there be applied to $BC$ the parallelogram $BE$ equal to the triangle $ABC$ [I 44] and to $CF$ the parallelogram $CM$ equal to $D$ in the angle $FCE$ which is equal to the angle $CBL$ [I 45]

Therefore $BC$ is in a straight line with $CF$ and $LE$ with $EM$

Now let $GH$ be taken a mean proportional to $BC$, $CF$ [VI 13] and on $GH$ let $KGH$ be described similar and similarly situated to $ABC$ [VI 18]

Then since, as $BC$ is to $GH$, so is $GH$ to $CF$,

and if three straight lines be proportional as the first is to the third so is the figure on the first to the similar and similarly situated figure described on the second [VI 19 Por]

therefore as $BC$ is to $CF$ so is the triangle $ABC$ to the triangle $KGH$

But, as $BC$ is to $CF$ so also is the parallelogram $BE$ to the parallelogram $EF$ [VI 1]

Therefore also as the triangle $ABC$ is to the triangle $KGH$ so is the parallelogram $BE$ to the parallelogram $EF$,

therefore alternately as the triangle $ABC$ is to the parallelogram $BE$ so is the triangle $KGH$ to the parallelogram $EF$ [V 16]

But the triangle $ABC$ is equal to the parallelogram $BE$

therefore the triangle $KGH$ is also equal to the parallelogram $EF$

But the parallelogram $EF$ is equal to $D$

therefore $KGH$ is also equal to $D$

And $KGH$ is also similar to $ABC$

Therefore one and the same figure $KGH$ has been constructed similar to the given rectilineal figure $ABC$ and equal to the other given figure $D$ Q E D

PROPOSITION 26

*If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it it is about the same diameter with the whole*

For from the parallelogram $ABCD$ let there be taken away the parallelo-

gram $AF$ similar and similarly situated to $ABCD$, and having the angle $DAB$ common with it,

I say that $ABCD$ is about the same diameter with $AF$

For suppose it is not but, if possible let $AHC$ be the diameter < of $ABCD$>, let $GF$ be produced and carried through to $H$, and let $HK$ be drawn through $H$ parallel to either of the straight lines $AD$, $BC$ [I 31]

Since, then $ABCD$ is about the same diameter with $KG$, therefore as $DA$ is to $AB$, so is $GA$ to $AK$ [VI 24]

But also because of the similarity of $ABCD$ $EG$,

as $DA$ is to $AB$, so is $GA$ to $AE$,

therefore also as $GA$ is to $AK$ so is $GA$ to $AE$ [V 11]

Therefore $GA$ has the same ratio to each of the straight lines $AK$, $AE$

Therefore $AE$ is equal to $AK$ [V 9] the less to the greater which is impossible

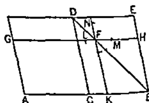
Therefore $ABCD$ cannot but be about the same diameter with $AF$, therefore the parallelogram $ABCD$ is about the same diameter with the parallelogram $AF$

Therefore etc Q E D

## Proposition 27

*Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect*

Let $AB$ be a straight line and let it be bisected at $C$, let there be applied to the straight line $AB$ the parallelogram $AD$ deficient by the parallelogrammic figure $DB$ described on the half of $AB$, that is $CB$,




I say that of the parallelograms applied to $AB$ and deficient by parallelogrammic figures similar and similarly situated to $DB$ $AD$ is greatest

For let there be applied to the straight line $AB$ the parallelogram $AF$ deficient by the parallelogrammic figure $FB$ similar and similarly situated to $DB$,

I say that $AD$ is greater than $AF$

For since the parallelogram $DB$ is similar to the parallelogram $FB$,

they are about the same diameter [VI 26]

Let their diameter $DB$ be drawn and let the figure be described

Then, since $CF$ is equal to $FE$ [I 43]

and $FB$ is common

therefore the whole $CH$ is equal to the whole $KE$

But $CH$ is equal to $CG$ since $AC$ is also equal to $CB$ [I 36]

Therefore $GC$ is also equal to $EK$

Let $CF$ be added to each

therefore the whole $AF$ is equal to the gnomon $LMN$,

so that the parallelogram $DB$, that is, $AD$, is greater than the parallelogram $AF$

Therefore etc Q E D

PROPOSITION 28

*To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect*

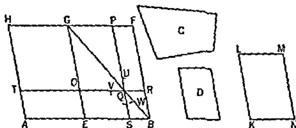
Let $AB$ be the given straight line $C$ the given rectilineal figure to which the figure to be applied to $AB$ is required to be equal not being greater than the parallelogram described on the half of $AB$ and similar to the defect and $D$ the parallelogram to which the defect is required to be similar,

thus it is required to apply to the given straight line $AB$ a parallelogram equal to the given rectilineal figure $C$ and deficient by a parallelogrammic figure which is similar to $D$

Let $AB$ be bisected at the point $E$, and on $EB$ let $EBFG$ be described similar and similarly situated to $D$, [VI 18]

let the parallelogram $AG$ be completed

If then $AG$ is equal to $C$ that which was enjoined will have been done for there has been applied to the given straight line $AB$ the parallelogram $AG$ equal to the given rectilineal figure $C$ and deficient by a parallelogrammic figure $GB$ which is similar to $D$




But if not let $HE$ be greater than $C$

Now $HE$ is equal to $GB$,

therefore $GB$ is also greater than $C$

Let $KLMN$ be constructed at once equal to the excess by which $GB$ is greater than $C$ and similar and similarly situated to $D$ [VI 25]

But $D$ is similar to $GB$,

therefore $KM$ is also similar to $GB$ [VI 21]

Let then, $KL$ correspond to $GE$ and $LM$ to $GF$

Now since $GB$ is equal to $C$, $KM$,

therefore $GB$ is greater than $KM$

therefore also $GE$ is greater than $KL$ and $GF$ than $LM$

Let $GO$ be made equal to $KL$ and $GP$ equal to $LM$ and let the parallelogram $OGPQ$ be completed

therefore it is equal and similar to $KM$

Therefore $GQ$ is also similar to $GB$ [VI 21]

therefore $GQ$ is about the same diameter with $GB$ [VI 26]

Let $GQB$ be their diameter, and let the figure be described

Then, since $BG$ is equal to $C$, $KM$,

and in them $GQ$ is equal to $KM$

therefore the remainder the gnomon $UWV$, is equal to the remainder $C$

And since $PR$ is equal to $OS$

let $QB$ be added to each

therefore the whole $PB$ is equal to the whole $OB$

But $OB$ is equal to $TE$ since the side $AE$ is also equal to the side $EB$, [I 36]

therefore $TE$ is also equal to $PB$

Let $OS$ be added to each,

therefore the whole $TS$ is equal to the whole the gnomon $VWU$

But the gnomon $VWU$ was proved equal to $C$,

therefore $TS$ is also equal to $C$

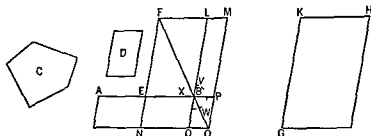
Therefore to the given straight line $AB$ there has been applied the parallelogram $ST$ equal to the given rectilineal figure $C$ and deficient by a parallelogrammic figure $QB$ which is similar to $D$ Q E F

PROPOSITION 29

*To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one*

Let $AB$ be the given straight line, $C$ the given rectilineal figure to which the figure to be applied to $AB$ is required to be equal, and $D$ that to which the excess is required to be similar

thus it is required to apply to the straight line $AB$ a parallelogram equal to the rectilineal figure $C$ and exceeding by a parallelogrammic figure similar to $D$




Let $AB$ be bisected at $E$

let there be described on $EB$ the parallelogram $BF$ similar and similarly situated to $D$

and let $GH$ be constructed at once equal to the sum of $BF$ $C$ and similar and similarly situated to $D$ [VI 25]

Let $KH$ correspond to $FL$ and $KG$ to $FE$

Now since $GH$ is greater than $FB$

therefore $KH$ is also greater than $FL$ and $KG$ than $FE$

Let $FL$, $FE$ be produced

let $FLM$ be equal to $KH$ and $FEN$ to $KG$

and let $MN$ be completed

therefore $MN$ is both equal and similar to $GH$

But $GH$ is similar to $EL$,

therefore $MN$ is also similar to $EL$ [VI 21]

therefore $EL$ is about the same diameter with $MN$ [VI 26]

Let their diameter $FO$ be drawn, and let the figure be described

Since $GH$ is equal to $EL$, $C$,

while $GH$ is equal to $MN$

therefore $MN$ is also equal to $EL$, $C$

Let $EL$ be subtracted from each

therefore the remainder, the gnomon $XWV$ is equal to $C$

Now, since $AE$ is equal to $EB$,

$AN$ is also equal to $NB$ [I 36] that is, to $LP$ [I 43]

Let $EO$ be added to each,

therefore the whole $AO$ is equal to the gnomon $VWX$

But the gnomon $VWX$ is equal to $C$

therefore $AO$ is also equal to $C$

Therefore to the given straight line $AB$ there has been applied the parallelogram $AO$ equal to the given rectilineal figure $C$ and exceeding by a parallelogrammic figure $QP$ which is similar to $D$ since $PQ$ is also similar to $EL$ [VI 24]

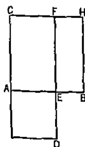
Q E F

## Proposition 30

*To cut a given finite straight line in extreme and mean ratio*

Let $AB$ be the given finite straight line

thus it is required to cut $AB$ in extreme and mean ratio




On $AB$ let the square $BC$ be described and let there be applied to $AC$ the parallelogram $CD$ equal to $BC$ and exceeding by the figure $AD$ similar to $BC$ [VI 29]

Now $BC$ is a square

therefore $AD$ is also a square

And since $BC$ is equal to $CD$

let $CE$ be subtracted from each,

therefore the remainder $BF$ is equal to the remainder $AD$

But it is also equiangular with it

therefore in $BF$, $AD$ the sides about the equal angles are reciprocally proportional, [VI 14]

therefore as $FE$ is to $ED$, so is $AE$ to $EB$

But $FE$ is equal to $AB$ and $ED$ to $AE$

Therefore, as $BA$ is to $AE$ so is $AE$ to $EB$

And $AB$ is greater than $AE$,

*therefore $AE$ is also greater than $EB$*

Therefore the straight line $AB$ has been cut in extreme and mean ratio at $E$ and the greater segment of it is $AE$ Q E F

## Proposition 31

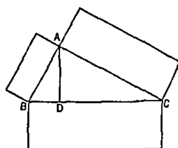
*In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle*

Let $ABC$ be a right angled triangle having the angle $BAC$ right

I say that the figure on $BC$ is equal to the similar and similarly described figures on $BA$ $AC$

Let $AD$ be drawn perpendicular

Then since in the right-angled triangle $ABC$, $AD$ has been drawn from the right angle at $A$ perpendicular to the base $BC$,




the triangles $ABD$, $ADC$ adjoining the perpendicular are similar both to the whole $ABC$ and to one another [VI 8]

And since $ABC$ is similar to $ABD$, therefore as $CB$ is to $BA$ so is $AB$ to $BD$ [VI Def 1]

And since three straight lines are proportional, as the first is to the third so is the figure on the first to the similar and similarly described figure on the second [VI 19 Por]

Therefore as $CB$ is to $BD$, so is the figure on $CB$ to the similar and similarly described figure on $BA$

For the same reason also,

as $BC$ is to $CD$ so is the figure on $BC$ to that on $CA$,

so that, in addition

as $BC$ is to $BD$, $DC$ so is the figure on $BC$ to the similar and similarly described figures on $BA$ $AC$

But $BC$ is equal to $BD$, $DC$,

therefore the figure on $BC$ is also equal to the similar and similarly described figures on $BA$, $AC$

Therefore etc Q E D

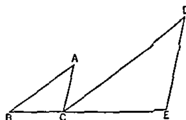
### PROPOSITION 32

*If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line*

Let $ABC$, $DCE$ be two triangles having the two sides $BA$ $AC$ proportional to the two sides $DC$ $DE$, so that as $AB$ is to $AC$ so is $DC$ to $DE$, and $AB$ parallel to $DC$, and $AC$ to $DE$

I say that $BC$ is in a straight line with $CE$

For since $AB$ is parallel to $DC$ and the straight line $AC$ has fallen upon them the alternate angles $BAC$ $ACD$ are equal to one another [I 29]




For the same reason the angle $CDE$ is also equal to the angle $ACD$,

so that the angle $BAC$ is equal to the angle $CDE$

And, since $ABC$, $DCE$ are two triangles having one angle the angle at $A$, equal to one angle the angle at $D$

and the sides about the equal angles proportional,

so that as $BA$ is to $AC$ so is $CD$ to $DE$

therefore the triangle $ABC$ is equiangular with the triangle $DCE$, [VI 6]

therefore the angle $ABC$ is equal to the angle $DCE$

*But the angle $ACD$ was also proved equal to the angle $BAC$,*

therefore the whole angle $ACE$ is equal to the two angles $ABC$, $BAC$

Let the angle $ACB$ be added to each,

therefore the angles $ACE$ $ACB$ are equal to the angles $BAC$, $ACB$ $CBA$

But the angles $BAC$, $ABC$ $ACB$ are equal to two right angles, [I 32]

therefore the angles $ACE$, $ACB$ are also equal to two right angles

Therefore with a straight line $AC$, and at the point $C$ on it the two straight lines $BC$ $CE$ not lying on the same side make the adjacent angles $ACE$, $ACB$ equal to two right angles,

therefore $BC$ is in a straight line with $CE$ [I 14]

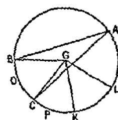
Therefore etc Q E D

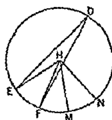
## Proposition 33

*In equal circles angles have the same ratio as the circumferences on which they stand whether they stand at the centres or at the circumferences*

Let $ABC$ $DEF$ be equal circles and let the angles $BGC$ $EHF$ be angles at their centres $G$ $H$, and the angles $BAC$ $EDF$ angles at the circumferences,

I say that, as the circumference $BC$ is to the circumference $EF$ so is the angle $BGC$ to the angle $EHF$, and the angle $BAC$ to the angle $EDF$




For let any number of consecutive circumferences $CK$, $KL$ be made equal to the circumference $BC$

and any number of consecutive circumferences $FM$ $MN$ equal to the circumference $EF$

and let $GK$, $GL$, $HM$ $HN$ be joined

Then since the circumferences $BC$ $CK$ $KL$ are equal to one another

the angles $BGC$ $CGK$ $KGL$ are also equal to one another [III 27]

therefore whatever multiple the circumference $BL$ is of $BC$, that multiple also is the angle $BGL$ of the angle $BGC$

For the same reason also

whatever multiple the circumference $NE$ is of $EF$ that multiple also is the angle $NHE$ of the angle $EHF$

If then the circumference $BL$ is equal to the circumference $EN$ the angle $BGL$ is also equal to the angle $EHN$ [III 27]

if the circumference $BL$ is greater than the circumference $EN$ the angle $BGL$ is also greater than the angle $EHN$

and if less less

There being then four magnitudes two circumferences $BC$ $EF$ and two angles $BGC$ $EHF$

there have been taken of the circumference $BC$ and the angle $BGC$ equimultiples namely the circumference $BL$ and the angle $BGL$

and of the circumference $EF$ and the angle $EHF$ equimultiples namely the circumference $EN$ and the angle $EHN$

And it has been proved that

if the circumference $BL$ is in excess of the circumference $EN$

the angle $BGL$ is also in excess of the angle $EHN$,
if equal, equal,
and if less less

Therefore as the circumference $BC$ is to $EF$, so is the angle $BGC$ to the angle $EHF$ [v Def 5]

But as the angle $BGC$ is to the angle $EHF$, so is the angle $BAC$ to the angle $EDF$, for they are doubles respectively

Therefore also, as the circumference $BC$ is to the circumference $EF$, so is the angle $BGC$ to the angle $EHF$, and the angle $BAC$ to the angle $EDF$

Therefore etc Q E D

# BOOK SEVEN

## DEFINITIONS

1 An *unit* is that by virtue of which each of the things that exist is called one

2 A *number* is a multitude composed of units

3 A number is *a part* of a number the less of the greater, when it measures the greater

4 but *parts* when it does not measure it

5 The greater number is a *multiple* of the less when it is measured by the less

6 An *even number* is that which is divisible into two equal parts

7 An *odd number* is that which is not divisible into two equal parts or that which differs by an unit from an even number

8 An *even times even number* is that which is measured by an even number according to an even number

9 An *even times odd number* is that which is measured by an even number according to an odd number

10 An *odd times odd number* is that which is measured by an odd number according to an odd number

11 A *prime number* is that which is measured by an unit alone

12 Numbers *prime to one another* are those which are measured by an unit alone as a common measure

13 A *composite number* is that which is measured by some number

14 Numbers *composite to one another* are those which are measured by some number as a common measure

15 A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other and thus some number is produced

16 And when two numbers having multiplied one another make some number the number so produced is called *plane* and its *sides* are the numbers which have multiplied one another

17 And when three numbers having multiplied one another make some number the number so produced is *solid* and its *sides* are the numbers which have multiplied one another

18 A *square number* is equal multiplied by equal, or a number which is contained by two equal numbers

19 And a *cube* is equal multiplied by equal and again by equal or a number which is contained by three equal numbers

20 Numbers are *proportional* when the first is the same multiple or the same part or the same parts of the second that the third is of the fourth

21 *Similar plane* and *solid* numbers are those which have their sides proportional

22 A *perfect number* is that which is equal to its own parts

## BOOK VII PROPOSITIONS

### Proposition 1

*Two unequal numbers being set out and the less being continually subtracted in turn from the greater if the number which is left never measures the one before it until an unit is left the original numbers will be prime to one another*

For the less of two unequal numbers *AB*, *CD* being continually subtracted from the greater let the number which is left never measure the one before it until an unit is left,

I say that *AB* *CD* are prime to one another that is that an unit alone measures *AB*, *CD*

For, if *AB*, *CD* are not prime to one another, some number will measure them

Let a number measure them, and let it be *E*, let *CD*, measuring *BF*, leave *F* 1 less than itself

let *AF* measuring *DG* leave *GC* less than itself,

and let *GC* measuring *FH* leave an unit *HA*

Since then *E* measures *CD* and *CD* measures *BF*,

therefore *E* also measures *BF*

But it also measures the whole *BA*,

therefore it will also measure the remainder *AF*

But *AF* measures *DG*,

therefore *E* also measures *DG*

But it also measures the whole *DC*

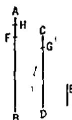
therefore it will also measure the remainder *CG*

But *CG* measures *FH*

therefore *E* also measures *FH*

But it also measures the whole *FA*

therefore it will also measure the remainder the unit *AH*, though it is a number which is impossible




Therefore no number will measure the numbers *AB* *CD* therefore *AB*, *CD* are prime to one another [VII Def 12]

Q E D

### Proposition 2

*Given two numbers not prime to one another to find their greatest common measure*

Let *AB* *CD* be the two given numbers not prime to one another

Thus it is required to find the greatest common measure of *AB*, *CD*

If now *CD* measures *AB*—and it also measures itself—*CD* is a common measure of *CD* *AB*

And it is manifest that it is also the greatest, for no greater number than *CD* will measure *CD*

But if *CD* does not measure *AB* then the less of the numbers *AB*, *CD* being continually subtracted from the greater, some number will be left which will measure the one before it

For an unit will not be left, otherwise $AB$, $CD$ will be prime to one another [VII 1] which is contrary to the hypothesis

Therefore some number will be left which will measure the one before it

Now let $CD$, measuring $BE$, leave $EA$ less than itself,
let $EA$, measuring $DF$, leave $FC$ less than itself,
and let $CF$ measure $AE$

Since then $CF$ measures $AE$, and $AE$ measures $DF$,
therefore $CF$ will also measure $DF$

But it also measures itself,
therefore it will also measure the whole $CD$

But $CD$ measures $BE$,
therefore $CF$ also measures $BE$

But it also measures $EA$,
therefore it will also measure the whole $BA$

But it also measures $CD$
therefore $CF$ measures $AB$, $CD$

Therefore $CF$ is a common measure of $AB$, $CD$

I say next that it is also the greatest

For if $CF$ is not the greatest common measure of $AB$ $CD$, some number which is greater than $CF$ will measure the numbers $AB$, $CD$

Let such a number measure them and let it be $G$

Now, since $G$ measures $CD$ while $CD$ measures $BE$, $G$ also measures $BE$

But it also measures the whole $BA$,
therefore it will also measure the remainder $AE$

But $AE$ measures $DF$,
therefore $G$ will also measure $DF$

But it also measures the whole $DC$,
therefore it will also measure the remainder $CF$ that is, the greater will measure the less which is impossible

Therefore no number which is greater than $CF$ will measure the numbers $AB$, $CD$

therefore $CF$ is the greatest common measure of $AB$, $CD$

PORISM From this it is manifest that if a number measure two numbers it will also measure their greatest common measure Q E D

PROPOSITION 3

*Given three numbers not prime to one another to find their greatest common measure*

Let $A$ $B$ $C$ be the three given numbers not prime to one another

thus it is required to find the greatest common measure of $A$, $B$, $C$

For let the greatest common measure $D$ of the two numbers $A$ $B$ be taken [VII 2]

then $D$ either measures or does not measure $C$

First let it measure it

But it measures $A$ $B$ also,
therefore $D$ measures $A$, $B$, $C$
therefore $D$ is a common measure of $A$ $B$ $C$

I say that it is also the greatest

For if $D$ is not the greatest common measure of $A$ $B$ $C$ some number which

21 *Similar plane* and *solid* numbers are those which have their sides proportional

22 A *perfect number* is that which is equal to its own parts

## BOOK VII PROPOSITIONS

### Proposition 1

*Two unequal numbers being set out and the less being continually subtracted in turn from the greater if the number which is left never measures the one before it until an unit is left the original numbers will be prime to one another*

For the less of two unequal numbers $AB$, $CD$ being continually subtracted from the greater let the number which is left never measure the one before it until an unit is left,

I say that $AB$, $CD$ are prime to one another, that is that an unit alone measures $AB$, $CD$

For if $AB$, $CD$ are not prime to one another, some number will measure them

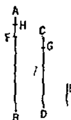
Let a number measure them, and let it be $E$, let $CD$, measuring $BF$, leave $FA$ less than itself

let $AF$, measuring $DG$, leave $GC$ less than itself

and let $GC$ measuring $FH$, leave an unit $HA$

Since then $E$ measures $CD$ and $CD$ measures $BF$,

therefore $E$ also measures $BF$




But it also measures the whole $BA$

therefore it will also measure the remainder $AF$

But $AF$ measures $DG$,

therefore $E$ also measures $DG$

But it also measures the whole $DC$

therefore it will also measure the remainder $CG$

But $CG$ measures $FH$

therefore $E$ also measures $FH$

But it also measures the whole $FA$,

therefore it will also measure the remainder the unit $AH$ though it is a number which is impossible

Therefore no number will measure the numbers $AB$, $CD$, therefore $AB$, $CD$ are prime to one another [VII Def 12]

Q E D

### Proposition 2

*Given two numbers not prime to one another to find their greatest common measure*

Let $AB$ $CD$ be the two given numbers not prime to one another

Thus it is required to find the greatest common measure of $AB$ $CD$

If now $CD$ measures $AB$—and it also measures itself—$CD$ is a common measure of $CD$, $AB$

And it is manifest that it is also the greatest, for no greater number than $CD$ will measure $CD$

But if $CD$ does not measure $AB$ then the less of the numbers $AB$ $CD$ being continually subtracted from the greater some number will be left which will measure the one before it

For an unit will not be left otherwise $AB$, $CD$ will be prime to one another [VII 1] which is contrary to the hypothesis

Therefore some number will be left which will measure the one before it

Now let $CD$ measuring $BE$, leave $EA$ less than itself,
let $EA$ measuring $DF$, leave $FC$ less than itself,
and let $CF$ measure $AE$

Since then $CF$ measures $AE$, and $AE$ measures $DF$,
therefore $CF$ will also measure $DF$

But it also measures itself
therefore it will also measure the whole $CD$

But $CD$ measures $BE$,
therefore $CF$ also measures $BE$

But it also measures $EA$
therefore it will also measure the whole $BA$

But it also measures $CD$,
therefore $CF$ measures $AB$, $CD$

Therefore $CF$ is a common measure of $AB$ $CD$

I say next that it is also the greatest

For if $CF$ is not the greatest common measure of $AB$ $CD$ some number which is greater than $CF$ will measure the numbers $AB$ $CD$

Let such a number measure them and let it be $G$

Now since $G$ measures $CD$ while $CD$ measures $BE$ $G$ also measures $BE$

But it also measures the whole $BA$
therefore it will also measure the remainder $AE$

But $AE$ measures $DF$,
therefore $G$ will also measure $DF$

But it also measures the whole $DC$
therefore it will also measure the remainder $CF$ that is the greater will measure the less which is impossible

Therefore no number which is greater than $CF$ will measure the numbers $AB$ $CD$

therefore $CF$ is the greatest common measure of $AB$ $CD$

PORISM From this it is manifest that if a number measure two numbers it will also measure their greatest common measure Q E D

### PROPOSITION 3

*Given three numbers not prime to one another to find their greatest common measure*

Let $A$ $B$ $C$ be the three given numbers not prime to one another
thus it is required to find the greatest common measure of $A$ $B$ $C$

For let the greatest common measure $D$ of the two numbers $A$, $B$ be taken [VII 2]
then $D$ either measures or does not measure $C$

First let it measure it

But it measures $A$ $B$ also
therefore $D$ measures $A$ $B$ $C$
therefore $D$ is a common measure of $A$ $B$ $C$

I say that it is also the greatest

For if $D$ is not the greatest common measure of $A$ $B$ $C$ some number which

is greater than $D$ will measure the numbers $A$, $B$, $C$

Let such a number measure them and let it be $E$

Since then $E$ measures $A$, $B$, $C$

it will also measure $A$, $B$,

therefore it will also measure the greatest common measure of $A$, $B$ [VII 2, Por]

But the greatest common measure of $A$, $B$ is $D$,

therefore $E$ measures $D$, the greater the less which is impossible

Therefore no number which is greater than $D$ will measure the numbers $A$, $B$, $C$,

therefore $D$ is the greatest common measure of $A$, $B$, $C$

Next let $D$ not measure $C$,

I say first that $C$, $D$ are not prime to one another

For since $A$, $B$, $C$ are not prime to one another some number will measure them

Now that which measures $A$ $B$ $C$ will also measure $A$ $B$, and will measure $D$ the greatest common measure of $A$ $B$ [VII 2 Por]

But it measures $C$ also,

therefore some number will measure the numbers $D$, $C$,

therefore $D$, $C$ are not prime to one another

Let then their greatest common measure $E$ be taken [VII 2]

Then since $E$ measures $D$

and $D$ measures $A$, $B$,

therefore $E$ also measures $A$, $B$

But it measures $C$ also

therefore $E$ measures $A$, $B$, $C$

therefore $E$ is a common measure of $A$, $B$, $C$

I say next that it is also the greatest

For if $E$ is not the greatest common measure of $A$, $B$ $C$, some number which is greater than $E$ will measure the numbers $A$ $B$, $C$

Let such a number measure them and let it be $F$

Now since $F$ measures $A$ $B$ $C$

it also measures $A$, $B$

therefore it will also measure the greatest common measure of $A$, $B$ [VII 2 Por]

But the greatest common measure of $A$, $B$ is $D$

therefore $F$ measures $D$

And it measures $C$ also

therefore $F$ measures $D$ $C$

therefore it will also measure the greatest common measure of $D$, $C$ [VII 2, Por]

But the greatest common measure of $D$, $C$ is $E$,

therefore $F$ measures $E$ the greater the less which is impossible

Therefore no number which is greater than $E$ will measure the numbers $A$, $B$, $C$

therefore $E$ is the greatest common measure of $A$ $B$ $C$

Q E D

### Proposition 4

*Any number is either a part or parts of any number, the less of the greater*

Let $A$, $BC$ be two numbers and let $BC$ be the less,

I say that $BC$ is either a part or parts of $A$

For $A$, $BC$ are either prime to one another or not

First, let $A$, $BC$ be prime to one another

Then if $BC$ be divided into the units in it, each unit of those in $BC$ will be some part of $A$, so that $BC$ is parts of $A$

Next let $A$, $BC$ not be prime to one another, then $BC$ either measures or does not measure $A$

If now $BC$ measures $A$, $BC$ is a part of $A$

But, if not let the greatest common measure $D$ of $A$, $BC$ be taken [VII 2]
and let $BC$ be divided into the numbers equal to $D$, namely $BE$ $EF$, $FC$

Now since $D$ measures $A$, $D$ is a part of $A$

But $D$ is equal to each of the numbers $BE$, $EF$, $FC$,
therefore each of the numbers $BE$, $EF$, $FC$ is also a part of $A$, so that $BC$ is parts of $A$

Therefore etc Q E D

### Proposition 5

*If a number be a part of a number and another be the same part of another, the sum will also be the same part of the sum that the one is of the one*

For let the number $A$ be a part of $BC$

and another $D$ the same part of another $EF$ that $A$ is of $BC$

I say that the sum of $A$, $D$ is also the same part of the sum of $BC$, $EF$ that $A$ is of $BC$

For since, whatever part $A$ is of $BC$, $D$ is also the same part of $EF$,
therefore, as many numbers as there are in $BC$ equal to $A$ so many numbers are there also in $EF$ equal to $D$

Let $BC$ be divided into the numbers equal to $A$ namely $BG$ $GC$,

and $EF$ into the numbers equal to $D$ namely $EH$ $HF$,
then the multitude of $BG$, $GC$ will be equal to the multitude of $EH$ $HF$

And since $BG$ is equal to $A$, and $EH$ to $D$,

therefore $BG$, $EH$ are also equal to $A$ $D$

For the same reason

$GC$ $HF$ are also equal to $A$ $D$

Therefore as many numbers as there are in $BC$ equal to $A$ so many are there also in $BC$, $EF$ equal to $A$ $D$

Therefore whatever multiple $BC$ is of $A$, the same multiple also is the sum of $BC$ $EF$ of the sum of $A$ $D$

Therefore whatever part $A$ is to $BC$, the same part also is the sum of $A$ $D$ of the sum of $BC$, $EF$ Q E D

### Proposition 6

*If a number be parts of a number and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one*

For let the number $AB$ be parts of the number $C$, and another, $DE$, the same parts of another, $F$, that $AB$ is of $C$,

I say that the sum of $AB$, $DE$ is also the same parts of the sum of $C$, $F$ that $AB$ is of $C$

For since whatever parts $AB$ is of $C$, $DE$ is also the same parts of $F$,

therefore, as many parts of $C$ as there are in $AB$, so many parts of $F$ are there also in $DE$

Let $AB$ be divided into the parts of $C$, namely $AG$, $GB$, and $DE$ into the parts of $F$, namely $DH$ $HE$,

*thus the multitude of $AG$, $GB$ will be equal to the multitude of $DH$, $HE$*

And since, whatever part $AG$ is of $C$, the same part is $DH$ of $F$ also, therefore, whatever part $AG$ is of $C$, the same part also is the sum of $AG$ $DH$ of the sum of $C$, $F$ [VII 5]

For the same reason

*whatever part $GB$ is of $C$, the same part also is the sum of $GB$ $HE$ of the sum of $C$, $F$*

Therefore, whatever parts $AB$ is of $C$, the same parts also is the sum of $AB$, $DE$ of the sum of $C$, $F$ Q E D

PROPOSITION 7

*If a number be that part of a number, which a number subtracted is of a number subtracted the remainder will also be the same part of the remainder that the whole is of the whole*

For let the number $AB$ be that part of the number $CD$ which $AE$ subtracted is of $CF$ subtracted

I say that the remainder $EB$ is also the same part of the remainder $FD$ that the whole $AB$ is of the whole $CD$




For whatever part $AE$ is of $CF$, the same part also let $EB$ be of $CG$

Now since whatever part $AE$ is of $CF$ the same part also is $EB$ of $CG$ therefore whatever part $AE$ is of $CF$ the same part also is $AB$ of $GF$ [VII 5]

But, whatever part $AE$ is of $CF$, the same part also, by hypothesis, is $AB$ of $CD$

therefore whatever part $AB$ is of $GF$, the same part is it of $CD$ also

therefore $GF$ is equal to $CD$

Let $CF$ be subtracted from each

therefore the remainder $GC$ is equal to the remainder $FD$

Now since whatever part $AE$ is of $CF$, the same part also is $EB$ of $GC$, while $GC$ is equal to $FD$

therefore whatever part $AE$ is of $CF$ the same part also is $EB$ of $FD$

But whatever part $AE$ is of $CF$ the same part also is $AB$ of $CD$,

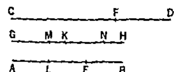
therefore also the remainder $EB$ is the same part of the remainder $FD$ that the whole $AB$ is of the whole $CD$ Q E D

PROPOSITION 8

*If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole*

For let the number $AB$ be the same parts of the number $CD$ that $AE$ subtracted is of $CF$ subtracted,

I say that the remainder $EB$ is also the same parts of the remainder $FD$ that the whole $AB$ is of the whole $CD$




For let $GH$ be made equal to $AB$

Therefore whatever parts $GH$ is of $CD$, the same parts also is $AE$ of $CF$

Let $GH$ be divided into the parts of $CD$ namely $GK$ $KH$, and $AE$ into the parts of $CF$, namely $AL$, $LE$

thus the multitude of $GK$, $KH$ will be equal to the multitude of $AL$ $LE$

Now since whatever part $GK$ is of $CD$ the same part also is $AL$ of $CF$,

while $CD$ is greater than $CF$

therefore $GK$ is also greater than $AL$

Let $GM$ be made equal to $AL$

Therefore, whatever part $GK$ is of $CD$, the same part also is $GM$ of $CF$

therefore also the remainder $MK$ is the same part of the remainder $FD$ that the whole $GK$ is of the whole $CD$ [VII 7]

Again since whatever part $KH$ is of $CD$ the same part also is $EL$ of $CF$,

while $CD$ is greater than $CF$,

therefore $HK$ is also greater than $EL$

Let $KN$ be made equal to $EL$

Therefore whatever part $KH$ is of $CD$, the same part also is $KN$ of $CF$,

therefore also the remainder $NH$ is the same part of the remainder $FD$ that the whole $KH$ is of the whole $CD$ [VII 7]

But the remainder $MK$ was also proved to be the same part of the remainder $FD$ that the whole $GK$ is of the whole $CD$,

therefore also the sum of $MK$, $NH$ is the same parts of $DF$ that the whole $HG$ is of the whole $CD$

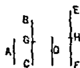
But the sum of $MK$, $NH$ is equal to $EB$,

and $HG$ is equal to $BA$,

therefore the remainder $EB$ is the same parts of the remainder $FD$ that the whole $AB$ is of the whole $CD$ Q E D

PROPOSITION 9

*If a number be a part of a number and another be the same part of another alternately also whatever part or parts the first is of the third the same part or the same parts will the second also be of the fourth*




For let the number $A$ be a part of the number $BC$, and an other $D$ the same part of another, $EF$ that $A$ is of $BC$,

I say that alternately also whatever part or parts $A$ is of $D$ the same part or parts is $BC$ of $EF$ also

For since whatever part $A$ is of $BC$ the same part also is $D$ of $EF$

therefore as many numbers as there are in $BC$ equal to $A$, so many also are there in $EF$ equal to $D$

Let $BC$ be divided into the numbers equal to $A$, namely $BG$ $GC$, and $EF$ into those equal to $D$ namely $EH$, $HF$,

thus the multitude of $BG$ $GC$ will be equal to the multitude of $EH$, $HF$

Now, since the numbers $BG$, $GC$ are equal to one another, and the numbers $EH$ $HF$ are also equal to one another

while the multitude of $BG$, $GC$ is equal to the multitude of $EH$ $HF$,

therefore whatever part or parts $BG$ is of $EH$, the same part or the same parts is $GC$ of $HF$ also,

so that in addition whatever part or parts $BG$ is of $EH$, the same part also or the same parts is the sum $BC$ of the sum $EF$ [VII 5 6]

But $BG$ is equal to $A$, and $EH$ to $D$,

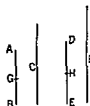
therefore whatever part or parts $A$ is of $D$ the same part or the same parts is $BC$ of $EF$ also Q E D

## Proposition 10

*If a number be parts of a number and another be the same parts of another alternately also whatever parts or part the first is of the third the same parts or the same part will the second also be of the fourth*

For let the number $AB$ be parts of the number $C$ and another, $DE$, the same parts of another, $F$,

I say that alternately also whatever parts or part $AB$ is of $DE$ the same parts or the same part is $C$ of $F$ also




For since whatever parts $AB$ is of $C$ the same parts also is $DE$ of $F$

therefore as many parts of $C$ as there are in $AB$ so many parts also of $F$ are there in $DE$

Let $AB$ be divided into the parts of $C$ namely $AG$, $GB$ and $DE$ into the parts of $F$, namely $DH$, $HE$

thus the multitude of $AG$ $GB$ will be equal to the multitude of $DH$, $HE$

Now since whatever part $AG$ is of $C$ the same part also is $DH$ of $F$,

alternately also whatever part or parts $AG$ is of $DH$, the same part or the same parts is $C$ of $F$ also [VII 9]

For the same reason also,

whatever part or parts $GB$ is of $HE$, the same part or the same parts is $C$ of $F$ also

so that in addition whatever parts or part $AB$ is of $DE$, the same parts also or the same part is $C$ of $F$ [VII 5 6]

Q E D

## Proposition 11

*If as whole is to whole so is a number subtracted to a number subtracted the remainder will also be to the remainder as whole to whole*

As the whole $AB$ is to the whole $CD$ so let $AE$ subtracted be to $CF$ subtracted,

I say that the remainder $EB$ is also to the remainder $FD$ as the whole $AB$ to the whole $CD$

Since, as $AB$ is to $CD$ so is $AE$ to $CF$,

whatever part or parts $AB$ is of $CD$, the same part or the same parts is $AE$ of $CF$ also, [VII Def 20]

Therefore also the remainder $EB$ is the same part or parts of $FD$ that $AB$ is of $CD$ [VII 7, 8]

Therefore as $EB$ is to $FD$, so is $AB$ to $CD$ [VII Def 20]

Q E D

### Proposition 12

*If there be as many numbers as we please in proportion, then as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents*

Let $A$, $B$, $C$, $D$ be as many numbers as we please in proportion so that,

as $A$ is to $B$, so is $C$ to $D$

I say that as $A$ is to $B$ so are $A$, $C$ to $B$, $D$

For since, as $A$ is to $B$ so is $C$ to $D$

whatever part or parts $A$ is of $B$, the same part or parts is $C$ of $D$ also [VII Def 20]

Therefore also the sum of $A$, $C$ is the same part or the same parts of the sum of $B$, $D$ that $A$ is of $B$ [VII 5, 6]

Therefore as $A$ is to $B$, so are $A$, $C$ to $B$, $D$ [VII Def 20]

Q E D

### Proposition 13

*If four numbers be proportional they will also be proportional alternately*

Let the four numbers $A$, $B$, $C$, $D$ be proportional so that

as $A$ is to $B$ so is $C$ to $D$

I say that they will also be proportional alternately so that

as $A$ is to $C$ so will $B$ be to $D$

For since as $A$ is to $B$ so is $C$ to $D$

therefore whatever part or parts $A$ is of $B$ the same part or the same parts is $C$ of $D$ also [VII Def 20]

Therefore, alternately whatever part or parts $A$ is of $C$ the same part or the same parts is $B$ of $D$ also [VII 10]

Therefore as $A$ is to $C$ so is $B$ to $D$ [VII Def 20]

Q E D

### Proposition 14

*If there be as many numbers as we please and others equal to them in multitude, which taken two and two are in the same ratio they will also be in the same ratio ex aequali*

Let there be as many numbers as we please $A$, $B$, $C$ and others equal to them in multitude $D$, $E$, $F$ which taken two and two are in the same ratio so that

as $A$ is to $B$ so is $D$ to $E$

and as $B$ is to $C$, so is $E$ to $F$

I say that *ex aequali*

as $A$ is to $C$ so also is $D$ to $F$

For since as $A$ is to $B$ so is $D$ to $E$

therefore alternately

as $A$ is to $D$ so is $B$ to $E$ [VII 13]

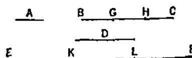
Again since as $B$ is to $C$ so is $E$ to $F$

therefore alternately,
as $B$ is to $E$, so is $C$ to $F$ [VII 13]
But as $B$ is to $F$ so is $A$ to $D$,
therefore also, as $A$ is to $D$, so is $C$ to $F$
Therefore alternately
as $A$ is to $C$, so is $D$ to $F$ [id]

PROPOSITION 15

*If an unit measure any number, and another number measure any other number the same number of times alternately also the unit will measure the third number the same number of times that the second measures the fourth*

For let the unit $A$ measure any number $BC$, and let another number $D$ measure any other number $EF$ the same number of times,




I say that alternately also the unit $A$ measures the number $D$ the same number of times that $BC$ measures $EF$

For, since the unit $A$ measures the number $BC$ the same number of times that $D$ measures $EF$

therefore as many units as there are in $BC$, so many numbers equal to $D$ are there in $EF$ also

Let $BC$ be divided into the units in it $BG$ $GH$, $HC$,
and $EF$ into the numbers $EK$, $KL$ $LF$ equal to $D$

Thus the multitude of $BG$, $GH$ $HC$ will be equal to the multitude of $EK$, $KL$, $LF$

And since the units $BG$, $GH$ $HC$ are equal to one another
and the numbers $EK$, $KL$, $LF$ are also equal to one another,
while the multitude of the units $BG$, $GH$, $HC$ is equal to the multitude of the numbers $EK$ $KL$, $LF$,
therefore as the unit $BG$ is to the number $EK$, so will the unit $GH$ be to the number $KL$ and the unit $HC$ to the number $LF$

Therefore also as one of the antecedents is to one of the consequents so will all the antecedents be to all the consequents [VII 12]

therefore as the unit $BG$ is to the number $EK$ so is $BC$ to $EF$

But the unit $BG$ is equal to the unit $A$,
and the number $EK$ to the number $D$

Therefore as the unit $A$ is to the number $D$ so is $BC$ to $EF$

Therefore the unit $A$ measures the number $D$ the same number of times that $BC$ measures $EF$ Q E D

PROPOSITION 16

*If two numbers by multiplying one another make certain numbers the numbers so produced will be equal to one another*

Let $A$ $B$ be two numbers and let $A$ by multiplying $B$ make $C$, and $B$ by multiplying $A$ make $D$,

I say that $C$ is equal to $D$

For since $A$ by multiplying $B$ has made $C$
therefore $B$ measures $C$ according to the units in $A$

But the unit $E$ also measures the number $A$ according to the units in it
therefore the unit $E$ measures $A$ the same number of times that $B$ measures $C$

Therefore alternately, the unit $E$ measures the number $B$ the same number of times that $A$ measures $C$ [VII 15]

Again, since $B$ by multiplying $A$ has made $D$,

therefore $A$ measures $D$ according to the units in $B$

But the unit $E$ also measures $B$ according to the units in it
therefore the unit $E$ measures the number $B$ the same number of times that $A$ measures $D$

But the unit $E$ measured the number $B$ the same number of times that $A$ measures $C$,
therefore $A$ measures each of the numbers $C$, $D$ the same number of times

Therefore $C$ is equal to $D$ Q E D

## Proposition 17

*If a number by multiplying two numbers make certain numbers the numbers so produced will have the same ratio as the numbers multiplied*

For let the number $A$ be multiplying the two numbers $B$ $C$ make $D$ $E$

I say that as $B$ is to $C$, so is $D$ to $E$

For since $A$ by multiplying $B$ has made $D$

therefore $B$ measures $D$ according to the units in $A$

But the unit $F$ also measures the number $A$ according to the units in it
therefore the unit $F$ measures the number $A$ the same number of times that $B$ measures $D$

Therefore as the unit $F$ is to the number $A$ so is $B$ to $D$ [VII Def 20]

For the same reason

as the unit $F$ is to the number $A$ so also is $C$ to $E$

therefore also as $B$ is to $D$ so is $C$ to $E$

Therefore alternately as $B$ is to $C$ so is $D$ to $E$ [VII 13]

Q E D

## Proposition 18

*If two numbers by multiplying any number make certain numbers the numbers so produced will have the same ratio as the multipliers*

For let two numbers $A$ $B$ by multiplying any number $C$ make $D$ $E$

I say that as $A$ is to $B$ so is $D$ to $E$

For since $A$ by multiplying $C$ has made $D$
therefore also $C$ by multiplying $A$ has made $D$ [VII 16]

For the same reason also

$C$ by multiplying $B$ has made $E$

Therefore the number $C$ by multiplying the two numbers $A$ $B$ has made $D$ $E$

Therefore as $A$ is to $B$ so is $D$ to $E$ [VII 17]

Q E D

## Proposition 19

*If four numbers be proportional the number produced from the first and fourth will be equal to the number produced from the second and third and, if the number produced from the first and fourth be equal to that produced from the second and third the four numbers will be proportional*

Let $A$, $B$, $C$, $D$ be four numbers in proportion so that

as $A$ is to $B$, so is $C$ to $D$,

and let $A$ by multiplying $D$ make $E$, and let $B$ by multiplying $C$ make $F$,

I say that $E$ is equal to $F$

For let $A$ by multiplying $C$ make $G$

Since then $A$ by multiplying $C$ has made $G$ and by multiplying $D$ has made $E$

the number $A$ by multiplying the two numbers $C$ $D$ has made $G$ $E$

Therefore, as $C$ is to $D$ so is $G$ to $E$ [VII 17]

But as $C$ is to $D$, so is $A$ to $B$,

therefore also as $A$ is to $B$, so is $G$ to $E$

Again since $A$ by multiplying $C$ has made $G$

but further, $B$ has also by multiplying $C$ made $F$,

the two numbers $A$, $B$ by multiplying a certain number $C$ have made $G$ $F$

Therefore as $A$ is to $B$, so is $G$ to $F$ [VII 18]

But further as $A$ is to $B$, so is $G$ to $E$ also,

therefore also, as $G$ is to $E$, so is $G$ to $F$

Therefore $G$ has to each of the numbers $E$, $F$ the same ratio,

therefore $E$ is equal to $F$ [cf V 9]

Again let $E$ be equal to $F$

I say that as $A$ is to $B$ so is $C$ to $D$

For, with the same construction

since $E$ is equal to $F$

therefore as $G$ is to $E$, so is $G$ to $F$ [cf V 7]

But as $G$ is to $E$ so is $C$ to $D$, [VII 17]

and as $G$ is to $F$ so is $A$ to $B$ [VII 18]

Therefore also, as $A$ is to $B$ so is $C$ to $D$ Q E D

## Proposition 20

*The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times the greater the greater and the less the less*

For let $CD$ $EF$ be the least numbers of those which have the same ratio with $A$, $B$,

I say that $CD$ measures $A$ the same number of times that $EF$ measures $B$

Now $CD$ is not parts of $A$

For, if possible let it be so

therefore $EF$ is also the same parts of $B$ that $CD$ is of $A$

[VII 13 and Def 20]

Therefore as many parts of $A$ as there are in $CD$ so many parts of $B$ are there also in $EF$

Let $CD$ be divided into the parts of $A$ namely $CG$ $GD$ and $EF$ into the parts of $B$ namely $EH$, $HF$

thus the multitude of $CG$, $GD$ will be equal to the multitude of $EH$ $HF$

Now since the numbers $CG$, $GD$ are equal to one another and the numbers $EH$, $HF$ are also equal to one another while the multitude of $CG$, $GD$ is equal to the multitude of $EH$, $HF$,

therefore as $CG$ is to $EH$ so is $GD$ to $HF$

Therefore also as one of the antecedents is to one of the consequents so will all the antecedents be to all the consequents [VII 12]

Therefore as $CG$ is to $EH$, so is $CD$ to $EF$

Therefore $CG$ $EH$ are in the same ratio with $CD$, $EF$, being less than they which is impossible, for by hypothesis $CD$ $EF$ are the least numbers of those which have the same ratio with them

Therefore $CD$ is not parts of $A$,

therefore it is a part of it [VII 4]

And $EF$ is the same part of $B$ that $CD$ is of $A$ [VII 13 and Def 20]

therefore $CD$ measures $A$ the same number of times that $EF$ measures $B$

Q E D

## Proposition 21

*Numbers prime to one another are the least of those which have the same ratio with them*

Let $A$ $B$ be numbers prime to one another

I say that $A$, $B$ are the least of those which have the same ratio with them

For if not there will be some numbers less than $A$ $B$ which are in the same ratio with $A$, $B$

Let them be $C$, $D$

Since then the least numbers of those which have the same ratio measure those which have the same ratio the same number of times the greater the greater and the less the less that is, the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore $C$ measures $A$ the same number of times that $D$ measures $B$

Now, as many times as $C$ measures $A$ so many units let there be in $E$

Therefore $D$ also measures $B$ according to the units in $E$

And since $C$ measures $A$ according to the units in $E$

therefore $E$ also measures $A$ according to the units in $C$ [VII 16]

For the same reason

$E$ also measures $B$ according to the units in $D$ [VII 16]

Therefore $E$ measures $A$, $B$ which are prime to one another which is impossible [VII Def 12]

Therefore there will be no numbers less than $A$, $B$ which are in the same ratio with $A$ $B$

Therefore $A$ $B$ are the least of those which have the same ratio with them

Q E D

PROPOSITION 22

*The least numbers of those which have the same ratio with them are prime to one another*

Let $A$, $B$ be the least numbers of those which have the same ratio with them,

I say that $A$ $B$ are prime to one another

For, if they are not prime to one another some number will measure them

Let some number measure them and let it be $C$

And as many times as $C$ measures $A$, so many units let there be in $D$

and as many times as $C$ measures $B$ so many units let there be in $E$

A———
B———
C———
D———
E———

Since $C$ measures $A$ according to the units in $D$

therefore $C$ by multiplying $D$ has made $A$ [VII Def 15]

For the same reason also

$C$ by multiplying $E$ has made $B$

Thus the number $C$ by multiplying the two numbers $D$, $E$ has made $A$ $B$,

therefore as $D$ is to $E$ so is $A$ to $B$, [VII 17]

therefore $D$ $E$ are in the same ratio with $A$, $B$ being less than they which is impossible

Therefore no number will measure the numbers $A$, $B$

Therefore $A$, $B$ are prime to one another Q E D

PROPOSITION 23

*If two numbers be prime to one another the number which measures the one of them will be prime to the remaining number*

Let $A$ $B$ be two numbers prime to one another, and let any number $C$ measure $A$,

I say that $C$ $B$ are also prime to one another

For if $C$, $B$ are not prime to one another

some number will measure $C$, $B$

Let a number measure them and let it be $D$

Since $D$ measures $C$ and $C$ measures $A$ therefore $D$ also measures $A$

But it also measures $B$

therefore $D$ measures $A$, $B$ which are prime to one another which is impossible [VII Def 12]

A B C D

Therefore no number will measure the numbers $C$ $B$

Therefore $C$ $B$ are prime to one another Q E D

PROPOSITION 24

*If two numbers be prime to any number their product also will be prime to the same*

For let the two numbers $A$ $B$ be prime to any number $C$ and let $A$ by multiplying $B$ make $D$

I say that $C$ $D$ are prime to one another

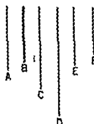
For if $C$ $D$ are not prime to one another some number will measure $C$ $D$

Let a number measure them and let it be $E$

Now, since $C$, $A$ are prime to one another

and a certain number $E$ measures $C$,
therefore $A$ $E$ are prime to one another [VII 23]

As many times then, as $E$ measures $D$ so many units let there be in $F$,




therefore $F$ also measures $D$ according to the units in $E$ [VII 16]

Therefore $E$ by multiplying $F$ has made $D$ [VII Def 15]

But further, $A$ by multiplying $B$ has also made $D$, therefore the product of $E$, $F$ is equal to the product of $A$, $B$

But if the product of the extremes be equal to that of the means the four numbers are proportional [VII 19]

therefore as $E$ is to $A$, so is $B$ to $F$

But $A$ $E$ are prime to one another numbers which are prime to one another are also the least of those which have the same ratio [VII 21]

and the least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent [VII 20]

therefore $E$ measures $B$

But it also measures $C$,

therefore $E$ measures $B$, $C$ which are prime to one another which is impossible [VII Def 12]

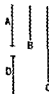
Therefore no number will measure the numbers $C$ $D$

Therefore $C$, $D$ are prime to one another Q E D

## Proposition 25

*If two numbers be prime to one another the product of one of them into itself will be prime to the remaining one*

Let $A$ $B$ be two numbers prime to one another and let $A$ by multiplying itself make $C$




I say that $B$ $C$ are prime to one another

For let $D$ be made equal to $A$

Since $A$ $B$ are prime to one another and $A$ is equal to $D$ therefore $D$ $B$ are also prime to one another

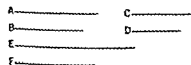
Therefore each of the two numbers $D$, $A$ is prime to $B$ therefore the product of $D$ $A$ will also be prime to $B$ [VII 24]

But the number which is the product of $D$ $A$ is $C$

Therefore $C$ $B$ are prime to one another Q E D

## Proposition 26

*If two numbers be prime to two numbers both to each their products also will be prime to one another*




For let the two numbers $A$ $B$ be prime to the two numbers $C$ $D$ both to each, and let $A$ by multiplying $B$ make $E$ and let $C$ by multiplying $D$ make $F$

I say that $E$ $F$ are prime to one another

For, since each of the numbers $A$ $B$ is prime to $C$
therefore the product of $A$ $B$ will also be prime to $C$ [VII 24]
But the product of $A$, $B$ is $E$
therefore $E$, $C$ are prime to one another
For the same reason
$E$ $D$ are also prime to one another
Therefore each of the numbers $C$, $D$ is prime to $E$
Therefore the product of $C$, $D$ will also be prime to $E$ [VII 24]
But the product of $C$, $D$ is $F$
Therefore $E$, $F$ are prime to one another Q E D

## PROPOSITION 27

*If two numbers be prime to one another and each by multiplying itself make a certain number, the products will be prime to one another and if the original numbers by multiplying the products make certain numbers the latter will also be prime to one another [and this is always the case with the extremes]*

Let $A$, $B$ be two numbers prime to one another let $A$ by multiplying itself make $C$, and by multiplying $C$ make $D$ and let $B$ by multiplying itself make $E$, and by multiplying $E$ make $F$,

I say that both $C$, $E$ and $D$ $F$ are prime to one another

For since $A$ $B$ are prime to one another and $A$ by multiplying itself has made $C$
therefore $C$, $B$ are prime to one another [VII 25]
Since then $C$, $B$ are prime to one another
and $B$ by multiplying itself has made $E$
therefore $C$ $E$ are prime to one another [id]
Again since $A$ $B$ are prime to one another
and $B$ by multiplying itself has made $E$,
therefore $A$ $E$ are prime to one another [id]

Since then the two numbers $A$ $C$ are prime to the two numbers $B$ $E$, both to each
therefore also the product of $A$ $C$ is prime to the product of $B$, $E$ [VII 26]

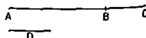
And the product of $A$ $C$ is $D$ and the product of $B$, $E$ is $F$
Therefore $D$ $F$ are prime to one another Q E D

## PROPOSITION 28

*If two numbers be prime to one another the sum will also be prime to each of them and if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another*

For let two numbers $AB$ $BC$ prime to one another be added

I say that the sum $AC$ is also prime to each of the numbers $AB$ $BC$




For if $CA$ $AB$ are not prime to one another
some number will measure $CA$ $AB$
Let a number measure them and let it be $D$
Since then $D$ measures $CA$ $AB$
therefore it will also measure the remainder $BC$

But it also measures $BA$,

therefore $D$ measures $AB$, $BC$ which are prime to one another which is impossible [VII Def 12]

Therefore no number will measure the numbers $CA$, $AB$, therefore $CA$, $AB$ are prime to one another

For the same reason

$AC$ $CB$ are also prime to one another

Therefore $CA$ is prime to each of the numbers $AB$, $BC$

Again let $CA$ $AB$ be prime to one another,

I say that $AB$ $BC$ are also prime to one another

For if $AB$ $BC$ are not prime to one another,

some number will measure $AB$ $BC$

Let a number measure them and let it be $D$

Now since $D$ measures each of the numbers $AB$ $BC$ it will also measure the whole $CA$

But it also measures $AB$

therefore $D$ measures $CA$ $AB$ which are prime to one another

which is impossible [VII Def 12]

Therefore no number will measure the numbers $AB$ $BC$

Therefore $AB$, $BC$ are prime to one another Q E D

PROPOSITION 29

*Any prime number is prime to any number which it does not measure*

Let $A$ be a prime number and let it not measure $B$

I say that $B$ $A$ are prime to one another

A

B

C

For if $B$ $A$ are not prime to one another,

some number will measure them

Let $C$ measure them

Since $C$ measures $B$

and $A$ does not measure $B$

therefore $C$ is not the same with $A$

Now since $C$ measures $B$ $A$

therefore it also measures $A$ which is prime though it is not the same with it

which is impossible

Therefore no number will measure $B$ $A$

Therefore $A$ $B$ are prime to one another Q E D

PROPOSITION 30

*If two numbers by multiplying one another make some number and any prime number measure the product it will also measure one of the original numbers*

For let the two numbers $A$ $B$ by multiplying one another make $C$ and let any prime number $D$ measure $C$

A

B

C

D

E

I say that $D$ measures one of the numbers $A$ $B$

For let it not measure $A$

Now $D$ is prime

therefore $A$ $D$ are prime to one another [VII 29]

And as many times as $D$ measures $C$ so many units let there be in $E$

Since then $D$ measures $C$ according to the units in $E$

therefore $D$ by multiplying $E$ has made $C$ [VII Def 15]

Further $A$ by multiplying $B$ has also made $C$,

therefore the product of $D$, $E$ is equal to the product of $A$, $B$

Therefore as $D$ is to $A$, so is $B$ to $E$ [VII 19]

But $D$ $A$ are prime to one another,

primes are also least [VII 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore $D$ measures $B$

Similarly we can also show that if $D$ does not measure $B$, it will measure $A$

Therefore $D$ measures one of the numbers $A$, $B$ Q E D

## Proposition 31

*Any composite number is measured by some prime number*

Let $A$ be a composite number,

I say that $A$ is measured by some prime number

For since $A$ is composite

some number will measure it

Let a number measure it and let it be $B$

Now, if $B$ is prime, what was enjoined will have been done

A———
B———
C———

But if it is composite some number will measure it

Let a number measure it, and let it be $C$

Then since $C$ measures $B$,

and $B$ measures $A$

therefore $C$ also measures $A$

And if $C$ is prime what was enjoined will have been done

But if it is composite some number will measure it

Thus if the investigation be continued in this way some prime number will be found which will measure the number before it which will also measure $A$

For if it is not found an infinite series of numbers will measure the number $A$ each of which is less than the other

which is impossible in numbers

Therefore some prime number will be found which will measure the one before it which will also measure $A$

Therefore any composite number is measured by some prime number

Q E D

## Proposition 32

*Any number either is prime or is measured by some prime number*

Let $A$ be a number

I say that $A$ either is prime or is measured by some prime number

A———

If now $A$ is prime that which was enjoined will have been done

But if it is composite some prime number will measure it [VII 31]

Therefore any number either is prime or is measured by some prime number

Q E D

PROPOSITION 33

*Given as many numbers as we please, to find the least of those which have the same ratio with them*

Let *A* *B*, *C* be the given numbers, as many as we please,

thus it is required to find the least of those which have the same ratio with *A* *B*, *C*

*A*, *B*, *C* are either prime to one another or not

Now, if *A*, *B*, *C* are prime to one another, they are the least of those which have the same ratio with them [VII 21]

But if not, let *D* the greatest common measure of *A*, *B*, *C* be taken [VII 3]

and, as many times as *D* measures the numbers *A*, *B* *C* respectively, so many units let there be in the numbers *E*, *F*, *G* respectively

Therefore the numbers *E*, *F*, *G* measure the numbers *A* *B*, *C* respectively according to the units in *D* [VII 16]

Therefore *E*, *F*, *G* measure *A* *B*, *C* the same number of times

therefore *E* *F*, *G* are in the same ratio with *A*, *B*, *C* [VII Def 20]

I say next that they are the least that are in that ratio

For if *E*, *F*, *G* are not the least of those which have the same ratio with *A*, *B*, *C*,

there will be numbers less than *E*, *F*, *G* which are in the same ratio with *A*, *B*, *C*

Let them be *H* *K* *L*,

therefore *H* measures *A* the same number of times that the numbers *K* *L* measure the numbers *B*, *C* respectively

Now, as many times as *H* measures *A*, so many units let there be in *M*

therefore the numbers *K* *L* also measure the numbers *B* *C* respectively according to the units in *M*

And since *H* measures *A* according to the units in *M*

therefore *M* also measures *A* according to the units in *H* [VII 16]

For the same reason

*M* also measures the numbers *B*, *C* according to the units in the numbers *K* *L* respectively,

Therefore *M* measures *A* *B*, *C*

Now since *H* measures *A* according to the units in *M*

therefore *H* by multiplying *M* has made *A* [VII Def 15]

For the same reason also

*E* by multiplying *D* has made *A*

Therefore the product of *E* *D* is equal to the product of *H*, *M*

Therefore as *E* is to *H* so is *M* to *D* [VII 19]

But *E* is greater than *H*

therefore *M* is also greater than *D*

And it measures *A* *B* *C*

which is impossible for by hypothesis *D* is the greatest common measure of *A*, *B* *C*

Therefore there cannot be any numbers less than $E$, $F$, $G$ which are in the same ratio with $A$, $B$, $C$

Therefore $E$ $F$, $G$ are the least of those which have the same ratio with $A$, $B$, $C$

Q E D

## Proposition 34

*Given two numbers to find the least number which they measure*

Let $A$ $B$ be the two given numbers,

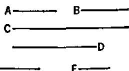
thus it is required to find the least number which they measure

Now $A$, $B$ are either prime to one another or not

First let $A$, $B$ be prime to one another, and let $A$ by multiplying $B$ make $C$,

therefore also $B$ by multiplying $A$ has made $C$

Therefore $A$, $B$ measure $C$




I say next that it is also the least number they measure

For if not, $A$ $B$ will measure some number which is less than $C$

Let them measure $D$

Then as many times as $A$ measures $D$ so many units let there be in $E$, and as many times as $B$ measures $D$, so many units let there be in $F$,

therefore $A$ by multiplying $E$ has made $D$

and $B$ by multiplying $F$ has made $D$ [VII Def 15]

therefore the product of $A$ $E$ is equal to the product of $B$ $F$

Therefore as $A$ is to $B$, so is $F$ to $E$ [VII 19]

But $A$ $B$ are prime

primes are also least [VII 21]

and the least measure the numbers which have the same ratio the same number of times the greater the greater and the less the less [VII 20]

therefore $B$ measures $E$ as consequent consequent

And since $A$ by multiplying $B$ $E$ has made $C$ $D$

therefore as $B$ is to $E$ so is $C$ to $D$ [VII 17]

But $B$ measures $E$,

therefore $C$ also measures $D$ the greater the less which is impossible

Therefore $A$, $B$ do not measure any number less than $C$

therefore $C$ is the least that is measured by $A$ $B$

Next, let $A$ $B$ not be prime to one another

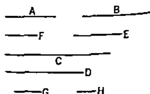
and let $F$ $E$ the least numbers of those which have the same ratio with $A$ $B$, be taken, [VII 33]

therefore the product of $A$ $E$ is equal to the product of $B$, $F$ [VII 19]

And let $A$ by multiplying $E$ make $C$

therefore also $B$ by multiplying $F$ has made $C$,

therefore $A$, $B$ measure $C$




I say next that it is also the least number that they measure

For, if not $A$ $B$ will measure some number which is less than $C$

Let them measure $D$

And, as many times as $A$ measures $D$, so many units let there be in $G$ and as many times as $B$ measures $D$ so many units let there be in $H$

Therefore $A$ by multiplying $G$ has made $D$,

and $B$ by multiplying $H$ has made $D$

Therefore the product of $A$, $G$ is equal to the product of $B$, $H$,

therefore, as $A$ is to $B$, so is $H$ to $G$ [VII 19]

But as $A$ is to $B$, so is $F$ to $E$

Therefore also as $F$ is to $E$ so is $H$ to $G$

But $F$, $E$ are least

and the least measure the numbers which have the same ratio the same number of times the greater the greater and the less the less [VII 20]

therefore $E$ measures $G$

And since $A$ by multiplying $E$ $G$ has made $C$, $D$

therefore as $E$ is to $G$, so is $C$ to $D$ [VII 17]

But $E$ measures $G$

therefore $C$ also measures $D$, the greater the less

which is impossible

Therefore $A$ $B$ will not measure any number which is less than $C$

Therefore $C$ is the least that is measured by $A$ $B$ Q E D

PROPOSITION 35

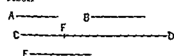
*If two numbers measure any number the least number measured by them will also measure the same*

For let the two numbers $A$ $B$ measure any number $CD$

and let $E$ be the least that they measure

I say that $E$ also measures $CD$

For if $E$ does not measure $CD$ let $E$ measuring $DF$ leave $CF$ less than itself




Now since $A$ $B$ measure $E$

and $E$ measures $DF$,

therefore $A$, $B$ will also measure $DF$

*But they also measure the whole $CD$*,

therefore they will also measure the remainder $CF$ which is less than $E$

which is impossible

Therefore $E$ cannot fail to measure $CD$

therefore it measures it Q E D

PROPOSITION 36

*Given three numbers to find the least number which they measure*

Let $A$, $B$, $C$ be the three given numbers,

thus it is required to find the least number which they measure

Let $D$ the least number measured by the two numbers $A$ $B$ be taken [VII 34]

Then $C$ either measures or does not measure $D$

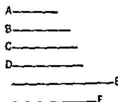
First let it measure it

But $A$ $B$ also measure $D$

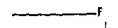
therefore $A$, $B$ $C$ measure $D$

I say next that it is also the least that they measure

For, if not, $A$, $B$ $C$ will measure some number which is less than $D$
Let them measure $E$
Since $A$, $B$, $C$ measure $E$,
therefore also $A$ $B$ measure $E$
Therefore the least number measured by $A$, $B$ will also measure $E$ [VII 35]
But $D$ is the least number measured by $A$, $B$,
therefore $D$ will measure $E$, the greater the less
which is impossible
Therefore $A$, $B$ $C$ will not measure any number which is less than $D$,
therefore $D$ is the least that $A$ $B$, $C$ measure
Again let $C$ not measure $D$,
and let $E$ the least number measured by $C$, $D$, be taken [VII 34]




Since $A$, $B$ measure $D$
and $D$ measures $E$
therefore also $A$, $B$ measure $E$
But $C$ also measures $E$
therefore also $A$ $B$, $C$ measure $E$




I say next that it is also the least that they measure
For if not $A$, $B$ $C$ will measure some number which is less than $E$
Let them measure $F$
Since $A$ $B$, $C$ measure $F$
therefore also $A$, $B$ measure $F$,
therefore the least number measured by $A$, $B$ will also measure $F$ [VII 35]
But $D$ is the least number measured by $A$, $B$,
therefore $D$ measures $F$
But $C$ also measures $F$,
therefore $D$ $C$ measure $F$
so that the least number measured by $D$ $C$ will also measure $F$
But $E$ is the least number measured by $C$, $D$,
therefore $E$ measures $F$ the greater the less
which is impossible
Therefore $A$ $B$ $C$ will not measure any number which is less than $E$
Therefore $E$ is the least that is measured by $A$, $B$ $C$ Q E D

PROPOSITION 37

*If a number be measured by any number the number which is measured will have a part called by the same name as the measuring number*

For let the number $A$ be measured by any number $B$

I say that $A$ has a part called by the same name as $B$

A
B
C
D

For as many times as $B$ measures $A$ so many units let there be in $C$

Since $B$ measures $A$ according to the units in $C$, and the unit $D$ also measures the number $C$ according to the units in it

therefore the unit $D$ measures the number $C$ the same number of times as $B$ measures $A$

Therefore alternately the unit $D$ measures the number $B$ the same number of times as $C$ measures $A$ [VII 15]

therefore, whatever part the unit $D$ is of the number $B$, the same part is $C$ of $A$ also

But the unit $D$ is a part of the number $B$ called by the same name as it,

therefore $C$ is also a part of $A$ called by the same name as $B$,

so that $A$ has a part $C$ which is called by the same name as $B$ Q E D

PROPOSITION 38

*If a number have any part whatever, it will be measured by a number called by the same name as the part*

For let the number $A$ have any part whatever, $B$

and let $C$ be a number called by the same name as the part $B$,

I say that $C$ measures $A$

For since $B$ is a part of $A$ called by the same name as $C$,

and the unit $D$ is also a part of $C$ called by the same name as it

therefore whatever part the unit $D$ is of the number $C$

the same part is $B$ of $A$ also

therefore the unit $D$ measures the number $C$ the same number of times that $B$ measures $A$

Therefore, alternately the unit $D$ measures the number $B$ the same number of times that $C$ measures $A$ [VII 15]

Therefore $C$ measures $A$ Q E D

PROPOSITION 39

*To find the number which is the least that will have given parts*

Let $A$, $B$, $C$ be the given parts

thus it is required to find the number which is the least that will have the parts $A$, $B$ $C$

Let $D$ $E$, $F$ be numbers called by the same name as the parts $A$, $B$ $C$

and let $G$ the least number measured by $D$, $E$, $F$, be taken [VII 36]

Therefore $G$ has parts called by the same name as $D$ $E$, $F$ [VII 37]

But $A$ $B$ $C$ are parts called by the same name as $D$ $E$ $F$

therefore $G$ has the parts $A$ $B$ $C$

I say next that it is also the least number that has

For if not there will be some number less than $G$ which will have the parts $A$, $B$ $C$

Let it be $H$

Since $H$ has the parts $A$ $B$ $C$

therefore $H$ will be measured by numbers called by the same name as the parts $A$ $B$, $C$ [VII 38]

But $D$ $E$, $F$ are numbers called by the same name as the parts $A$, $B$ $C$,

therefore $H$ is measured by $D$ $E$ $F$

And it is less than $G$ which is impossible

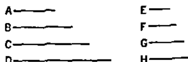
Therefore there will be no number less than $G$ that will have the parts $A$ $B$ $C$ Q E D

# BOOK EIGHT

## Proposition 1

*If there be as many numbers as we please in continued proportion and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them*

Let there be as many numbers as we please A, B C D, in continued proportion and let the extremes of them A D be prime to one another




I say that A B C D are the least of those which have the same ratio with them

For if not let E F G H be less than A B C, D and in the same ratio with them

Now since A B C D are in the same ratio with E F, G H and the multitude of the numbers A, B C D is equal to the multitude of the numbers E F, G, H,

therefore *ex aequali*
as A is to D so is F to H [VII 14]

But A D are prime

primes are also least [VII 21]

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent [VII 20]

Therefore A measures E, the greater the less

which is impossible

Therefore E F G H which are less than A B C, D are not in the same ratio with them

Therefore A B, C D are the least of those which have the same ratio with them

Q E D

## Proposition 2

*To find numbers in continued proportion as many as may be prescribed and the least that are in a given ratio*

Let the ratio of A to B be the given ratio in least numbers,

thus it is required to find numbers in continued proportion as many as may be prescribed and the least that are in the ratio of A to B

Let four be prescribed

let A by multiplying itself make C and by multiplying B let it make D

let B by multiplying itself make E,

further let A by multiplying C D E make F, G, H

and let $B$ by multiplying $E$ make $K$.

Now, since $A$ by multiplying itself has made $C$, and by multiplying $B$ has made $D$, therefore, as $A$ is to $B$, so is $C$ to $D$. [VII. 17]

Again, since $A$ by multiplying $B$ has made $D$, and $B$ by multiplying itself has made $E$, therefore the numbers $A$, $B$ by multiplying $B$ have made the numbers $D$, $E$ respectively.

Therefore, as $A$ is to $B$, so is $D$ to $E$. [VII. 18]

But, as $A$ is to $B$, so is $C$ to $D$;

therefore also, as $C$ is to $D$, so is $D$ to $E$.

And, since $A$ by multiplying $C$, $D$ has made $F$, $G$,

therefore, as $C$ is to $D$, so is $F$ to $G$. [VII. 17]

But, as $C$ is to $D$, so was $A$ to $B$;

therefore also, as $A$ is to $B$, so is $F$ to $G$.

Again, since $A$ by multiplying $D$, $E$ has made $G$, $H$,

therefore, as $D$ is to $E$, so is $G$ to $H$. [VII. 17]

But, as $D$ is to $E$, so is $A$ to $B$.

Therefore also, as $A$ is to $B$, so is $G$ to $H$.

And, since $A$, $B$ by multiplying $E$ have made $H$, $K$,

therefore, as $A$ is to $B$, so is $H$ to $K$. [VII. 18]

But, as $A$ is to $B$, so is $F$ to $G$, and $G$ to $H$.

Therefore also, as $F$ is to $G$, so is $G$ to $H$, and $H$ to $K$;

*therefore $C$, $D$, $E$, and $F$, $G$, $H$, $K$ are proportional in the ratio of $A$ to $B$.*

I say next that they are the least numbers that are so.

For, since $A$, $B$ are the least of those which have the same ratio with them, and the least of those which have the same ratio are prime to one another, [VII. 22]

therefore $A$, $B$ are prime to one another.

And the numbers $A$, $B$ by multiplying themselves respectively have made the numbers $C$, $E$, and by multiplying the numbers $C$, $E$ respectively have made the numbers $F$, $K$;

therefore $C$, $E$ and $F$, $K$ are prime to one another respectively. [VII. 27]

But, if there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, they are the least of those which have the same ratio with them. [VIII. 1]

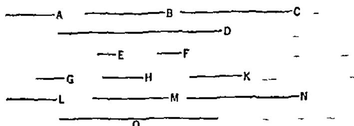
Therefore $C$, $D$, $E$ and $F$, $G$, $H$, $K$ are the least of those which have the same ratio with $A$, $B$. Q. E. D.

PORISM. From this it is manifest that, if three numbers in continued proportion be the least of those which have the same ratio with them, the extremes of them are squares, and, if four numbers, cubes.

PROPOSITION 3.

*If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.*

Let as many numbers as we please $A, B, C, D$ in continued proportion be the least of those which have the same ratio with them,




I say that the extremes of them $A, D$ are prime to one another

For let two numbers $E, F$, the least that are in the ratio of $A, B, C, D$ be taken, [VII 33]

then three others $G$ $H$ $K$ with the same property,

and others more by one continually [VIII 2]

until the multitude taken becomes equal to the multitude of the numbers $A$ $B, C$ $D$

Let them be taken and let them be $L$ $M$ $N$ $O$

Now, since $E$ $F$ are the least of those which have the same ratio with them they are prime to one another [VII 22]

And since the numbers $E, F$ by multiplying themselves respectively have made the numbers $G$ $K$ and by multiplying the numbers $G, K$ respectively have made the numbers $L, O$ [VIII 2 Por]

therefore both $G$ $K$ and $L, O$ are prime to one another [VII 27]

And since $A$ $B, C$ $D$ are the least of those which have the same ratio with them

while $L$ $M$ $N$ $O$ are the least that are in the same ratio with $A, B, C$ $D$ and the multitude of the numbers $A$ $B, C, D$ is equal to the multitude of the numbers $L$ $M$ $N$ $O$

therefore the numbers $A$ $B$ $C$ $D$ are equal to the numbers $L$ $M$ $N, O$ respectively

therefore $A$ is equal to $L$ and $D$ to $O$

And $L$ $O$ are prime to one another

Therefore $A$ $D$ are also prime to one another Q E D

PROPOSITION 4

*Given as many ratios as we please in least numbers to find numbers in continued proportion which are the least in the given ratios*

Let the given ratios in least numbers be that of $A$ to $B$, that of $C$ to $D$ and that of $E$ to $F$

thus it is required to find numbers in continued proportion which are the least that are in the ratio of $A$ to $B$ in the ratio of $C$ to $D$ and in the ratio of $E$ to $F$

Let $G$ the least number measured by $B$ $C$ be taken [VII 34]

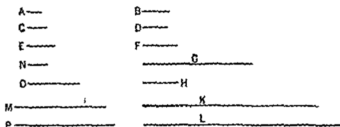
And as many times as $B$ measures $G$ so many times also let $A$ measure $H$, and as many times as $C$ measures $G$ so many times also let $D$ measure $K$

Now $E$ either measures or does not measure $K$

First let it measure it

And as many times as $E$ measures $K$, so many times let $F$ measure $L$ also

Now since $A$ measures $H$ the same number of times that $B$ measures $G$ therefore as $A$ is to $B$, so is $H$ to $G$ [VII Def 20, VII 13]




For the same reason also

as $C$ is to $D$ so is $G$ to $K$

and further as $E$ is to $F$, so is $K$ to $L$,

therefore $H$, $G$ $K$, $L$ are continuously proportional in the ratio of $A$ to $B$, in the ratio of $C$ to $D$ and in the ratio of $E$ to $F$

I say next that they are also the least that have this property

For, if $H$, $G$, $K$, $L$ are not the least numbers continuously proportional in the ratios of $A$ to $B$, of $C$ to $D$ and of $E$ to $F$, let them be $N$, $O$ $M$ $P$

Then since, as $A$ is to $B$, so is $N$ to $O$

while $A$ $B$ are least

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent,

therefore $B$ measures $O$ [VII 20]

For the same reason

$C$ also measures $O$,

therefore $B$ $C$ measure $O$

therefore the least number measured by $B$, $C$ will also measure $O$ [VII 35]

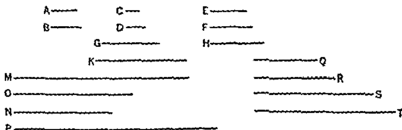
But $G$ is the least number measured by $B$ $C$,

therefore $G$ measures $O$ the greater the less

which is impossible

Therefore there will be no numbers less than $H$ $G$ $K$, $L$ which are continuously in the ratio of $A$ to $B$, of $C$ to $D$ and of $E$ to $F$

Next let $E$ not measure $K$




Let $M$ the least number measured by $E$ $K$ be taken

And as many times as $K$ measures $M$ so many times let $H$ $G$ measure $N$ $O$ respectively

and as many times as $E$ measures $M$ so many times let $F$ measure $P$ also

Since $H$ measures $N$ the same number of times that $G$ measures $O$,

therefore, as $H$ is to $G$, so is $N$ to $O$ [VII 13 and Def 20]

But as $H$ is to $G$, so is $A$ to $B$,

therefore also as $A$ is to $B$, so is $N$ to $O$

For the same reason also

as $C$ is to $D$, so is $O$ to $M$

Again since $E$ measures $M$ the same number of times that $F$ measures $P$,

therefore as $L$ is to $F$ so is $M$ to $P$ [VII 13 and Def 20]

therefore $N$, $O$ $M$ $P$ are continuously proportional in the ratios of $A$ to $B$, of $C$ to $D$ and of $E$ to $F$

I say next that they are also the least that are in the ratios $A$ $B$, $C$ $D$, $E$ $F$

For if not there will be some numbers less than $N$, $O$, $M$ $P$ continuously proportional in the ratios $A$ $B$, $C$ $D$ $E$ $F$

Let them be $Q$ $R$, $S$, $T$

Now since as $Q$ is to $R$, so is $A$ to $B$

while $A$, $B$ are least

and the least numbers measure those which have the same ratio with them the same number of times the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore $B$ measures $R$

For the same reason $C$ also measures $R$,

therefore $B$, $C$ measure $R$

Therefore the least number measured by $B$ $C$ will also measure $R$ [VII 35]

But $G$ is the least number measured by $B$, $C$,

therefore $G$ measures $R$

And as $G$ is to $R$ so is $K$ to $S$ [VII 13]

therefore $K$ also measures $S$

But $E$ also measures $S$

therefore $E$ $K$ measure $S$

Therefore the least number measured by $E$ $K$ will also measure $S$ [VII 35]

But $M$ is the least number measured by $E$ $K$,

therefore $M$ measures $S$ the greater the less

which is impossible

Therefore there will not be any numbers less than $N$ $O$ $M$ $P$ continuously proportional in the ratios of $A$ to $B$ of $C$ to $D$ and of $E$ to $F$,

therefore $N$ $O$ $M$ $P$ are the least numbers continuously proportional in the ratios $A$ $B$ $C$ $D$ $E$ $F$ Q E D

## Proposition 5

*Plane numbers have to one another the ratio compounded of the ratios of their sides*

Let $A$ $B$ be plane numbers and let the numbers $C$ $D$ be the sides of $A$ and $E$ $F$ of $B$

I say that $A$ has to $B$ the ratio compounded of the ratios of the sides

For the ratios being given which $C$ has to $E$ and $D$ to $F$, let the least numbers $G$ $H$, $K$ that are continuously in the ratios $C$ $E$ $D$ $F$ be taken so that

as $C$ is to $E$ so is $G$ to $H$

and as $D$ is to $F$ so is $H$ to $K$ [VIII 4]

And let $D$ by multiplying $E$ make $L$

Now since $D$ by multiplying $C$ has made $A$, and by multiplying $E$ has made $L$,

therefore as $C$ is to $E$, so is $A$ to $L$ [VII 17]

But as $C$ is to $E$ so is $G$ to $H$,

therefore also as $G$ is to $H$, so is $A$ to $L$

Again since $E$ by multiplying $D$ has made $L$, and further by multiplying $F$ has made $B$

therefore, as $D$ is to $F$ so is $L$ to $B$ [VII 17]

But as $D$ is to $F$, so is $H$ to $K$,

therefore also as $H$ is to $K$, so is $L$ to $B$

But it was also proved that

as $G$ is to $H$, so is $A$ to $L$,

therefore *ex aequali*,

as $G$ is to $K$ so is $A$ to $B$ [VII 14]

But $G$ has to $K$ the ratio compounded of the ratios of the sides

therefore $A$ also has to $B$ the ratio compounded of the ratios of the sides

Q E D

PROPOSITION 6

*If there be as many numbers as we please in continued proportion, and the first do not measure the second neither will any other measure any other*

Let there be as many numbers as we please $A$, $B$ $C$, $D$, $E$ in continued proportion and let $A$ not measure $B$,

I say that neither will any other measure any other

Now it is manifest that $A$ $B$, $C$ $D$ $E$ do not measure one another in order for $A$ does not even measure $B$

I say then that neither will any other measure any other

For if possible let $A$ measure $C$

And however many $A$ $B$, $C$ are let as many numbers $F$, $G$ $H$ the least of those which have the same ratio with $A$ $B$ $C$ be taken [VII 33]

Now, since $F$ $G$ $H$ are in the same ratio with $A$, $B$, $C$, and the multitude of the numbers $A$ $B$ $C$ is equal to the multitude of the numbers $F$, $G$ $H$

therefore, *ex aequali* as $A$ is to $C$ so is $F$ to $H$ [VII 14]

And since, as $A$ is to $B$ so is $F$ to $G$

while $A$ does not measure $B$,

therefore neither does $F$ measure $G$, [VII Def 20]

therefore $F$ is not an unit for the unit measures any number

Now $F$ $H$ are prime to one another [VIII 3]

And as $F$ is to $H$ so is $A$ to $C$

therefore neither does $A$ measure $C$

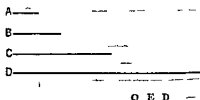
Similarly we can prove that neither will any other measure any other

Q E D

PROPOSITION 7

*If there be as many numbers as we please in continued proportion, and the first measure the last it will measure the second also*

Let there be as many numbers as we please, *A*, *B*, *C*, *D*, in continued proportion, and let *A* measure *D*,




I say that *A* also measures *B*

For if *A* does not measure *B*, neither will any other of the numbers measure any other [VIII 6]

But *A* measures *D*

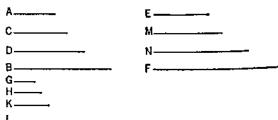
Therefore *A* also measures *B* Q E D

PROPOSITION 8

*If between two numbers there fall numbers in continued proportion with them then however many numbers fall between them in continued proportion so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers*

Let the numbers *C*, *D* fall between the two numbers *A*, *B* in continued proportion with them, and let *E* be made in the same ratio to *F* as *A* is to *B*,

I say that as many numbers as have fallen between *A* *B* in continued proportion so many will also fall between *E* *F* in continued proportion




For as many as *A* *B* *C* *D* are in multitude let so many numbers *G* *H* *K*, *L* the least of those which have the same ratio with *A*, *C* *D* *B*, be taken [VII 33]

therefore the extremes of them *G*, *L* are prime to one another [VIII 3]

Now since *A*, *C*, *D* *B* are in the same ratio with *G* *H* *K* *L* and the multitude of the numbers *A*, *C* *D* *B* is equal to the multitude of the numbers *G*, *H* *K*, *L*

therefore *ex aequali* as *A* is to *B* so is *G* to *L* [VII 14]

But, as *A* is to *B* so is *E* to *F*

therefore also as *G* is to *L* so is *E* to *F*

But *G* *L* are prime,

primes are also least [VII 21]

and the least numbers measure those which have the same ratio the same number of times the greater the greater and the less the less that is the antecedent the antecedent and the consequent the consequent [VII 20]

Therefore *G* measures *E* the same number of times as *L* measures *F*

Next as many times as *G* measures *E* so many times let *H*, *K* also measure *M*, *N* respectively

therefore *G* *H*, *K* *L* measure *E* *M*, *N*, *F* the same number of times

Therefore *G* *H* *K* *L* are in the same ratio with *E* *M* *N*, *F* [VII Def 20]

But *G*, *H*, *K* *L* are in the same ratio with *A* *C* *D* *B*

therefore *A*, *C* *D* *B* are also in the same ratio with *E* *M*, *N*, *F*

But $A$ $C$ $D$ $B$ are in continued proportion

therefore $E$, $M$, $N$, $F$ are also in continued proportion

Therefore, as many numbers as have fallen between $A$, $B$ in continued proportion with them, so many numbers have also fallen between $E$, $F$ in continued proportion Q E D

PROPOSITION 9

*If two numbers be prime to one another and numbers fall between them in continued proportion, then however many numbers fall between them in continued proportion so many will also fall between each of them and an unit in continued proportion*

Let $A$, $B$ be two numbers prime to one another and let $C$, $D$ fall between them in continued proportion

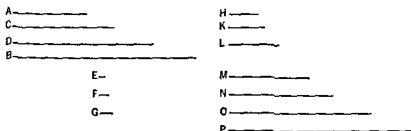
and let the unit $E$ be set out,

I say that as many numbers as fall between $A$, $B$ in continued proportion so many will also fall between either of the numbers $A$, $B$ and the unit in continued proportion

For let two numbers $F$, $G$, the least that are in the ratio of $A$, $C$, $D$, $B$ be taken

three numbers $H$, $K$, $L$ with the same property,

and others more by one continually until their multitude is equal to the multitude of $A$, $C$ $D$, $B$ [VIII 2]




Let them be taken and let them be $M$ $N$, $O$ $P$

It is now manifest that $F$ by multiplying itself has made $H$ and by multiplying $H$ has made $M$ while $G$ by multiplying itself has made $L$ and by multiplying $L$ has made $P$ [VIII 2, Por]

And since $M$ $N$ $O$ $P$ are the least of those which have the same ratio with $F$ $G$

and $A$, $C$ $D$ $B$ are also the least of those which have the same ratio with $F$ $G$ [VIII 1]

while the multitude of the numbers $M$, $N$, $O$ $P$ is equal to the multitude of the numbers $A$ $C$ $D$ $B$

therefore $M$ $N$ $O$ $P$ are equal to $A$ $C$ $D$ $B$ respectively,

therefore $M$ is equal to $A$, and $P$ to $B$

Now since $F$ by multiplying itself has made $H$

therefore $F$ measures $H$ according to the units in $F$

But the unit $E$ also measures $F$ according to the units in it

therefore the unit $E$ measures the number $F$ the same number of times as $F$ measures $H$

Therefore as the unit $\Gamma$ is to the number $F$ so is $\Gamma$ to $H$ [VII Def 20]

Again, since $F$ by multiplying $H$ has made $M$,

*therefore $H$ measures $M$ according to the units in $F$*

But the unit $L$ also measures the number $F$ according to the units in it, therefore the unit $E$ measures the number $F$ the same number of times as $H$ measures $M$

Therefore, as the unit $E$ is to the number $F$, so is $H$ to $M$

But it was also proved that as the unit $E$ is to the number $F$ so is $F$ to $H$, therefore also as the unit $E$ is to the number $F$, so is $F$ to $H$, and $H$ to $M$

But $M$ is equal to $A$

therefore, as the unit $E$ is to the number $F$, so is $F$ to $H$ and $H$ to $A$

For the same reason also

as the unit $E$ is to the number $G$ so is $G$ to $L$ and $L$ to $B$

Therefore as many numbers as have fallen between $A$, $B$ in continued proportion so many numbers also have fallen between each of the numbers $A$, $B$ and the unit $E$ in continued proportion Q E D

## Proposition 10

*If numbers fall between each of two numbers and an unit in continued proportion however many numbers fall between each of them and an unit in continued proportion so many also will fall between the numbers themselves in continued proportion*

For let the numbers $D$ $E$ and $F$ $G$ respectively fall between the two numbers $A$ $B$ and the unit $C$ in continued proportion

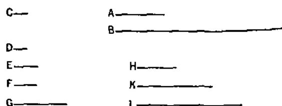
I say that as many numbers as have fallen between each of the numbers $A$ $B$ and the unit $C$ in continued proportion so many numbers will also fall between $A$ $B$ in continued proportion

For let $D$ by multiplying $F$ make $H$ and let the numbers $D$ $F$ by multiplying $H$ make $K$ $L$ respectively

Now since as the unit $C$ is to the number $D$ so is $D$ to $E$

therefore the unit $C$ measures the number $D$ the same number of times as $D$ measures $E$ [VII Def 20]

But the unit $C$ measures the number $D$ according to the units in $D$




therefore the number $D$ also measures $E$ according to the units in $D$,

therefore $D$ by multiplying itself has made $E$

Again since as $C$ is to the number $D$ so is $E$ to $A$

therefore the unit $C$ measures the number $D$ the same number of times as $E$ measures $A$

But the unit $C$ measures the number $D$ according to the units in $D$,

therefore $E$ also measures $A$ according to the units in $D$

therefore $D$ by multiplying $E$ has made $A$

For the same reason also

$F$ by multiplying itself has made $G$ and by multiplying $G$ has made $B$

And since $D$ by multiplying itself has made $E$ and by multiplying $F$ has made $H$,

therefore as $D$ is to $F$, so is $E$ to $H$ [VII 17]

For the same reason also

as $D$ is to $F$, so is $H$ to $G$ [VII 18]

Therefore also as $E$ is to $H$ so is $H$ to $G$

Again, since $D$ by multiplying the numbers $E$ $H$ has made $A$ $K$ respectively,

therefore as $E$ is to $H$, so is $A$ to $K$ [VII 17]

But, as $E$ is to $H$, so is $D$ to $F$,

therefore also, as $D$ is to $F$ so is $A$ to $K$

Again since the numbers $D$ $F$ by multiplying $H$ have made $K$, $L$ respectively,

therefore, as $D$ is to $F$, so is $K$ to $L$ [VII 18]

But as $D$ is to $F$, so is $A$ to $K$

therefore also as $A$ is to $K$, so is $K$ to $L$

Further, since $F$ by multiplying the numbers $H$ $G$ has made $L$, $B$ respectively

therefore as $H$ is to $G$, so is $L$ to $B$, [VII 17]

But, as $H$ is to $G$ so is $D$ to $F$,

therefore also as $D$ is to $F$, so is $L$ to $B$

But it was also proved that,

as $D$ is to $F$ so is $A$ to $K$ and $K$ to $L$

therefore also as $A$ is to $K$, so is $K$ to $L$ and $L$ to $B$

Therefore $A$ $K$, $L$, $B$ are in continued proportion

Therefore as many numbers as fall between each of the numbers $A$, $B$ and the unit $C$ in continued proportion so many also will fall between $A$, $B$ in continued proportion Q E D

PROPOSITION 11

*Between two square numbers there is one mean proportional number and the square has to the square the ratio duplicate of that which the side has to the side*

Let $A$, $B$ be square numbers

and let $C$ be the side of $A$ and $D$ of $B$

I say that between $A$ $B$ there is one mean proportional number and $A$ has to $B$ the ratio duplicate of that which $C$ has to $D$

A———

B————

C—— D——

E————

For let $C$ by multiplying $D$ make $E$

Now since $A$ is a square and $C$ is its side

therefore $C$ by multiplying itself has made $A$

For the same reason also

$D$ by multiplying itself has made $B$

Since then $C$ by multiplying the numbers $C$ $D$ has made $A$ $E$ respectively

therefore as $C$ is to $D$, so is $A$ to $E$ [VII 17]

For the same reason also

as $C$ is to $D$ so is $E$ to $B$ [VII 18]

Therefore also as $A$ is to $E$ so is $E$ to $B$

Therefore between $A$ $B$ there is one mean proportional number

I say next that $A$ also has to $B$ the ratio duplicate of that which $C$ has to $D$

For since $A$ $E$ $B$ are three numbers in proportion

therefore $A$ has to $B$ the ratio duplicate of that which $A$ has to $E$ [v Def 9]

But as $A$ is to $E$, so is $C$ to $D$

Therefore $A$ has to $B$ the ratio duplicate of that which the side $C$ has to $D$

Q E D

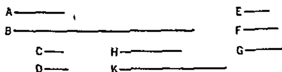
## Proposition 12

*Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side*

Let $A$, $B$ be cube numbers,

and let $C$ be the side of $A$, and $D$ of $B$,

I say that between $A$, $B$ there are two mean proportional numbers and $A$ has to $B$ the ratio triplicate of that which $C$ has to $D$




For let $C$ by multiplying itself make $E$ and by multiplying $D$ let it make $F$ let $D$ by multiplying itself make $G$ and let the numbers $C$, $D$ by multiplying $F$ make $H$, $K$ respectively

Now since $A$ is a cube and $C$ its side

and $C$ by multiplying itself has made $E$,

therefore $C$ by multiplying itself has made $E$ and by multiplying $E$ has made $A$

For the same reason also

$D$ by multiplying itself has made $G$ and by multiplying $G$ has made $B$

And, since $C$ by multiplying the numbers $C$, $D$ has made $E$, $F$ respectively

therefore, as $C$ is to $D$ so is $E$ to $F$ [vii 17]

For the same reason also,

as $C$ is to $D$ so is $F$ to $G$ [vii 18]

Again since $C$ by multiplying the numbers $E$ $F$ has made $A$, $H$ respectively

therefore as $E$ is to $F$, so is $A$ to $H$ [vii 17]

But as $E$ is to $F$ so is $C$ to $D$

Therefore also as $C$ is to $D$, so is $A$ to $H$

Again since the numbers $C$ $D$ by multiplying $F$ have made $H$, $K$ respectively,

therefore as $C$ is to $D$, so is $H$ to $K$ [vii 18]

Again since $D$ by multiplying each of the numbers $F$ $G$ has made $K$, $B$ respectively

therefore as $F$ is to $G$, so is $K$ to $B$ [vii 17]

But, as $F$ is to $G$, so is $C$ to $D$

therefore also as $C$ is to $D$, so is $A$ to $H$, $H$ to $K$ and $K$ to $B$

Therefore $H$, $K$ are two mean proportionals between $A$ $B$

I say next that $A$ also has to $B$ the ratio triplicate of that which $C$ has to $D$

For since $A$ $H$ $K$, $B$ are four numbers in proportion,

therefore $A$ has to $B$ the ratio triplicate of that which $A$ has to $H$ [v Def 10]

But, as $A$ is to $H$ so is $C$ to $D$,

therefore $A$ also has to $B$ the ratio triplicate of that which $C$ has to $D$

Q E D

PROPOSITION 13

*If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional and, if the original numbers by multiplying the products make certain numbers the latter will also be proportional*

Let there be as many numbers as we please, $A$, $B$, $C$, in continued proportion so that as $A$ is to $B$ so is $B$ to $C$,

let $A$, $B$, $C$ by multiplying themselves make $D$, $E$ $F$, and by multiplying $D$, $E$, $F$ let them make $G$, $H$, $K$

I say that $D$, $E$, $F$ and $G$ $H$, $K$ are in continued proportion

For let $A$ by multiplying $B$ make $L$

and let the numbers $A$, $B$ by multiplying $L$ make $M$ $N$ respectively

And again let $B$ by multiplying $C$ make $O$,

and let the numbers $B$, $C$ by multiplying $O$ make $P$, $Q$ respectively

Then in manner similar to the foregoing we can prove that $D$ $L$, $E$ and $G$, $M$ $N$ $H$ are continuously proportional in the ratio of $A$ to $B$ and further $E$, $O$ $F$ and $H$, $P$, $Q$, $K$ are continuously proportional in the ratio of $B$ to $C$

Now, as $A$ is to $B$, so is $B$ to $C$,

therefore $D$ $L$, $E$ are also in the same ratio with $E$ $O$ $F$,

and further $G$ $M$ $N$ $H$ in the same ratio with $H$ $P$ $Q$ $K$

And the multitude of $D$ $L$ $E$ is equal to the multitude of $E$, $O$ $F$ and that of $G$, $M$, $N$, $H$ to that of $H$, $P$ $Q$ $K$,

therefore *ex aequali*

as $D$ is to $E$ so is $E$ to $F$

and as $G$ is to $H$, so is $H$ to $K$ [VII 14]

Q E D

PROPOSITION 14

*If a square measure a square the side will also measure the side and if the side measure the side the square will also measure the square*

Let $A$, $B$ be square numbers let $C$ $D$ be their sides and let $A$ measure $B$

I say that $C$ also measures $D$

For let $C$ by multiplying $D$ make $E$,

therefore $A$ $E$ $B$ are continuously proportional in the ratio of $C$ to $D$ [VIII 11]

And since $A$, $E$ $B$ are continuously proportional and $A$ measures $B$

therefore $A$ also measures $E$ [VIII 7]

therefore $A$ has to $B$ the ratio duplicate of that which $A$ has to $E$ [V Def 9]

But, as $A$ is to $E$, so is $C$ to $D$

Therefore $A$ has to $B$ the ratio duplicate of that which the side $C$ has to $D$

Q E D

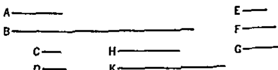
## Proposition 12

*Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side*

Let $A$, $B$ be cube numbers,

and let $C$ be the side of $A$, and $D$ of $B$,

I say that between $A$, $B$ there are two mean proportional numbers and $A$ has to $B$ the ratio triplicate of that which $C$ has to $D$




For let $C$ by multiplying itself make $E$ and by multiplying $D$ let it make $F$, let $D$ by multiplying itself make $G$ and let the numbers $C$, $D$ by multiplying $F$ make $H$, $K$ respectively

Now, since $A$ is a cube and $C$ its side

and $C$ by multiplying itself has made $E$,

therefore $C$ by multiplying itself has made $E$ and by multiplying $E$ has made $A$

For the same reason also

$D$ by multiplying itself has made $G$ and by multiplying $G$ has made $B$

And since $C$ by multiplying the numbers $C$, $D$ has made $E$, $F$ respectively

therefore, as $C$ is to $D$ so is $E$ to $F$ [VII 17]

For the same reason also

as $C$ is to $D$ so is $F$ to $G$ [VII 18]

Again since $C$ by multiplying the numbers $E$, $F$ has made $A$, $H$ respectively

therefore as $E$ is to $F$, so is $A$ to $H$ [VII 17]

But as $E$ is to $F$, so is $C$ to $D$

Therefore also as $C$ is to $D$, so is $A$ to $H$

Again since the numbers $C$, $D$ by multiplying $F$ have made $H$, $K$ respectively

therefore, as $C$ is to $D$, so is $H$ to $K$ [VII 18]

Again since $D$ by multiplying each of the numbers $F$, $G$ has made $K$, $B$ respectively

therefore, as $F$ is to $G$, so is $K$ to $B$ [VII 17]

But as $F$ is to $G$ so is $C$ to $D$,

therefore also as $C$ is to $D$ so is $A$ to $H$, $H$ to $K$, and $K$ to $B$

Therefore $H$, $K$ are two mean proportionals between $A$, $B$

I say next that $A$ also has to $B$ the ratio triplicate of that which $C$ has to $D$

For since $A$, $H$, $K$, $B$ are four numbers in proportion

therefore $A$ has to $B$ the ratio triplicate of that which $A$ has to $H$ [V Def 10]

But, as $A$ is to $H$ so is $C$ to $D$

therefore $A$ also has to $B$ the ratio triplicate of that which $C$ has to $D$

Q E D

### Proposition 13

*If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional; and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.*

Let there be as many numbers as we please, $A$, $B$, $C$, in continued proportion, so that, as $A$ is to $B$, so is $B$ to $C$;

let $A$, $B$, $C$ by multiplying themselves make $D$, $E$, $F$, and by multiplying $D$, $E$, $F$ let them make $G$, $H$, $K$.

I say that $D$, $E$, $F$ and $G$, $H$, $K$ are in continued proportion.

For let $A$ by multiplying $B$ make $L$,

and let the numbers $A$, $B$ by multiplying $L$ make $M$, $N$ respectively.

And again let $B$ by multiplying $C$ make $O$,

and let the numbers $B$, $C$ by multiplying $O$ make $P$, $Q$ respectively.

Then, in manner similar to the foregoing, we can prove that

$D$, $L$, $E$ and $G$, $M$, $N$, $H$ are continuously proportional in the ratio of $A$ to $B$, and further $E$, $O$, $F$ and $H$, $P$, $Q$, $K$ are continuously proportional in the ratio of $B$ to $C$.

Now, as $A$ is to $B$, so is $B$ to $C$;

therefore $D$, $L$, $E$ are also in the same ratio with $E$, $O$, $F$,

and further $G$, $M$, $N$, $H$ in the same ratio with $H$, $P$, $Q$, $K$.

And the multitude of $D$, $L$, $E$ is equal to the multitude of $E$, $O$, $F$, and that of $G$, $M$, $N$, $H$ to that of $H$, $P$, $Q$, $K$;

therefore, *ex aequali*,

as $D$ is to $E$, so is $E$ to $F$,

and, as $G$ is to $H$, so is $H$ to $K$. [VII. 14]

Q. E. D.

### Proposition 14

*If a square measure a square, the side will also measure the side; and, if the side measure the side, the square will also measure the square.*

Let $A$, $B$ be square numbers, let $C$, $D$ be their sides, and let $A$ measure $B$;

I say that $C$ also measures $D$.

For let $C$ by multiplying $D$ make $E$;

therefore $A$, $E$, $B$ are continuously proportional in the ratio of $C$ to $D$. [VIII. 11]

And since $A$, $E$, $B$ are continuously proportional, and $A$ measures $B$,

therefore $A$ also measures $E$. [VIII. 7]

And as $A$ is to $E$, so is $C$ to $D$,

therefore also $C$ measures $D$ [VII Def 20]

Again, let $C$ measure $D$,

I say that $A$ also measures $B$

For, with the same construction we can in a similar manner prove that $A$, $E$ $B$ are continuously proportional in the ratio of $C$ to $D$

And since as $C$ is to $D$, so is $A$ to $E$,

and $C$ measures $D$

therefore $A$ also measures $E$ [VII Def 20]

And $A$, $E$ $B$ are continuously proportional,

therefore $A$ also measures $B$

Therefore etc Q E D

## Proposition 15

*If a cube number measure a cube number the side will also measure the side and, if the side measure the side the cube will also measure the cube*

For let the cube number $A$ measure the cube $B$,

and let $C$ be the side of $A$ and $D$ of $B$,

I say that $C$ measures $D$

For let $C$ by multiplying itself make $E$

and let $D$ by multiplying itself make $G$

further let $C$ by multiplying $D$ make $F$

and let $C$, $D$ by multiplying $F$ make $H$, $K$ respectively

Now it is manifest that $E$, $F$ $G$ and $A$ $H$ $K$, $B$ are continuously proportional in the ratio of $C$ to $D$ [VIII 11 12]

A——
B————————————
C–
D——
E——
G————
F——
H————
K——————

And since $A$, $H$ $K$, $B$ are continuously proportional,

and $A$ measures $B$,

therefore it also measures $H$ [VIII 7]

And as $A$ is to $H$ so is $C$ to $D$

therefore $C$ also measures $D$ [VII Def 20]

Next let $C$ measure $D$,

I say that $A$ will also measure $B$

For with the same construction, we can prove in a similar manner that $A$, $H$, $K$, $B$ are continuously proportional in the ratio of $C$ to $D$

And since $C$ measures $D$

and as $C$ is to $D$ so is $A$ to $H$,

therefore $A$ also measures $H$ [VII Def 20]

so that $A$ measures $B$ also Q E D

## Proposition 16

*If a square number do not measure a square number, neither will the side measure the side and if the side do not measure the side, neither will the square measure the square*

Let $A$, $B$ be square numbers and let $C$ $D$ be their sides and let $A$ not measure $B$,

A
B
C
D

I say that neither does $C$ measure $D$
For if $C$ measures $D$ $A$ will also measure $B$ [VIII 14]
But $A$ does not measure $B$
therefore neither will $C$ measure $D$
Again let $C$ not measure $D$,
I say that neither will $A$ measure $B$
For if $A$ measures $B$ $C$ will also measure $D$ [VIII 14]
But $C$ does not measure $D$
therefore neither will $A$ measure $B$ Q E D

PROPOSITION 17

*If a cube number do not measure a cube number neither will the side measure the side and if the side do not measure the side, neither will the cube measure the cube*

For let the cube number $A$ not measure the cube number $B$,
and let $C$ be the side of $A$ and $D$ of $B$,

A
B
C
D

I say that $C$ will not measure $D$
For if $C$ measures $D$, $A$ will also measure $B$ [VIII 15]
But $A$ does not measure $B$,
therefore neither does $C$ measure $D$
Again let $C$ not measure $D$,
I say that neither will $A$ measure $B$
For if $A$ measures $B$ $C$ will also measure $D$ [VIII 15]
But $C$ does not measure $D$
therefore neither will $A$ measure $B$ Q E D

PROPOSITION 18

*Between two similar plane numbers there is one mean proportional number and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side*

Let $A$, $B$ be two similar plane numbers and let the numbers $C$, $D$ be the sides of $A$ and $E$ $F$ of $B$

A C
B D
E
G F

Now since similar plane numbers are those which have their sides proportional [VII Def 21]
therefore as $C$ is to $D$ so is $E$ to $F$
I say then that between $A$ $B$ there is one mean proportional number and $A$ has to $B$ the ratio duplicate of that which $C$ has to $E$ or $D$ to $F$, that is of that which the corresponding side has to the corresponding side
Now since as $C$ is to $D$ so is $E$ to $F$
therefore alternately as $C$ is to $E$ so is $D$ to $F$ [VII 13]
And since $A$ is plane and $C$, $D$ are its sides
therefore $D$ by multiplying $C$ has made $A$
For the same reason also

$E$ by multiplying $\Gamma$ has made $B$

Now let $D$ by multiplying $L$ make $G$

Then since $D$ by multiplying $C$ has made $A$, and by multiplying $E$ has made $G$

therefore, as $C$ is to $E$, so is $A$ to $G$ [VII 17]

But, as $C$ is to $E$, so is $D$ to $F$,

therefore also as $D$ is to $F$, so is $A$ to $G$

Again since $E$ by multiplying $D$ has made $G$, and by multiplying $F$ has made $B$,

therefore as $D$ is to $F$ so is $G$ to $B$ [VII 17]

But it was also proved that

as $D$ is to $F$, so is $A$ to $G$,

therefore also as $A$ is to $G$ so is $G$ to $B$

Therefore $A$ $G$, $B$ are in continued proportion

Therefore between $A$, $B$ there is one mean proportional number

I say next that $A$ also has to $B$ the ratio duplicate of that which the corresponding side has to the corresponding side that is of that which $C$ has to $E$ or $D$ to $F$

For since $A$ $G$ $B$ are in continued proportion

$A$ has to $B$ the ratio duplicate of that which it has to $G$ [V Def 9]

And as $A$ is to $G$ so is $C$ to $E$ and so is $D$ to $F$

Therefore $A$ also has to $B$ the ratio duplicate of that which $C$ has to $E$ or $D$ to $F$ Q E D

## Proposition 19

*Between two similar solid numbers there fall two mean proportional numbers and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side*

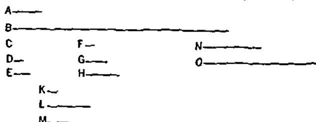
Let $A$ $B$ be two similar solid numbers and let $C$ $D$ $E$ be the sides of $A$ and $F$ $G$, $H$ of $B$

Now since similar solid numbers are those which have their sides proportional [VII Def 21]

therefore as $C$ is to $D$ so is $F$ to $G$

and as $D$ is to $E$ so is $G$ to $H$

I say that between $A$ $B$ there fall two mean proportional numbers, and $A$ has to $B$ the ratio triplicate of that which $C$ has to $F$, $D$ to $G$, and also $E$ to $H$




For let $C$ by multiplying $D$ make $K$ and let $F$ by multiplying $G$ make $L$

Now since $C$ $D$ are in the same ratio with $F$ $G$

and $K$ is the product of $C$ $D$, and $L$ the product of $F$, $G$ $K$ $L$ are similar plane numbers, [VII Def 21]

*therefore between K, L there is one mean proportional number* [VIII 18]

Let it be M

Therefore M is the product of D F, as was proved in the theorem preceding this [VIII 18]

Now, since D by multiplying C has made K, and by multiplying F has made M,

therefore as C is to F so is K to M [VII 17]

But, as K is to M so is M to L

Therefore K M L are continuously proportional in the ratio of C to F

And since as C is to D so is F to G

alternately therefore, as C is to F, so is D to G [VII 13]

For the same reason also

as D is to G so is E to H

Therefore K M L are continuously proportional in the ratio of C to F in the ratio of D to G and also in the ratio of E to H

Next let E, H by multiplying M make N O respectively

Now, since A is a solid number and C, D E are its sides

*therefore* E by multiplying the product of C, D has made A

But the product of C D is K,

therefore E by multiplying K has made A

For the same reason also

H by multiplying L has made B

Now since E by multiplying K has made A and further also by multiplying M has made N,

therefore as K is to M, so is A to N [VII 17]

But, as K is to M so is C to F D to G and also E to H,

therefore also as C is to F, D to G, and E to H, so is A to N

Again, since E, H by multiplying M have made N O respectively

therefore as E is to H, so is N to O [VII 18]

But, as E is to H so is C to F and D to G,

therefore also, as C is to F D to G and E to H so is A to N and N to O

Again since H by multiplying M has made O and further also by multiplying L has made B,

therefore as M is to L so is O to B [VII 17]

But as M is to L *so is C to F D to G and E to H*

Therefore also as C is to F D to G and E to H so not only is O to B but also A to N and N to O

Therefore A, N O B are continuously proportional in the aforesaid ratios of the sides

I say that A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side that is of the ratio which the number C has to F or D to G and also E to H

For since A, N O B are four numbers in continued proportion

*therefore A has to B the ratio triplicate of that which A has to N* [V Def 10]

But as A is to N so it was proved that C is to F D to G and also E to H

Therefore A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side that is of the ratio which the number C has to F, D to G and also E to H

Q E D

PROPOSITION 20

*If one mean proportional number fall between two numbers the numbers will be similar plane numbers*

For let one mean proportional number $C$ fall between the two numbers $A$, $B$,

I say that $A$, $B$ are similar plane numbers

Let $D$ $E$, the least numbers of those which have the same ratio with $A$ $C$ be taken, [VII 33]

therefore $D$ measures $A$ the same number of times that $E$ measures $C$ [VII 20]

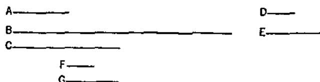
Now, as many times as $D$ measures $A$ so many units let there be in $F$,

therefore $F$ by multiplying $D$ has made $A$

so that $A$ is plane and $D$ $F$ are its sides

Again since $D$, $E$ are the least of the numbers which have the same ratio with $C$, $B$

therefore $D$ measures $C$ the same number of times that $E$ measures $B$ [VII 20]




As many times then as $E$ measures $B$ so many units let there be in $G$

therefore $E$ measures $B$ according to the units in $G$,

therefore $G$ by multiplying $E$ has made $B$

Therefore $B$ is plane and $E$ $G$ are its sides

Therefore $A$, $B$ are plane numbers

I say next that they are also similar

For since $F$ by multiplying $D$ has made $A$ and by multiplying $E$ has made $C$

therefore as $D$ is to $E$ so is $A$ to $C$ that is $C$ to $B$ [VII 17]

Again since $E$ by multiplying $F$ $G$ has made $C$, $B$ respectively,

therefore as $F$ is to $G$, so is $C$ to $B$ [VII 17]

But as $C$ is to $B$ so is $D$ to $E$

therefore also as $D$ is to $E$ so is $F$ to $G$

And alternately as $D$ is to $F$ so is $E$ to $G$ [VII 13]

Therefore $A$, $B$ are similar plane numbers for their sides are proportional

Q E D

PROPOSITION 21

*If two mean proportional numbers fall between two numbers the numbers are similar solid numbers*

For let two mean proportional numbers $C$ $D$ fall between the two numbers $A$, $B$,

I say that $A$, $B$ are similar solid numbers

For let three numbers $E$ $F$ $G$ the least of those which have the same ratio with $A$ $C$, $D$ be taken [VII 33 or VIII 2]

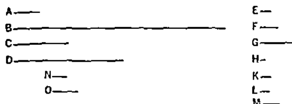
therefore the extremes of them $E$, $G$ are prime to one another [VIII 3]

Now, since one mean proportional number $F$ has fallen between $E$ $G$,

therefore $E$, $G$ are similar plane numbers [VIII 20]

Let then $H$, $K$ be the sides of $E$ and $L$ $M$ of $G$

Therefore it is manifest from the theorem before this that $E$ $F$, $G$ are continuously proportional in the ratio of $H$ to $L$ and that of $K$ to $M$




Now since $E$, $F$, $G$ are the least of the numbers which have the same ratio with $A$, $C$, $D$,

and the multitude of the numbers $E$, $F$ $G$ is equal to the multitude of the numbers $A$ $C$, $D$,

therefore *ex aequali* as $E$ is to $G$ so is $A$ to $D$ [VII 14]

But $E$, $G$ are prime

primes are also least [VII 21]

and the least measure those which have the same ratio with them the same number of times the greater the greater and the less the less that is, the antecedent the antecedent and the consequent the consequent [VII 20]

therefore $E$ measures $A$ the same number of times that $G$ measures $D$

Now, as many times as $E$ measures $A$, so many units let there be in $N$

Therefore $N$ by multiplying $E$ has made $A$

But $E$ is the product of $H$ $K$

therefore $N$ by multiplying the product of $H$ $K$ has made $A$

Therefore $A$ is solid and $H$ $K$ $N$ are its sides

Again since $E$ $F$ $G$ are the least of the numbers which have the same ratio as $C$, $D$ $B$

therefore $E$ measures $C$ the same number of times that $G$ measures $B$

Now as many times as $E$ measures $C$ so many units let there be in $O$

Therefore $G$ measures $B$ according to the units in $O$

therefore $O$ by multiplying $G$ has made $B$

But $G$ is the product of $L$ $M$

therefore $O$ by multiplying the product of $L$ $M$ has made $B$

Therefore $B$ is solid and $L$ $M$ $O$ are its sides

therefore $A$, $B$ are solid

I say that they are also similar

For since $N$ $O$ by multiplying $E$ have made $A$ $C$

therefore as $N$ is to $O$ so is $A$ to $C$ that is $E$ to $F$ [VII 18]

But as $E$ is to $F$ so is $H$ to $L$ and $K$ to $M$

therefore also as $H$ is to $L$ so is $K$ to $M$ and $N$ to $O$

And $H$ $K$ $N$ are the sides of $A$ and $O$ $L$ $M$ the sides of $B$

Therefore $A$ $B$ are similar solid numbers Q E D

### Proposition 22

*If three numbers be in continued proportion, and the first be square, the third will also be square*

Let $A$, $B$, $C$ be three numbers in continued proportion, and let $A$ the first be square

I say that $C$ the third is also square

For since between $A$, $C$ there is one mean proportional number, $B$

therefore $A$, $C$ are similar plane numbers [VIII 20]

But $A$ is square,

therefore $C$ is also square Q E D

### Proposition 23

*If four numbers be in continued proportion and the first be cube the fourth will also be cube*

Let $A$, $B$ $C$, $D$ be four numbers in continued proportion and let $A$ be cube,

I say that $D$ is also cube

For, since between $A$, $D$ there are two mean proportional numbers $B$ $C$

therefore $A$ $D$ are similar solid numbers [VIII 21]

But $A$ is cube

therefore $D$ is also cube Q E D

### Proposition 24

*If two numbers have to one another the ratio which a square number has to a square number and the first be square the second will also be square*

For let the two numbers $A$ $B$ have to one another the ratio which the square number $C$ has to the square number $D$ and let $A$ be square

I say that $B$ is also square

For since $C$ $D$ are square

$C$ $D$ are similar plane numbers

Therefore one mean proportional number falls between $C$, $D$ [VIII 18]

And as $C$ is to $D$, so is $A$ to $B$

therefore one mean proportional number falls between $A$, $B$ also [VIII 8]

And $A$ is square

therefore $B$ is also square [VIII 22]

Q E D

### Proposition 25

*If two numbers have to one another the ratio which a cube number has to a cube number and the first be cube the second will also be cube*

For let the two numbers $A$, $B$ have to one another the ratio which the cube number $C$ has to the cube number $D$ and let $A$ be cube

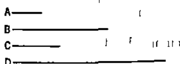
I say that $B$ is also cube

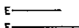
For since $C$ $D$ are cube

$C$ $D$ are similar solid numbers

Therefore two mean proportional numbers fall between $C$, $D$ [VIII 19]

And, as many numbers as fall between $C$ $D$ in continued proportion, so many will also fall between those which have the same ratio with them, [VIII 8] so that two mean proportional numbers fall between $A$, $B$ also




Let $E$, $F$ so fall

Since, then, the four numbers $A$, $E$ $F$, $B$ are in continued proportion and $A$ is cube therefore $B$ is also cube [VIII 23]

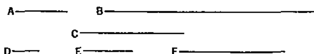
Q E D

## Proposition 26

*Similar plane numbers have to one another the ratio which a square number has to a square number*

Let $A$, $B$ be similar plane numbers,

I say that $A$ has to $B$ the ratio which a square number has to a square number




For since $A$, $B$ are similar plane numbers

therefore one mean proportional number falls between $A$, $B$ [VIII 18]

Let it so fall and let it be $C$,

and let $D$ $E$ $F$, the least numbers of those which have the same ratio with $A$, $C$, $B$, be taken [VII 33 or VIII 2]

therefore the extremes of them $D$ $F$ are square [VIII 2 Por ]

And since as $D$ is to $F$, so is $A$ to $B$,

and $D$ $F$ are square

therefore $A$ has to $B$ the ratio which a square number has to a square number

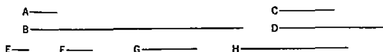
Q E D

## Proposition 27

*Similar solid numbers have to one another the ratio which a cube number has to a cube number*

Let $A$ $B$ be similar solid numbers

I say that $A$ has to $B$ the ratio which a cube number has to a cube number




For since $A$ $B$ are similar solid numbers

therefore two mean proportional numbers fall between $A$ $B$ [VIII 19]

Let $C$, $D$ so fall;

and let $E$, $F$, $G$, $H$ the least numbers of those which have the same ratio with $A$, $C$, $D$, $B$, and equal with them in multitude be taken, [VII. 33 or VIII. 2]

therefore the extremes of them $E$, $H$ are cube. [VIII. 2, Por.]

And as $E$ is to $H$, so is $A$ to $B$;

therefore $A$ also has to $B$ the ratio which a cube number has to a cube number.

Q. E. D.

# BOOK NINE

## Proposition 1

*If two similar plane numbers by multiplying one another make some number, the product will be square*

Let $A$, $B$ be two similar plane numbers and let $A$ by multiplying $B$ make $C$,

A
B
C
D

I say that $C$ is square

For let $A$ by multiplying itself make $D$

Therefore $D$ is square

Since then $A$ by multiplying itself has made $D$ and by multiplying $B$ has made $C$

therefore as $A$ is to $B$, so is $D$ to $C$ [VII 17]

And since $A$, $B$ are similar plane numbers

therefore one mean proportional number falls between $A$, $B$ [VIII 18]

But if numbers fall between two numbers in continued proportion as many as fall between them so many also fall between those which have the same ratio, [VIII 8]

so that one mean proportional number falls between $D$, $C$ also

And $D$ is square

therefore $C$ is also square [VIII 22]

Q E D

## Proposition 2

*If two numbers by multiplying one another make a square number they are similar plane numbers*

Let $A$, $B$ be two numbers and let $A$ by multiplying $B$ make the square number $C$

A
B
C
D

I say that $A$, $B$ are similar plane numbers

For let $A$ by multiplying itself make $D$,

therefore $D$ is square

Now since $A$ by multiplying itself has made $D$ and by multiplying $B$ has made $C$

therefore as $A$ is to $B$, so is $D$ to $C$ [VII 17]

And since $D$ is square and $C$ is so also

therefore $D$, $C$ are similar plane numbers

Therefore one mean proportional number falls between $D$, $C$ [VIII 18]

And as $D$ is to $C$ so is $A$ to $B$

therefore one mean proportional number falls between $A$, $B$ also [VIII 8]

But if one mean proportional number fall between two numbers they are similar plane numbers [VIII 20]

therefore $A$, $B$ are similar plane numbers Q E D

### Proposition 3

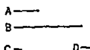
*If a cube number by multiplying itself make some number, the product will be cube*

For let the cube number $A$ by multiplying itself make $B$,

I say that $B$ is cube

For let $C$, the side of $A$, be taken and let $C$ by multiplying itself make $D$

It is then manifest that $C$ by multiplying $D$ has made $A$




Now since $C$ by multiplying itself has made $D$, therefore $C$ measures $D$ according to the units in itself

But further the unit also measures $C$ according to the units in it

therefore as the unit is to $C$ so is $C$ to $D$ [VII Def 20]

Again, since $C$ by multiplying $D$ has made $A$,

therefore $D$ measures $A$ according to the units in $C$

But the unit also measures $C$ according to the units in it,

therefore as the unit is to $C$ so is $D$ to $A$

But as the unit is to $C$ so is $C$ to $D$,

therefore also as the unit is to $C$ so is $C$ to $D$ and $D$ to $A$

Therefore between the unit and the number $A$ two mean proportional numbers $C$, $D$ have fallen in continued proportion

Again since $A$ by multiplying itself has made $B$

therefore $A$ measures $B$ according to the units in itself

But the unit also measures $A$ according to the units in it,

therefore as the unit is to $A$, so is $A$ to $B$ [VII Def 20]

But between the unit and $A$ two mean proportional numbers have fallen,

therefore two mean proportional numbers will also fall between $A$ $B$ [VIII 8]

But if two mean proportional numbers fall between two numbers and the first be cube the second will also be cube [VIII 23]

And $A$ is cube

therefore $B$ is also cube Q E D

### Proposition 4

*If a cube number by multiplying a cube number make some number, the product will be cube*

For let the cube number $A$ by multiplying the cube number $B$ make $C$,

I say that $C$ is cube

For let $A$ by multiplying itself make $D$,

therefore $D$ is cube [IX 3]

And since $A$ by multiplying itself has made $D$ and by multiplying $B$ has made $C$

therefore as $A$ is to $B$ so is $D$ to $C$ [VII 17]

And, since $A$ $B$ are cube numbers

$A$ $B$ are similar solid numbers

Therefore two mean proportional numbers fall between $A$ $B$ [VIII 19]

so that two mean proportional numbers will fall between $D$ $C$ also [VIII 8]

And $D$ is cube

therefore $C$ is also cube [VIII 23]

Q E D

PROPOSITION 5

*If a cube number by multiplying any number make a cube number, the multiplied number will also be cube*

For let the cube number $A$ by multiplying any number $B$ make the cube number $C$,

A———
B————
C————————
D—————

I say that $B$ is cube

For let $A$ by multiplying itself make $D$,

therefore $D$ is cube [IX 3]

Now since $A$ by multiplying itself has made $D$ and by multiplying $B$ has made $C$,

therefore as $A$ is to $B$ so is $D$ to $C$ [VII 17]

And since $D$, $C$ are cube,

they are similar solid numbers

Therefore two mean proportional numbers fall between $D$, $C$ [VIII 19]

And, as $D$ is to $C$, so is $A$ to $B$,

therefore two mean proportional numbers fall between $A$, $B$ also [VIII 8]

And $A$ is cube,

therefore $B$ is also cube [VIII 23]

PROPOSITION 6

*If a number by multiplying itself make a cube number, it will itself also be cube*

For let the number $A$ by multiplying itself make the cube number $B$,

I say that $A$ is also cube

A———
B————
C—————

For let $A$ by multiplying $B$ make $C$

Since then, $A$ by multiplying itself has made $B$, and by multiplying $B$ has made $C$

therefore $C$ is cube

And since $A$ by multiplying itself has made $B$,

therefore $A$ measures $B$ according to the units in itself

But the unit also measures $A$ according to the units in it

Therefore as the unit is to $A$, so is $A$ to $B$ [VII Def 20]

And since $A$ by multiplying $B$ has made $C$,

therefore $B$ measures $C$ according to the units in $A$

But the unit also measures $A$ according to the units in it

Therefore as the unit is to $A$, so is $B$ to $C$ [VII Def 20]

But as the unit is to $A$ so is $A$ to $B$,

therefore also as $A$ is to $B$, so is $B$ to $C$

And since $B$, $C$ are cube

they are similar solid numbers

Therefore there are two mean proportional numbers between $B$, $C$ [VIII 19]

And as $B$ is to $C$, so is $A$ to $B$

Therefore there are two mean proportional numbers between $A$, $B$ also [VIII 8

And $B$ is cube

therefore $A$ is also cube [cf VIII 23]

Q E D

## Proposition 7

*If a composite number by multiplying any number make some number, the product will be solid*

For, let the composite number $A$ by multiplying any number $B$ make $C$,

I say that $C$ is solid

For since $A$ is composite, it will be measured by some number [VII Def 13]

Let it be measured by $D$ and as many times as $D$ measures $A$, so many units let there be in $E$

Since then $D$ measures $A$ according to the units in $E$,

therefore $E$ by multiplying $D$ has made $A$ [VII Def 15]

And since $A$ by multiplying $B$ has made $C$,

and $A$ is the product of $D$ $E$,

therefore the product of $D$ $E$ by multiplying $B$ has made $C$

Therefore $C$ is solid and $D$, $E$, $B$ are its sides Q E D

## Proposition 8

*If as many numbers as we please beginning from an unit be in continued proportion the third from the unit will be square as will also those which successively leave out one the fourth will be cube as will also all those which leave out two and the seventh will be at once cube and square as will also those which leave out five*

Let there be as many numbers as we please, $A$ $B$, $C$ $D$, $E$, $F$, beginning from an unit and in continued proportion

I say that $B$ the third from the unit is square as are also all those which leave out one $C$ the fourth is cube as are also all those which leave out two and $F$, the seventh is at once cube and square as are also all those which leave out five

For since as the unit is to $A$, so is $A$ to $B$,

therefore the unit measures the number $A$ the same number of times that $A$ measures $B$ [VII Def 20]

But the unit measures the number $A$ according to the units in it,

therefore $A$ also measures $B$ according to the units in $A$

Therefore $A$ by multiplying itself has made $B$,

therefore $B$ is square

And since $B$ $C$ $D$ are in continued proportion and $B$ is square

therefore $D$ is also square [VIII 22]

For the same reason

$F$ is also square

Similarly we can prove that all those which leave out one are square

I say next that $C$ the fourth from the unit is cube as are also all those which leave out two

For since as the unit is to $A$ so is $B$ to $C$

therefore the unit measures the number $A$ the same number of times that $B$ measures $C$

But the unit measures the number $A$ according to the units in $A$,

therefore $B$ also measures $C$ according to the units in $A$

Therefore $A$ by multiplying $B$ has made $C$

Since then $A$ by multiplying itself has made $B$ and by multiplying $B$ has made $C$,

therefore $C$ is cube

And since $C$, $D$, $E$, $F$ are in continued proportion and $C$ is cube,

therefore $F$ is also cube [VIII 23]

But it was also proved square,

therefore the seventh from the unit is both cube and square

Similarly we can prove that all the numbers which leave out five are also both cube and square Q E D

## Proposition 9

*If as many numbers as we please beginning from an unit be in continued proportion and the number after the unit be square, all the rest will also be square And if the number after the unit be cube, all the rest will also be cube*

Let there be as many numbers as we please, $A$ $B$, $C$, $D$, $E$, $F$, beginning from an unit and in continued proportion and let $A$, the number after the unit be square,

A
B
C
D
E
F

I say that all the rest will also be square

Now it has been proved that $B$, the third from the unit is square as are also all those which leave out one [IX 8]

I say that all the rest are also square

For since $A$ $B$, $C$ are in continued proportion,

and $A$ is square

therefore $C$ is also square [VIII 22]

Again since $B$ $C$, $D$ are in continued proportion

and $B$ is square

$D$ is also square [VIII 22]

Similarly we can prove that all the rest are also square

Next, let $A$ be cube

I say that all the rest are also cube

Now it has been proved that $C$, the fourth from the unit is cube as also are all those which leave out two, [IX 8]

I say that all the rest are also cube

For, since as the unit is to $A$ so is $A$ to $B$,

therefore the unit measures $A$ the same number of times as $A$ measures $B$

But the unit measures $A$ according to the units in it

therefore $A$ also measures $B$ according to the units in itself,

therefore $A$ by multiplying itself has made $B$

And $A$ is cube

But if a cube number by multiplying itself make some number the product is cube [IX 3]

Therefore $B$ is also cube

And since the four numbers $A$ $B$ $C$ $D$ are in continued proportion

and $A$ is cube

$D$ also is cube [VIII 23]

For the same reason

$E$ is also cube and similarly all the rest are cube Q E D

## Proposition 10

*If as many numbers as we please beginning from an unit be in continued proportion and the number after the unit be not square neither will any other be square except the third from the unit and all those which leave out one And if the number after the unit be not cube neither will any other be cube except the fourth from the unit and all those which leave out two*

Let there be as many numbers as we please $A$ $B$ $C$, $D$ $E$ $F$, beginning from an unit and in continued proportion

and let $A$ the number after the unit not be square,

I say that neither will any other be square except the third from the unit <and those which leave out one>

For if possible let $C$ be square

But $B$ is also square [IX 8]

{therefore $B$, $C$ have to one another the ratio which a square number has to a square number}

And as $B$ is to $C$ so is $A$ to $B$

therefore $A$ $B$ have to one another the ratio which a square number has to a square number

{so that $A$ $B$ are similar plane numbers} [VIII 26 converse]

And $B$ is square,

therefore $A$ is also square

which is contrary to the hypothesis.

Therefore $C$ is not square

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one

Next let $A$ not be cube

I say that neither will any other be cube except the fourth from the unit and those which leave out two

For if possible let $D$ be cube

Now $C$ is also cube for it is fourth from the unit [IX 8]

And as $C$ is to $D$ so is $B$ to $C$

therefore $B$ also has to $C$ the ratio which a cube has to a cube

And $C$ is cube

therefore $B$ is also cube [VIII 25]

And since as the unit is to $A$ so is $A$ to $B$

and the unit measures $A$ according to the units in it

therefore $A$ also measures $B$ according to the units in itself

therefore $A$ by multiplying itself has made the cube number $B$

But if a number by multiplying itself make a cube number it is also itself cube [IX 6]

Therefore $A$ is also cube

which is contrary to the hypothesis

Therefore $D$ is not cube

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two Q E D

PROPOSITION 11

*If as many numbers as we please beginning from an unit be in continued proportion the less measures the greater according to some one of the numbers which have place among the proportional numbers*

Let there be as many numbers as we please, $B$ $C$ $D$, $E$ beginning from the unit $A$ and in continued proportion

I say that $B$, the least of the numbers $B$ $C$, $D$ $E$ measures $E$ according to some one of the numbers $C$ $D$

For since as the unit $A$ is to $B$, so is $D$ to $E$

therefore the unit $A$ measures the number $B$ the same number of times as $D$ measures $E$,

therefore alternately, the unit $A$ measures $D$ the same number of times as $B$ measures $E$ [VII 15]

But the unit $A$ measures $D$ according to the units in it,

therefore $B$ also measures $E$ according to the units in $D$

so that $B$ the less measures $E$ the greater according to some number of those which have place among the proportional numbers —

PORISM And it is manifest that whatever place the measuring number has reckoned from the unit the same place also has the number according to which it measures reckoned from the number measured in the direction of the number before it — Q E D

PROPOSITION 12

*If as many numbers as we please beginning from an unit be in continued proportion by however many prime numbers the last is measured the next to the unit will also be measured by the same*

Let there be as many numbers as we please $A$ $B$ $C$, $D$ beginning from an unit and in continued proportion,

I say that by however many prime numbers $D$ is measured $A$ will also be measured by the same

For let $D$ be measured by any prime number $E$

I say that $E$ measures $A$

For suppose it does not

now $E$ is prime and any prime number is prime to any which it does not measure [VII 29]

therefore $E$ $A$ are prime to one another

And since $E$ measures $D$ let it measure it according to $F$

therefore $E$ by multiplying $F$ has made $D$

Again since $A$ measures $D$ according to the units in $C$ [IX 11 and Por]

therefore $A$ by multiplying $C$ has made $D$

But further $E$ has also by multiplying $F$ made $D$

therefore the product of $A$ $C$ is equal to the product of $E$ $F$

Therefore as $A$ is to $E$ so is $F$ to $C$ [VII 19]

But $A$, $E$ are prime

primes are also least [VII 21]

and the least measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent [VII 20]

therefore $E$ measures $C$

Let it measure it according to $G$,

therefore $E$ by multiplying $G$ has made $C$

But, further, by the theorem before this

$A$ has also by multiplying $B$ made $C$ [IX 11 and Por]

Therefore the product of $A$, $B$ is equal to the product of $E$, $G$

Therefore as $A$ is to $E$, so is $G$ to $B$ [VII 19]

But $A$, $E$ are prime,

primes are also least, [VII 21]

and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent [VII 20]

therefore $E$ measures $B$

Let it measure it according to $H$,

therefore $E$ by multiplying $H$ has made $B$

But further, $A$ has also by multiplying itself made $B$, [IX 8]

therefore the product of $E$ $H$ is equal to the square on $A$

Therefore as $E$ is to $A$, so is $A$ to $H$ [VII 19]

But $A$ $E$ are prime,

primes are also least [VII 21]

and the least measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore $E$ measures $A$ as antecedent antecedent

But again it also does not measure it

which is impossible

Therefore $E$ $A$ are not prime to one another

Therefore they are composite to one another

But numbers composite to one another are measured by some number [VII Def 14]

And since $E$ is by hypothesis prime

and the prime is not measured by any number other than itself,

therefore $E$ measures $A$, $E$,

so that $E$ measures $A$

[But it also measures $D$

therefore $E$ measures $A$ $D$]

Similarly we can prove that by however many prime numbers $D$ is measured $A$ will also be measured by the same Q E D

### Proposition 13

*If as many numbers as we please beginning from an unit be in continued proportion and the number after the unit be prime the greatest will not be measured by any except those which have a place among the proportional numbers*

Let there be as many numbers as we please $A$ $B$ $C$ $D$ beginning from an unit and in continued proportion and let $A$ the number after the unit be prime

I say that $D$ the greatest of them will not be measured by any other number except $A$ $B$ $C$

For, if possible let it be measured by $E$ and let $E$ not be the same with any of the numbers $A$ $B$ $C$

It is then manifest that $E$ is not prime

For, if $E$ is prime and measures $D$

it will also measure $A$ [IX 12], which is prime, though it is not the same with it

which is impossible

Therefore $E$ is not prime

Therefore it is composite

But any composite number is measured by some prime number, [VII 31]

therefore $E$ is measured by some prime number

I say next that it will not be measured by any other prime except $A$

For, if $E$ is measured by another

and $E$ measures $D$

that other will also measure $D$

so that it will also measure $A$ [IX 12] which is prime though it is not the same with it

which is impossible

Therefore $A$ measures $E$

And since $E$ measures $D$, let it measure it according to $F$

I say that $F$ is not the same with any of the numbers $A$ $B$, $C$

For if $F$ is the same with one of the numbers $A$ $B$ $C$

and measures $D$ according to $E$

therefore one of the numbers $A$, $B$ $C$ also measures $D$ according to $E$

But one of the numbers $A$, $B$, $C$ measures $D$ according to some one of the numbers $A$ $B$, $C$, [IX 11]

therefore $E$ is also the same with one of the numbers $A$ $B$, $C$

which is contrary to the hypothesis

Therefore $F$ is not the same as any one of the numbers $A$ $B$, $C$

Similarly we can prove that $F$ is measured by $A$, by proving again that $F$ is not prime

For if it is and measures $D$

it will also measure $A$ [IX 12] which is prime though it is not the same with it

which is impossible

therefore $F$ is not prime

Therefore it is composite

But any composite number is measured by some prime number [VII 31]

therefore $F$ is measured by some prime number

I say next that it will not be measured by any other prime except $A$

For if any other prime number measures $F$

*and $F$ measures $D$*

that other will also measure $D$

so that it will also measure $A$ [IX 12] which is prime though it is not the same with it

which is impossible

Therefore $A$ measures $F$

And since $E$ measures $D$ according to $F$

therefore $E$ by multiplying $F$ has made $D$

But further $A$ has also by multiplying $C$ made $D$ [IX 11]

therefore the product of $A$ $C$ is equal to the product of $E$ $F$

Therefore proportionally, as $A$ is to $E$, so is $F$ to $C$ [VII 19]

But $A$ measures $E$,

therefore $F$ also measures $C$

Let it measure it according to $G$

Similarly then we can prove that $G$ is not the same with any of the numbers $A$ $B$, and that it is measured by $A$

And since $F$ measures $C$ according to $G$

therefore $F$ by multiplying $G$ has made $C$

But further $A$ has also by multiplying $B$ made $C$ [IX 11]

therefore the product of $A$ $B$ is equal to the product of $F$ $G$

Therefore proportionally, as $A$ is to $F$ so is $G$ to $B$ [VII 19]

But $A$ measures $F$

therefore $G$ also measures $B$

Let it measure it according to $H$

Similarly then we can prove that $H$ is not the same with $A$

And since $G$ measures $B$ according to $H$

therefore $G$ by multiplying $H$ has made $B$

But further $A$ has also by multiplying itself made $B$ [IX 8]

therefore the product of $H$ $G$ is equal to the square on $A$

Therefore as $H$ is to $A$, so is $A$ to $G$ [VII 19]

But $A$ measures $G$

therefore $H$ also measures $A$ which is prime though it is not the same with it which is absurd

Therefore $D$ the greatest will not be measured by any other number except $A$ $B$ $C$ Q E D

PROPOSITION 14

*If a number be the least that is measured by prime numbers it will not be measured by any other prime number except those originally measuring it*

For let the number $A$ be the least that is measured by the prime numbers $B$ $C$ $D$

I say that $A$ will not be measured by any other prime number except $B$, $C$ $D$

A———— B——
E———— C——
F———— D——

For if possible let it be measured by the prime number $E$ and let $E$ not be the same with any one of the numbers $B$ $C$ $D$

Now since $E$ measures $A$ let it measure it according to $F$,

therefore $E$ by multiplying $F$ has made $A$

And $A$ is measured by the prime numbers $B$ $C$ $D$

But if two numbers by multiplying one another make some number and any prime number measure the product it will also measure one of the original numbers [VII 30]

therefore $B$ $C$ $D$ will measure one of the numbers $E$ $F$

Now they will not measure $E$

for $E$ is prime and not the same with any one of the numbers $B$ $C$, $D$

Therefore they will measure $F$ which is less than $A$

which is impossible for $A$ is by hypothesis the least number measured by $B$ $C$ $D$

Therefore no prime number will measure $A$ except $B$ $C$ $D$ Q E D

PROPOSITION 15

*If three numbers in continued proportion be the least of those which have the same ratio with them any two whatever added together will be prime to the remaining number*

Let $A$, $B$, $C$, three numbers in continued proportion be the least of those which have the same ratio with them

A B C D E F

I say that any two of the numbers $A, B, C$ whatever added together are prime to the remaining number namely $A, B$ to $C$, $B, C$ to $A$, and further, $A$ $C$ to $B$

For let two numbers $DE$, $EF$, the least of those which have the same ratio with $A$, $B$ $C$, be taken [VIII 2]

It is then manifest that $DE$ by multiplying itself has made $A$, and by multiplying $EF$ has made $B$, and further $EF$ by multiplying itself has made $C$ [VIII 2]

Now since $DE$, $EF$ are least,

they are prime to one another [VII 22]

But, if two numbers be prime to one another

their sum is also prime to each [VII 28]

therefore $DF$ is also prime to each of the numbers $DE$, $EF$

But further, $DE$ is also prime to $EF$,

therefore $DF$, $DE$ are prime to $EF$

But, if two numbers be prime to any number

their product is also prime to the other [VII 24]

so that the product of $FD$ $DE$ is prime to $EF$

hence the product of $FD$, $DE$ is also prime to the square on $EF$ [VII 25]

But the product of $FD$ $DE$ is the square on $DE$ together with the product of $DE$, $EF$ [II 3]

therefore the square on $DE$ together with the product of $DE$ $EF$ is prime to the square on $EF$

And the square on $DE$ is $A$

the product of $DE$ $EF$ is $B$,

and the square on $EF$ is $C$,

therefore $A$, $B$ added together are prime to $C$

Similarly we can prove that $B$, $C$ added together are prime to $A$

I say next that $A$ $C$ added together are also prime to $B$

For since $DF$ is prime to each of the numbers $DE$ $EF$

the square on $DF$ is also prime to the product of $DE$ $EF$ [VII 24 25]

But the squares on $DE$ $EF$ together with twice the product of $DE$ $EF$ are equal to the square on $DF$ [II 4]

therefore the squares on $DE$ $EF$ together with twice the product of $DE$ $EF$ are prime to the product of $DE$ $EF$

*Separando* the squares on $DE$ $EF$ together with once the product of $DE$ $EF$ are prime to the product of $DE$ $EF$

Therefore *separando* again the squares on $DE$, $EF$ are prime to the product of $DE$ $EF$

And the square on $DE$ is $A$

the product of $DE$ $EF$ is $B$

and the square on $EF$ is $C$

Therefore *A*, *C* added together are prime to *B* Q E D

## Proposition 16

*If two numbers be prime to one another, the second will not be to any other number as the first is to the second*

For let the two numbers *A*, *B* be prime to one another

I say that *B* is not to any other number as *A* is to *B*

For if possible as *A* is to *B*, so let *B* be to *C*

Now *A*, *B* are prime

primes are also least [VII 21]

and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, [VII 20]

therefore *A* measures *B* as antecedent antecedent

But it also measures itself

therefore *A* measures *A* *B* which are prime to one another which is absurd

Therefore *B* will not be to *C*, as *A* is to *B* Q E D

## Proposition 17

*If there be as many numbers as we please in continued proportion and the extremes of them be prime to one another, the last will not be to any other number as the first to the second*

For let there be as many numbers as we please *A* *B*, *C*, *D* in continued proportion

and let the extremes of them, *A* *D*, be prime to one another

I say that *D* is not to any other number as *A* is to *B*

For if possible as *A* is to *B* so let *D* be to *E*

therefore alternately as *A* is to *D* so is *B* to *E* [VII 13]

But *A* *D* are prime

primes are also least [VII 21]

and the least numbers measure those which have the same ratio the same number of times the antecedent the antecedent and the consequent the consequent [VII 20]

Therefore *A* measures *B*

And as *A* is to *B* so is *B* to *C*

Therefore *B* also measures *C*

so that *A* also measures *C*

And since as *B* is to *C* so is *C* to *D*

and *B* measures *C*

therefore *C* also measures *D*

But *A* measured *C*

so that *A* also measures *D*

But it also measures it elf,

therefore *A* measures *A* *D* which are prime to one another which is impossible

Therefore *D* will not be to any other number as *A* is to *B* Q E D

## Proposition 18

*Given two numbers, to investigate whether it is possible to find a third proportional to them*

Let $A$, $B$ be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them

Now $A$, $B$ are either prime to one another or not

And if they are prime to one another, it has been proved that it is impossible to find a third proportional to them [IX 16]

Next let $A$, $B$ not be prime to one another,

and let $B$ by multiplying itself make $C$

Then $A$ either measures $C$ or does not measure it

First let it measure it according to $D$,

therefore $A$ by multiplying $D$ has made $C$

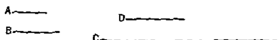
But further, $B$ has also by multiplying itself made $C$

therefore the product of $A$, $D$ is equal to the square on $B$

Therefore as $A$ is to $B$ so is $B$ to $D$ [VII 19]

therefore a third proportional number $D$ has been found to $A$, $B$

Next let $A$ not measure $C$,




I say that it is impossible to find a third proportional number to $A$, $B$

For if possible let $D$, such third proportional have been found

Therefore the product of $A$, $D$ is equal to the square on $B$

But the square on $B$ is $C$

therefore the product of $A$, $D$ is equal to $C$

Hence $A$ by multiplying $D$ has made $C$

therefore $A$ measures $C$ according to $D$

But by hypothesis it also does not measure it

which is absurd

Therefore it is not possible to find a third proportional number to $A$, $B$ when $A$ does not measure $C$ Q E D

## Proposition 19

*Given three numbers to investigate when it is possible to find a fourth proportional to them*

Let $A$, $B$, $C$ be the given three numbers and let it be required to investigate when it is possible to find a fourth proportional to them

[The Greek text of this proposition is corrupt However analagously to Proposition 18 the condition that a fourth proportional to A B C exists is that A measure the product of B and C]

## Proposition 20

*Prime numbers are more than any assigned multitude of prime numbers*

Let $A$, $B$, $C$ be the assigned prime numbers

I say that there are more prime numbers than $A$, $B$, $C$

For let the least number measured by $A$ $B$ $C$ be taken,
and let it be $DE$
let the unit $DF$ be added to $DE$

Then $EF$ is either prime or not

First let it be prime

then the prime numbers $A$ $B$, $C$ $EF$ have been found which are more than $A$ $B$ $C$

Next, let $EF$ not be prime

therefore it is measured by some prime number [VII 31]

Let it be measured by the prime number $G$

I say that $G$ is not the same with any of the numbers $A$, $B$, $C$

For if possible let it be so

Now $A$ $B$ $C$ measure $DE$

therefore $G$ also will measure $DE$

But it also measures $EF$

Therefore $G$ being a number will measure the remainder the unit $DF$ which is absurd

Therefore $G$ is not the same with any one of the numbers $A$ $B$, $C$

And by hypothesis it is prime

Therefore the prime numbers $A$ $B$ $C$ $G$ have been found which are more than the assigned multitude of $A$ $B$ $C$ Q E D

## Proposition 21

*If as many even numbers as we please be added together the whole is even*

For let as many even numbers as we please $AB$, $BC$ $CD$ $DE$ be added together

I say that the whole $AE$ is even

For since each of the numbers $AB$ $BC$ $CD$ $DE$ is even it has a half part [VII Def 6]

so that the whole $AE$ also has a half part

But an even number is that which is divisible into two equal parts [id]

therefore $AE$ is even Q E D

## Proposition 22

*If as many odd numbers as we please be added together and their multitude be even, the whole will be even*

For let as many odd numbers as we please $AB$ $BC$ $CD$ $DE$ even in multitude be added together,

I say that the whole $AE$ is even

For since each of the numbers $AB$ $BC$ $CD$ $DE$ is odd if an unit be subtracted from each each of the remainders will be even [VII Def 7]

so that the sum of them will be even [IX 21]

*But the multitude of the units is also even*

Therefore the whole $AE$ is also even [IX 21]

Q E D

## Proposition 23

*If as many odd numbers as we please be added together, and their multitude be odd the whole will also be odd*

For let as many odd numbers as we please, $AB$, $BC$, $CD$, the multitude of which is odd be added together,

A B C E D

I say that the whole $AD$ is also odd

Let the unit $DE$ be subtracted from $CD$,

therefore the remainder $CE$ is even [VII Def 7]

But $CA$ is also even, [IX 22]

therefore the whole $AE$ is also even [IX 21]

And $DE$ is an unit

Therefore $AD$ is odd [VII Def 7]

Q E D

## Proposition 24

*If from an even number an even number be subtracted, the remainder will be even*

For from the even number $AB$ let the even number $BC$ be subtracted

A C B

I say that the remainder $CA$ is even

For since $AB$ is even it has a half part [VII Def 6]

For the same reason $BC$ also has a half part

so that the remainder [$CA$ also has a half part and] $AC$ is therefore even

Q E D

## Proposition 25

*If from an even number an odd number be subtracted, the remainder will be odd*

For from the even number $AB$ let the odd number $BC$ be subtracted

A C D B

I say that the remainder $CA$ is odd

For let the unit $CD$ be subtracted from $BC$,

therefore $DB$ is even [VII Def 7]

But $AB$ is also even,

therefore the remainder $AD$ is also even [IX 24]

And $CD$ is an unit

therefore $CA$ is odd [VII Def 7]

Q E D

## Proposition 26

*If from an odd number an odd number be subtracted the remainder will be even*

For from the odd number $AB$ let the odd number $BC$ be subtracted

A C D B

I say that the remainder $CA$ is even

For since $AB$ is odd let the unit $BD$ be subtracted

therefore the remainder $AD$ is even [VII Def 7]

For the same reason $CD$ is also even [VII Def 7]

so that the remainder $CA$ is also even [IX 24]

Q E D

## Proposition 27

*If from an odd number an even number be subtracted the remainder will be odd*

For from the odd number $AB$ let the even number $BC$ be subtracted

I say that the remainder $CA$ is odd

Let the unit $AD$ be subtracted,

therefore $DB$ is even [VII Def 7]

A D C B

But $BC$ is also even,

therefore the remainder $CD$ is even [IX 24]

Therefore $CA$ is odd [VII Def 7]

Q E D

## Proposition 28

*If an odd number by multiplying an even number make some number, the product will be even*

For let the odd number $A$ by multiplying the even number $B$ make $C$,

I say that $C$ is even

A

B

C

For since $A$ by multiplying $B$ has made $C$, therefore $C$ is made up of as many numbers equal to $B$ as there are units in $A$ [VII Def 15]

And $B$ is even

therefore $C$ is made up of even numbers

But if as many even numbers as we please be added together, the whole is even [IX 21]

Therefore $C$ is even Q E D

## Proposition 29

*If an odd number by multiplying an odd number make some number, the product will be odd*

For let the odd number $A$ by multiplying the odd number $B$ make $C$,

I say that $C$ is odd

A

B

C

For since $A$ by multiplying $B$ has made $C$ therefore $C$ is made up of as many numbers equal to $B$ as there are units in $A$ [VII Def 15]

And each of the numbers $A$ $B$ is odd

therefore $C$ is made up of odd numbers the multitude of which is odd

Thus $C$ is odd [IX 23]

Q E D

## Proposition 30

*If an odd number measure an even number, it will also measure the half of it*

For let the odd number $A$ measure the even number $B$,

I say that it will also measure the half of it

A

B

C

For since $A$ measures $B$

let it measure it according to $C$

I say that $C$ is not odd

For if possible let it be so

Then since $A$ measures $B$ according to $C$

therefore $A$ by multiplying $C$ has made $B$

Therefore $B$ is made up of odd numbers the multitude of which is odd

Therefore $B$ is odd [IX 23]

which is absurd for by hypothesis it is even

Therefore $C$ is not odd

therefore $C$ is even

Thus $A$ measures $B$ an even number of times

For this reason then it also measures the half of it Q E D

PROPOSITION 31

*If an odd number be prime to any number, it will also be prime to the double of it*

For let the odd number $A$ be prime to any number $B$,

and let $C$ be double of $B$,

A, B, C, D

I say that $A$ is prime to $C$

For if they are not prime to one another some number will measure them

Let a number measure them and let it be $D$

Now $A$ is odd,

therefore $D$ is also odd

And since $D$ which is odd measures $C$

and $C$ is even

therefore [$D$] will measure the half of $C$ also [IX 30]

But $B$ is half of $C$,

therefore $D$ measures $B$

But it also measures $A$

therefore $D$ measures $A$, $B$ which are prime to one another

which is impossible

Therefore $A$ cannot but be prime to $C$

Therefore $A$, $C$ are prime to one another Q E D

PROPOSITION 32

*Each of the numbers which are continually doubled beginning from a dyad is even times even only*

For let as many numbers as we please $B$, $C$, $D$ have been continually doubled beginning from the dyad $A$,

A, B, C, D

I say that $B$, $C$, $D$ are even times even only

Now that each of the numbers $B$, $C$, $D$ is even times even is manifest for it is doubled from a dyad

I say that it is also even times even only

For let an unit be set out

Since then as many numbers as we please beginning from an unit are in continued proportion

and the number $A$ after the unit is prime

therefore $D$ the greatest of the numbers $A$, $B$, $C$, $D$ will not be measured by any other number except $A$, $B$, $C$ [IX 13]

And each of the numbers $A$, $B$, $C$ is even

therefore $D$ is even times even only [VII Def 8]

Similarly we can prove that each of the numbers $B$, $C$ is even times even only Q E D

PROPOSITION 33

*If a number have its half odd it is even times odd only*

For let the number $A$ have its half odd

I say that $A$ is even times odd only

Now that it is even times odd is manifest for the half of it being odd measures it an even number of times [VII Def 9]

I say next that it is also even times odd only

For if $A$ is even times even also

it will be measured by an even number according to an even number [VII Def 8]

so that the half of it will also be measured by an even number though it is odd which is absurd

Therefore $A$ is even times odd only Q E D

PROPOSITION 34

*If a number neither be one of those which are continually doubled from a dyad, nor have its half odd it is both even-times even and even-times odd*

For let the number $A$ neither be one of those doubled from a dyad nor have its half odd

I say that $A$ is both even times even and even times odd

Now that $A$ is even times even is manifest

for it has not its half odd [VII Def 8]

I say next that it is also even times odd

For if we bisect $A$ then bisect its half and do this continually we shall come upon some odd number which will measure $A$ according to an even number

For if not we shall come upon a dyad

and $A$ will be among those which are doubled from a dyad

which is contrary to the hypothesis

Thus $A$ is even times odd

But it was also proved even times even

Therefore $A$ is both even times even and even times odd Q E D

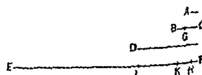
PROPOSITION 35

*If as many numbers as we please be in continued proportion and there be subtracted from the second and the last numbers equal to the first then as the excess of the second is to the first so will the excess of the last be to all those before it*

Let there be as many numbers as we please in continued proportion $A$, $BC$ $D$ $EF$ beginning from $A$ as least

and let there be subtracted from $BC$ and $EF$ the numbers $BG$ $FH$ each equal to $A$

I say that as $GC$ is to $A$ so is $EH$ to $A$ $BC$ $D$




For let $FK$ be made equal to $BC$ and $FL$ equal to $D$

Then since $FK$ is equal to $BC$

and of these the part $FH$ is equal to the part $BG$

therefore the remainder $HK$ is equal to the remainder $GC$

And since as $EF$ is to $D$ so is $D$ to $BC$ and $BC$ to $A$

while $D$ is equal to $FL$ $BC$ to $FK$ and $A$ to $FH$

therefore as $EF$ is to $FL$ so is $LF$ to $FK$ and $FK$ to $FH$

*Separando* as $EL$ is to $LF$ so is $LK$ to $FK$, and $KH$ to $FH$ [VII 11 13]

Therefore also as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [VII 12]

therefore as $KH$ is to $FH$ so are $EL$, $LK$, $KH$ to $LF$, $FK$, $HF$

But $KH$ is equal to $CG$, $FH$ to $A$ and $LF$, $FK$, $HF$ to $D$, $BC$, $A$,

therefore as $CG$ is to $A$, so is $EH$ to $D$, $BC$, $A$

Therefore as the excess of the second is to the first so is the excess of the last to all those before it Q E D

PROPOSITION 36

*If as many numbers as we please beginning from an unit be set out continuously in double proportion until the sum of all becomes prime and if the sum multiplied into the last make some number the product will be perfect*

For let as many numbers as we please $A$, $B$, $C$ $D$ beginning from an unit be set out in double proportion until the sum of all becomes prime

let $E$ be equal to the sum and let $E$ by multiplying $D$ make $FG$,

I say that $FG$ is perfect

For however many $A$ $B$ $C$ $D$ are in multitude let so many $E$, $HK$ $L$ $M$ be taken in double proportion beginning from $E$,

therefore *ex aequali* as $A$ is to $D$ so is $E$ to $M$ [VII 14]

Therefore the product of $E$ $D$ is equal to the product of $A$ $M$ [VII 19]

And the product of $E$ $D$ is $FG$

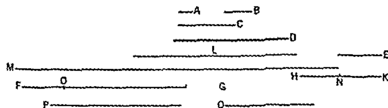
therefore the product of $A$ $M$ is also $FG$

Therefore $A$ by multiplying $M$ has made $FG$

therefore $M$ measures $FG$ according to the units in $A$

And $A$ is a dyad

therefore $FG$ is double of $M$




But $M$ $L$ $HK$ $E$ are continuously double of each other

therefore $E$ $HK$ $L$ $M$, $FG$ are continuously proportional in double proportion

Now let there be subtracted from the second $HK$ and the last $FG$ the numbers $HN$ $FO$ each equal to the first $E$

therefore as the excess of the second is to the first so is the excess of the last to all those before it [IX 35]

Therefore as $NK$ is to $E$ so is $OG$ to $M$ $L$ $KH$ $E$

And $NK$ is equal to $E$

therefore $OG$ is also equal to $M$ $L$ $HK$ $E$

But $FO$ is also equal to $E$

and $E$ is equal to $A$ $B$ $C$ $D$ and the unit

Therefore the whole $FG$ is equal to $E$ $HK$ $L$ $M$ and $A$, $B$ $C$ $D$ and the unit

and it is measured by them

I say also that $FG$ will not be measured by any other number except $A$, $B$, $C$, $D$, $E$, $HK$, $L$, $M$ and the unit.

For, if possible, let some number $P$ measure $FG$,
and let $P$ not be the same with any of the numbers $A$, $B$, $C$, $D$, $E$, $HK$, $L$, $M$.

And, as many times as $P$ measures $FG$, so many units let there be in $Q$;
therefore $Q$ by multiplying $P$ has made $FG$.

But, further, $E$ has also by multiplying $D$ made $FG$;
therefore, as $E$ is to $Q$, so is $P$ to $D$. [VII. 19]

And, since $A$, $B$, $C$, $D$ are continuously proportional beginning from an unit,
therefore $D$ will not be measured by any other number except $A$, $B$, $C$. [IX. 13]

And, by hypothesis, $P$ is not the same with any of the numbers $A$, $B$, $C$;
therefore $P$ will not measure $D$.

But, as $P$ is to $D$, so is $E$ to $Q$;
therefore neither does $E$ measure $Q$. [VII. Def. 20]

And $E$ is prime;
and any prime number is prime to any number which it does not measure. [VII. 29]

Therefore $E$, $Q$ are prime to one another.

But primes are also least, [VII. 21]
and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]

and, as $E$ is to $Q$, so is $P$ to $D$;
therefore $E$ measures $P$ the same number of times that $Q$ measures $D$.

But $D$ is not measured by any other number except $A$, $B$, $C$;
therefore $Q$ is the same with one of the numbers $A$, $B$, $C$.

Let it be the same with $B$.

And, however many $B$, $C$, $D$ are in multitude, let so many $E$, $HK$, $L$ be taken beginning from $E$.

Now $E$, $HK$, $L$ are in the same ratio with $B$, $C$, $D$;
therefore, *ex aequali*, as $B$ is to $D$, so is $E$ to $L$. [VII. 14]

Therefore the product of $B$, $L$ is equal to the product of $D$, $E$. [VII. 19]

But the product of $D$, $E$ is equal to the product of $Q$, $P$;
therefore the product of $Q$, $P$ is also equal to the product of $B$, $L$.

Therefore, as $Q$ is to $B$, so is $L$ to $P$. [VII. 19]

And $Q$ is the same with $B$;
therefore $L$ is also the same with $P$:
which is impossible, for by hypothesis $P$ is not the same with any of the numbers set out.

Therefore no number will measure $FG$ except $A$, $B$, $C$, $D$, $E$, $HK$, $L$, $M$ and the unit.

And $FG$ was proved equal to $A$, $B$, $C$, $D$, $E$, $HK$, $L$, $M$ and the unit;
and a perfect number is that which is equal to its own parts; [VII. Def. 22]
therefore $FG$ is perfect.

Q. E. D.

# BOOK TEN

## DEFINITIONS I

1 Those magnitudes are said to be *commensurable* which are measured by the same measure and those *incommensurable* which cannot have any common measure

2 Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure

3 With these hypotheses it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively some in length only and others in square also with an assigned straight line Let then the assigned straight line be called *rational* and those straight lines which are commensurable with it whether in length and in square or in square only, *rational* but those which are incommensurable with it *irrational*

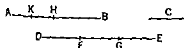
4 And let the square on the assigned straight line be called *rational* and those areas which are commensurable with it *rational* but those which are incommensurable with it *irrational* and the straight lines which produce them *irrational* that is in case the areas are squares the sides themselves but in case they are any other rectilineal figures the straight lines on which are described squares equal to them

## BOOK X PROPOSITIONS

### PROPOSITION 1

*Two unequal magnitudes being set out if from the greater there be subtracted a magnitude greater than its half and from that which is left a magnitude greater than its half and if this process be repeated continually there will be left some magnitude which will be less than the lesser magnitude set out*

Let $AB$ $C$ be two unequal magnitudes of which $AB$ is the greater




I say that if from $AB$ there be subtracted a magnitude greater than its half and from that which is left a magnitude greater than its half and if this process be repeated continually there will be left some magnitude which will be less than the magnitude $C$

For $C$ if multiplied will sometime be greater than $AB$ [cf v Def 4]

Let it be multiplied and let $DE$ be a multiple of $C$ and greater than $AB$

let $DE$ be divided into the parts $DF$ $FG$ $GE$ equal to $C$

from $AB$ let there be subtracted $BH$ greater than its half

and from $AH$ $HK$ greater than its half

and let this process be repeated continually until the divisions in $AB$ are equal in multitude with the divisions in $DE$

Let then $AK$, $KH$, $HB$ be divisions which are equal in multitude with $DF$, $FG$, $GE$

Now since $DE$ is greater than $AB$

and from $DE$ there has been subtracted $EG$ less than its half,

and from $AB$, $BH$ greater than its half

therefore the remainder $GD$ is greater than the remainder $HA$

And since $GD$ is greater than $HA$,

and there has been subtracted, from $GD$, the half $GF$,

and from $HA$, $HK$ greater than its half

therefore the remainder $DF$ is greater than the remainder $AK$

But $DF$ is equal to $C$

therefore $C$ is also greater than $AK$

Therefore $AK$ is less than $C$

Therefore there is left of the magnitude $AB$ the magnitude $AK$ which is less than the lesser magnitude set out, namely $C$ Q E D

And the theorem can be similarly proved even if the parts subtracted be halves

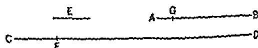
## Proposition 2

*If when the less of two unequal magnitudes is continually subtracted in turn from the greater that which is left never measures the one before it the magnitudes will be incommensurable*

For there being two unequal magnitudes $AB$, $CD$, and $AB$ being the less, when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it

I say that the magnitudes $AB$, $CD$ are incommensurable

For if they are commensurable some magnitude will measure them




Let a magnitude measure them if possible and let it be $E$,

let $AB$ measuring $FD$ leave $CF$ less than itself,

let $CF$ measuring $BG$ leave $AG$ less than itself

and let this process be repeated continually until there is left some magnitude which is less than $E$

Suppose this done and let there be left $AG$ less than $E$

Then since $E$ measures $AB$

while $AB$ measures $DF$

therefore $E$ will also measure $FD$

But it measures the whole $CD$ also

therefore it will also measure the remainder $CF$

But $CF$ measures $BG$

therefore $E$ also measures $BG$

But it measures the whole $AB$ also

therefore it will also measure the remainder $AG$, the greater the less which is impossible

Therefore no magnitude will measure the magnitudes $AB$ $CD$,
therefore the magnitudes $AB$, $CD$ are incommensurable [x Def 1]
Therefore etc Q E D

PROPOSITION 3

*Given two commensurable magnitudes, to find their greatest common measure*

Let the two given commensurable magnitudes be $AB$, $CD$ of which $AB$ is the less

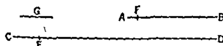
thus it is required to find the greatest common measure of $AB$, $CD$

Now the magnitude $AB$ either measures $CD$ or it does not

If then it measures it—and it measures itself also—$AB$ is a common measure of $AB$, $CD$

And it is manifest that it is also the greatest,
for a greater magnitude than the magnitude $AB$ will not measure $AB$

Next, let $AB$ not measure $CD$




Then if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because $AB$, $CD$ are not incommensurable, [cf x 2]

let $AB$, measuring $ED$ leave $EC$ less than itself
let $EC$ measuring $FB$ leave $AF$ less than itself
and let $AF$ measure $CE$

Since, then $AF$ measures $CE$,
while $CE$ measures $FB$,
therefore $AF$ will also measure $FB$

But it measures itself also,
therefore $AF$ will also measure the whole $AB$

But $AB$ measures $DE$
therefore $AF$ will also measure $ED$

But it measures $CE$ also,
therefore it also measures the whole $CD$

Therefore $AF$ is a common measure of $AB$ $CD$

I say next that it is also the greatest

For if not there will be some magnitude greater than $AF$ which will measure $AB$ $CD$

Let it be $G$

Since then $G$ measures $AB$
while $AB$ measures $ED$
therefore $G$ will also measure $ED$

But it measures the whole $CD$ also
therefore $G$ will also measure the remainder $CE$

But $CE$ measures $FB$
therefore $G$ will also measure $FB$

But it measures the whole $AB$ also
and it will therefore measure the remainder $AF$ the greater the less
which is impossible

Therefore no magnitude greater than $AF$ will measure $AB$ $CD$
therefore $AF$ is the greatest common measure of $AB$ $CD$

Therefore the greatest common measure of the two given commensurable magnitudes $AB$, $CD$ has been found Q E D

PORISM From this it is manifest that if a magnitude measure two magnitudes it will also measure their greatest common measure

PROPOSITION 4

*Given three commensurable magnitudes, to find their greatest common measure*

Let $A$ $B$ $C$ be the three given commensurable magnitudes,
thus it is required to find the greatest common measure of $A$, $B$, $C$

Let the greatest common measure of the two magnitudes $A$ $B$ be taken and let it be $D$ [x 3]
then $D$ either measures $C$ or does not measure it

First let it measure it

A
B
C
D E F

Since then $D$ measures $C$
while it also measures $A$ $B$
therefore $D$ is a common measure of $A$ $B$ $C$

And it is manifest that it is also the greatest
for a greater magnitude than the magnitude $D$ does not measure $A$ $B$

Next let $D$ not measure $C$

I say first that $C$ $D$ are commensurable

For since $A$ $B$ $C$ are commensurable
some magnitude will measure them
and this will of course measure $A$ $B$ also
so that it will also measure the greatest common measure of $A$ $B$ namely $D$ [x 3 Por]

But it also measures $C$
so that the said magnitude will measure $C$ $D$
therefore $C$ $D$ are commensurable

Now let their greatest common measure be taken and let it be $E$ [x 3]

Since then $E$ measures $D$
while $D$ measures $A$ $B$
therefore $E$ will also measure $A$ $B$

But it measures $C$ also
therefore $E$ measures $A$ $B$ $C$
therefore $E$ is a common measure of $A$, $B$ $C$

I say next that it is also the greatest

For if possible let there be some magnitude $F$ greater than $E$ and let it measure $A$ $B$ $C$

Now since $F$ measures $A$ $B$ $C$
it will also measure $A$ $B$
and will measure the greatest common measure of $A$, $B$ [x 3 Por]

But the greatest common measure of $A$ $B$ is $D$
therefore $F$ measures $D$

But it measures $C$ also
therefore $F$ measures $C$ $D$
therefore $F$ will also measure the greatest common measure of $C$ $D$ [x 3 Por]

But that is $E$,

therefore $F$ will measure $E$, the greater the less which is impossible

Therefore no magnitude greater than the magnitude $E$ will measure $A$, $B$ $C$, therefore $E$ is the greatest common measure of $A$ $B$ $C$ if $D$ do not measure $C$, and if it measure it $D$ is itself the greatest common measure

Therefore the greatest common measure of the three given commensurable magnitudes has been found

PORISM From this it is manifest that if a magnitude measure three magnitudes it will also measure their greatest common measure

Similarly too, with more magnitudes the greatest common measure can be found and the porism can be extended Q E D

PROPOSITION 5

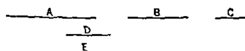
*Commensurable magnitudes have to one another the ratio which a number has to a number*

Let $A$, $B$ be commensurable magnitudes,

I say that $A$ has to $B$ the ratio which a number has to a number

For, since $A$, $B$ are commensurable some magnitude will measure them

Let it measure them and let it be $C$




And as many times as $C$ measures $A$, so many units let there be in $D$,

and as many times as $C$ measures $B$, so many units let there be in $E$

Since then $C$ measures $A$ according to the units in $D$

while the unit also measures $D$ according to the units in it

therefore the unit measures the number $D$ the same number of times as the magnitude $C$ measures $A$

therefore as $C$, is to $A$ so is the unit to $D$, [VII Def 20]

therefore, inversely as $A$ is to $C$, so is $D$ to the unit [cf V 7 Por]

Again since $C$ measures $B$ according to the units in $E$

while the unit also measures $E$ according to the units in it

therefore the unit measures $E$ the same number of times as $C$ measures $B$,

therefore as $C$ is to $B$ so is the unit to $E$

But it was also proved that

as $A$ is to $C$, so is $D$ to the unit

therefore *ex aequali*

as $A$ is to $B$ so is the number $D$ to $E$ [V 22]

Therefore the commensurable magnitudes $A$ $B$ have to one another the ratio which the number $D$ has to the number $E$ Q E D

PROPOSITION 6

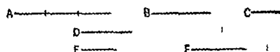
*If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable*

For let the two magnitudes $A$ $B$ have to one another the ratio which the number $D$ has to the number $E$

I say that the magnitudes $A$ $B$ are commensurable

For let $A$ be divided into as many equal parts as there are units in $D$ and let $C$ be equal to one of them,

and let $F$ be made up of as many magnitudes equal to $C$ as there are units in $E$




Since then there are in $A$ as many magnitudes equal to $C$ as there are units in $D$

whatever part the unit is of $D$, the same part is $C$ of $A$ also

therefore as $C$ is to $A$ so is the unit to $D$ [VII Def 20]

But the unit measures the number $D$,

therefore $C$ also measures $A$

And since as $C$ is to $A$, so is the unit to $D$

therefore inversely as $A$ is to $C$, so is the number $D$ to the unit [cf V 7 Por]

Again since there are in $F$ as many magnitudes equal to $C$ as there are units in $E$

therefore as $C$ is to $F$ so is the unit to $E$ [VII Def 20]

But it was also proved that

as $A$ is to $C$ so is $D$ to the unit

therefore *ex aequali* as $A$ is to $F$ so is $D$ to $E$ [V 22]

But as $D$ is to $E$ so is $A$ to $B$

therefore also as $A$ is to $B$ so is it to $F$ also [V 11]

*Therefore $A$ has the same ratio to each of the magnitudes $B$, $F$,*

therefore $B$ is equal to $F$ [V 9]

But $C$ measures $F$

therefore it measures $B$ also

Further it measures $A$ also

therefore $C$ measures $A$ $B$

Therefore $A$ is commensurable with $B$

Therefore etc

PORISM From this it is manifest that if there be two numbers as $D$ $E$ and a straight line as $A$ it is possible to make a straight line $[F]$ such that the given straight line is to it as the number $D$ is to the number $E$

And if a mean proportional be also taken between $A$ $F$ as $B$,

as $A$ is to $F$ so will the square on $A$ be to the square on $B$ that is as the first is to the third so is the figure on the first to that which is similar and similarly described on the second [VI 19 Por]

But as $A$ is to $F$ so is the number $D$ to the number $E$

therefore it has been contrived that as the number $D$ is to the number $E$ so also is the figure on the straight line $A$ to the figure on the straight line $B$

Q E D

PROPOSITION 7

*Incommensurable magnitudes have not to one another the ratio which a number has to a number*

Let $A$, $B$ be incommensurable magnitudes

I say that $A$ has not to $B$ the ratio which a number has to a number

For, if $A$ has to $B$ the ratio which a number has to a number, $A$ will be com

mensurable with $B$ [x 6]

But it is not,

A

B

therefore $A$ has not to $B$ the ratio which a number has to a number

Therefore etc Q E D

## Proposition 8

*If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable*

For let the two magnitudes $A$, $B$ not have to one another the ratio which a number has to a number,

A

B

I say that the magnitudes $A$, $B$ are incommensurable

For if they are commensurable $A$ will have to $B$ the ratio which a number has to a number [x 5]

But it has not,

therefore the magnitudes $A$, $B$ are incommensurable

Therefore etc Q E D

## Proposition 9

*The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number, and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either*

For let $A$, $B$ be commensurable in length

A B

C

D

I say that the square on $A$ has to the square on $B$ the ratio which a square number has to a square number

For since $A$ is commensurable in length with $B$,

therefore $A$ has to $B$ the ratio which a number has to a number [x 5]

Let it have to it the ratio which $C$ has to $D$

Since then, as $A$ is to $B$, so is $C$ to $D$

while the ratio of the square on $A$ to the square on $B$ is duplicate of the ratio of $A$ to $B$,

for similar figures are in the duplicate ratio of their corresponding sides [vi 20 Por ]

and the ratio of the square on $C$ to the square on $D$ is duplicate of the ratio of $C$ to $D$

for between two square numbers there is one mean proportional number and the square number has to the square number the ratio duplicate of that which the side has to the side [viii 11]

therefore also as the square on $A$ is to the square on $B$ so is the square on $C$ to the square on $D$

Next as the square on $A$ is to the square on $B$, so let the square on $C$ be to the square on $D$,

I say that $A$ is commensurable in length with $B$

For since as the square on $A$ is to the square on $B$, so is the square on $C$ to the square on $D$,

while the ratio of the square on $A$ to the square on $B$ is duplicate of the ratio of $A$ to $B$,

and the ratio of the square on $C$ to the square on $D$ is duplicate of the ratio of $C$ to $D$,

therefore also as $A$ is to $B$, so is $C$ to $D$

Therefore $A$ has to $B$ the ratio which the number $C$ has to the number $D$

therefore $A$ is commensurable in length with $B$ [x 6]

Next let $A$ be incommensurable in length with $B$,

I say that the square on $A$ has not to the square on $B$ the ratio which a square number has to a square number

For if the square on $A$ has to the square on $B$ the ratio which a square number has to a square number, $A$ will be commensurable with $B$

But it is not

therefore the square on $A$ has not to the square on $B$ the ratio which a square number has to a square number

Again let the square on $A$ not have to the square on $B$ the ratio which a square number has to a square number,

I say that $A$ is incommensurable in length with $B$

For, if $A$ is commensurable with $B$ the square on $A$ will have to the square on $B$ the ratio which a square number has to a square number

But it has not,

therefore $A$ is not commensurable in length with $B$

Therefore etc

PORISM And it is manifest from what has been proved that straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always commensurable in length also

[LEMMA It has been proved in the arithmetical books that similar plane numbers have to one another the ratio which a square number has to a square number [VIII 26]

and that if two numbers have to one another the ratio which a square number has to a square number they are similar plane numbers [Converse of VIII 26]

And it is manifest from these propositions that numbers which are not similar plane numbers that is those which have not their sides proportional have not to one another the ratio which a square number has to a square number

For if they have they will be similar plane numbers which is contrary to the hypothesis

Therefore numbers which are not similar plane numbers have not to one another the ratio which a square number has to a square number ]

PROPOSITION 10

*To find two straight lines incommensurable the one in length only and the other in square also with an assigned straight line*

Let $A$ be the assigned straight line,

thus it is required to find two straight lines incommensurable the one in length only and the other in square also with $A$

Let two numbers $B$ $C$ be set out which have not to one another the ratio

which a square number has to a square number that is which are not similar plane numbers,

and let it be contrived that,
as $B$ is to $C$, so is the square on $A$ to the square on $D$
—for we have learnt how to do this— [X 6 Por]
therefore the square on $A$ is commensurable with the square on $D$ [X 6]
And since $B$ has not to $C$ the ratio which a square number has to a square number,
therefore neither has the square on $A$ to the square on $D$ the ratio which a square number has to a square number
therefore $A$ is incommensurable in length with $D$ [X 9]
Let $E$ be taken a mean proportional between $A$ $D$,
therefore as $A$ is to $D$, so is the square on $A$ to the square on $E$ [V Def 9]
But $A$ is incommensurable in length with $D$,
therefore the square on $A$ is also incommensurable with the square on $E$ [X 11]
therefore $A$ is incommensurable in square with $E$
Therefore two straight lines $D$ $E$ have been found incommensurable $D$ in length only and $E$ in square and of course in length also with the assigned straight line $A$ Q E D

## Proposition 11

*If four magnitudes be proportional, and the first be commensurable with the second the third will also be commensurable with the fourth and if the first be incommensurable with the second the third will also be incommensurable with the fourth*

Let $A$, $B$, $C$ $D$ be four magnitudes in proportion so that as $A$ is to $B$ so is $C$ to $D$

and let $A$ be commensurable with $B$,
I say that $C$ will also be commensurable with $D$
For since $A$ is commensurable with $B$
therefore $A$ has to $B$ the ratio which a number has to a number [X 5]
And as $A$ is to $B$ so is $C$ to $D$,
therefore $C$ also has to $D$ the ratio which a number has to a number,
therefore $C$ is commensurable with $D$ [X 6]
Next let $A$ be incommensurable with $B$
I say that $C$ will also be incommensurable with $D$
For since $A$ is incommensurable with $B$
therefore $A$ has not to $B$ the ratio which a number has to a number [X 7]
And as $A$ is to $B$ so is $C$ to $D$
therefore neither has $C$ to $D$ the ratio which a number has to a number
therefore $C$ is incommensurable with $D$ [X 8]
Therefore etc Q E D

## Proposition 12

*Magnitudes commensurable with the same magnitude are commensurable with one another also*

For let each of the magnitudes $A$ $B$ be commensurable with $C$

I say that $A$ is also commensurable with $B$

For since $A$ is commensurable with $C$, therefore $A$ has to $C$ the ratio which a number has to a number [x 5]

Let it have the ratio which $D$ has to $E$

Again since $C$ is commensurable with $B$, therefore $C$ has to $B$ the ratio which a number has to a number [x 5]

Let it have the ratio which $F$ has to $G$

And given any number of ratios we please namely the ratio which $D$ has to $E$ and that which $F$ has to $G$ let the numbers $H$, $K$ $L$ be taken continuously in the given ratios, [cf viii 4]

so that as $D$ is to $E$, so is $H$ to $K$,

and, as $F$ is to $G$, so is $K$ to $L$

Since then, as $A$ is to $C$, so is $D$ to $E$,

while as $D$ is to $E$ so is $H$ to $K$,

therefore also, as $A$ is to $C$ so is $H$ to $K$ [v 11]

Again since as $C$ is to $B$ so is $F$ to $G$,

while as $F$ is to $G$ so is $K$ to $L$

therefore also as $C$ is to $B$ so is $K$ to $L$ [v 11]

But also as $A$ is to $C$, so is $H$ to $K$

therefore *ex aequali* as $A$ is to $B$, so is $H$ to $L$ [v 22]

Therefore $A$ has to $B$ the ratio which a number has to a number,

therefore $A$ is commensurable with $B$ [x 6]

Therefore etc Q E D

## Proposition 13

*If two magnitudes be commensurable and the one of them be incommensurable with any magnitude the remaining one will also be incommensurable with the same*

Let $A$, $B$ be two commensurable magnitudes and let one of them $A$ be incommensurable with any other magnitude $C$

I say that the remaining one $B$ will also be incommensurable with $C$

For if $B$ is commensurable with $C$

while $A$ is also commensurable with $B$

$A$ is also commensurable with $C$ [x 12]

But it is also incommensurable with it

which is impossible

Therefore $B$ is not commensurable with $C$

therefore it is incommensurable with it

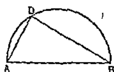
Therefore etc Q E D

## Lemma

*Given two unequal straight lines to find by what square the square on the greater is greater than the square on the less*

Let $AB$, $C$ be the given two unequal straight lines and let $AB$ be the greater of them,

thus it is required to find by what square the square on $AB$ is greater than the square on $C$




Let the semicircle $ADB$ be described on $AB$ and let $AD$ be fitted into it equal to $C$, [IV 1] let $DB$ be joined

It is then manifest that the angle $ADB$ is right [III 31] and that the square on $AB$ is greater than the square on $AD$, that is, $C$ by the square on $DB$ [I 47]

Similarly also if two straight lines be given, the straight line the square on which is equal to the sum of the squares on them is found in this manner

Let $AD$ $DB$ be the given two straight lines, and let it be required to find the straight line the square on which is equal to the sum of the squares on them

Let them be placed so as to contain a right angle that formed by $AD$ $DB$, and let $AB$ be joined

It is again manifest that the straight line the square on which is equal to the sum of the squares on $AD$, $DB$ is $AB$ [I 47]

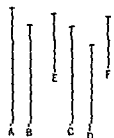
Q E D

## Proposition 14

*If four straight lines be proportional and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third*

*And if the square on the first be greater than the square on the second by the square on a straight line incommensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line incommensurable with the third*

Let $A$ $B$ $C$ $D$ be four straight lines in proportion so that as $A$ is to $B$ so is $C$ to $D$




and let the square on $A$ be greater than the square on $B$ by the square on $E$

and let the square on $C$ be greater than the square on $D$ by the square on $F$,

I say that if $A$ is commensurable with $E$, $C$ is also commensurable with $F$

and, if $A$ is incommensurable with $E$ $C$ is also incommensurable with $F$

For since as $A$ is to $B$ so is $C$ to $D$ therefore also as the square on $A$ is to the square on $B$ so is the square on $C$ to the square on $D$ [VI 22]

But the squares on $E$ $B$ are equal to the square on $A$ and the squares on $D$ $F$ are equal to the square on $C$

Therefore as the squares on $E$ $B$ are to the square on $B$ so are the squares on $D$ $F$ to the square on $D$

therefore *separando* as the square on $E$ is to the square on $B$ so is the square on $F$ to the square on $D$, [V 17]

therefore also as $E$ is to $B$ so is $F$ to $D$ [VI 22]

therefore inversely as $B$ is to $E$ so is $D$ to $F$

But as $A$ is to $B$, so also is $C$ to $D$,

therefore *ex aequali*, as $A$ is to $E$, so is $C$ to $F$ [v 22]

Therefore if $A$ is commensurable with $E$ $C$ is also commensurable with $F$, and if $A$ is incommensurable with $E$, $C$ is also incommensurable with $F$ [x 11]

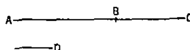
Therefore etc Q E D

PROPOSITION 15

*If two commensurable magnitudes be added together, the whole will also be commensurable with each of them and, if the whole be commensurable with one of them the original magnitudes will also be commensurable*

For let the two commensurable magnitudes $AB$, $BC$ be added together

I say that the whole $AC$ is also commensurable with each of the magnitudes $AB$ $BC$




For since $AB$, $BC$ are commensurable, some magnitude will measure them

Let it measure them and let it be $D$

Since then $D$ measures $AB$, $BC$, it will also measure the whole $AC$

But it measures $AB$ $BC$ also,

therefore $D$ measures $AB$ $BC$ $AC$

therefore $AC$ is commensurable with each of the magnitudes $AB$, $BC$ [x Def 1]

Next, let $AC$ be commensurable with $AB$,

I say that $AB$, $BC$ are also commensurable

For since $AC$, $AB$ are commensurable some magnitude will measure them

Let it measure them and let it be $D$

Since then $D$ measures $CA$ $AB$ it will also measure the remainder $BC$

But it measures $AB$ also

therefore $D$ will measure $AB$ $BC$,

therefore $AB$ $BC$ are commensurable [x Def 1]

Therefore etc Q E D

PROPOSITION 16

*If two incommensurable magnitudes be added together the whole will also be incommensurable with each of them and if the whole be incommensurable with one of them the original magnitudes will also be incommensurable*




For let the two incommensurable magnitudes $AB$ $BC$ be added together I say that the whole $AC$ is also incommensurable with each of the magnitudes $AB$ $BC$

For if $CA$ $AB$ are not incommensurable some magnitude will measure them

Let it measure them if possible and let it be $D$

Since then $D$ measures $CA$ $AB$

therefore it will also measure the remainder $BC$

But it measures $AB$ also

therefore $D$ measures $AB$ $BC$

Therefore $AB$ $BC$ are commensurable

but they were also by hypothesis incommensurable

which is impossible

Therefore no magnitude will measure $CA$, $AB$,

therefore $CA$ $AB$ are incommensurable [x Def 1]

Similarly we can prove that $AC$, $CB$ are also incommensurable

Therefore $AC$ is incommensurable with each of the magnitudes $AB$ $BC$

Next let $AC$ be incommensurable with one of the magnitudes $AB$, $BC$

First let it be incommensurable with $AB$,

I say that $AB$, $BC$ are also incommensurable

For, if they are commensurable some magnitude will measure them

Let it measure them, and let it be $D$

Since then, $D$ measures $AB$, $BC$,

therefore it will also measure the whole $AC$

But it measures $AB$ also

therefore $D$ measures $CA$ $AB$

Therefore $CA$, $AB$ are commensurable,

but they were also by hypothesis incommensurable

which is impossible

Therefore no magnitude will measure $AB$ $BC$

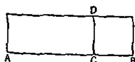
therefore $AB$, $BC$ are incommensurable [x Def 1]

Therefore etc Q E D

## Lemma

*If to any straight line there be applied a parallelogram deficient by a square figure the applied parallelogram is equal to the rectangle contained by the segments of the straight line resulting from the application*

For let there be applied to the straight line $AB$ the parallelogram $AD$ deficient by the square figure $DB$,




I say that $AD$ is equal to the rectangle contained by $AC$, $CB$

This is indeed at once manifest

for since $DB$ is a square

$DC$ is equal to $CB$

and $AD$ is the rectangle $AC$ $CD$, that is the rectangle $AC$ $CB$

Therefore etc Q E D

## Proposition 17

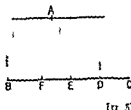
*If there be two unequal straight lines and to the greater there be applied to a parallelogram equal to the fourth part of the square on the less and deficient by a square figure and if it divide it into parts which are commensurable in length then the square on the greater will be greater than the square on the less by the square on a straight line commensurable with the greater*

*And if the square on the greater be greater than the square on the less by the square on a straight line commensurable with the greater and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it will divide it into parts which are commensurable in length*

Let $A$ $BC$ be two unequal straight lines of which $BC$ is the greater and let there be applied to $BC$ a parallelogram equal to the fourth part of the square on the less $A$ that is equal to the square on the half of $A$ and deficient by a square figure Let this be the rectangle $BD$ $DC$ [cf Lemma]

and let $BD$ be commensurable in length with $DC$,

I say that the square on $BC$ is greater than the square on $A$ by the square on a straight line commensurable with $BC$




For let $BC$ be bisected at the point $E$

and let $EF$ be made equal to $DE$

Therefore the remainder $DC$ is equal to $BF$

And since the straight line $BC$ has been cut into equal parts at $E$ and into unequal parts at $D$ therefore the rectangle contained by $BD$, $DC$, together with the square on $ED$, is equal to the square on $EC$, [II 5]

And the same is true of their quadruples,

therefore four times the rectangle $BD$ $DC$, together with four times the square on $DE$ is equal to four times the square on $EC$

But the square on $A$ is equal to four times the rectangle $BD$ $DC$ and the square on $DF$ is equal to four times the square on $DE$, for $DF$ is double of $DE$

And the square on $BC$ is equal to four times the square on $EC$, for again $BC$ is double of $CE$

Therefore the squares on $A$ $DF$ are equal to the square on $BC$,

so that the square on $BC$ is greater than the square on $A$ by the square on $DF$

It is to be proved that $BC$ is also commensurable with $DF$

Since $BD$ is commensurable in length with $DC$

therefore $BC$ is also commensurable in length with $CD$ [x 15]

But $CD$ is commensurable in length with $CD$, $BF$ for $CD$ is equal to $BF$ [x 6]

Therefore $BC$ is also commensurable in length with $BF$ $CD$ [x 12]

so that $BC$ is also commensurable in length with the remainder $FD$ [x 15]

therefore the square on $BC$ is greater than the square on $A$ by the square on a straight line commensurable with $BC$

Next let the square on $BC$ be greater than the square on $A$ by the square on a straight line commensurable with $BC$

let a parallelogram be applied to $BC$ equal to the fourth part of the square on $A$ and deficient by a square figure and let it be the rectangle $BD$, $DC$

It is to be proved that $BD$ is commensurable in length with $DC$

With the same construction we can prove similarly that the square on $BC$ is greater than the square on $A$ by the square on $FD$

But the square on $BC$ is greater than the square on $A$ by the square on a straight line commensurable with $BC$

Therefore $BC$ is commensurable in length with $FD$

so that $BC$ is also commensurable in length with the remainder the sum of $BF$ $DC$ [x 15]

But the sum of $BF$ $DC$ is commensurable with $DC$ [x 6]

so that $BC$ is also commensurable in length with $CD$ [x 12]

and therefore *separando* $BD$ is commensurable in length with $DC$ [x 15]

Therefore etc Q E D

## Proposition 18

*If there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square*

*figure, and if it divide it into parts which are incommensurable the square on the greater will be greater than the square on the less by the square on a straight line incommensurable with the greater*

*And if the square on the greater be greater than the square on the less by the square on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure it divides it into parts which are incommensurable*

Let $A$, $BC$ be two unequal straight lines of which $BC$ is the greater, and to $BC$ let there be applied a parallelogram equal to the fourth part of the square on the less $A$, and deficient by a square figure Let this be the rectangle $BD$, $DC$, [cf Lemma before x 17] and let $BD$ be incommensurable in length with $DC$,

I say that the square on $BC$ is greater than the square on $A$ by the square on a straight line incommensurable with $BC$

For with the same construction as before, we can prove similarly that the square on $BC$ is greater than the square on $A$ by the square on $FD$

It is to be proved that $BC$ is incommensurable in length with $DF$

Since $BD$ is incommensurable in length with $DC$,

therefore $BC$ is also incommensurable in length with $CD$ [x 16]

But $DC$ is commensurable with the sum of $BF$, $DC$, [x 6]

therefore $BC$ is also incommensurable with the sum of $BF$, $DC$ [x 13]

so that $BC$ is also incommensurable in length with the remainder $FD$ [x 16]

And the square on $BC$ is greater than the square on $A$ by the square on $FD$, therefore the square on $BC$ is greater than the square on $A$ by the square on a straight line incommensurable with $BC$

Again, let the square on $BC$ be greater than the square on $A$ by the square on a straight line incommensurable with $BC$ and let there be applied to $BC$ a parallelogram equal to the fourth part of the square on $A$ and deficient by a square figure Let this be the rectangle $BD$, $DC$

It is to be proved that $BD$ is incommensurable in length with $DC$

For with the same construction, we can prove similarly that the square on $BC$ is greater than the square on $A$ by the square on $FD$

But the square on $BC$ is greater than the square on $A$ by the square on a straight line incommensurable with $BC$,

therefore $BC$ is incommensurable in length with $FD$,

so that $BC$ is also incommensurable with the remainder the sum of $BF$ $DC$ [x 16]

But the sum of $BF$ $DC$ is commensurable in length with $DC$, [x 6]

therefore $BC$ is also incommensurable in length with $DC$ [x 13]

so that, *separando* $BD$ is also incommensurable in length with $DC$ [x 16]

Therefore etc Q E D

LEMMA

Since it has been proved that straight lines commensurable in length are always commensurable in square also while those commensurable in square are not always commensurable in length also but can of course be either commensurable or incommensurable in length it is manifest that if any straight line be *commensurable in length with a given rational straight line, it is called*

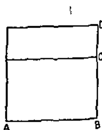
rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also

But, if any straight line be commensurable in square with a given rational straight line then, if it is also commensurable in length with it it is called in this case also rational and commensurable with it both in length and in square but if again any straight line being commensurable in square with a given rational straight line be incommensurable in length with it, it is called in this case also rational but commensurable in square only

PROPOSITION 19

*The rectangle contained by rational straight lines commensurable in length is rational*

For let the rectangle $AC$ be contained by the rational straight lines $AB$, $BC$ commensurable in length,



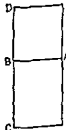

I say that $AC$ is rational
For on $AB$ let the square $AD$ be described,
therefore $AD$ is rational [x Def 4]
And since $AB$ is commensurable in length with $BC$,
while $AB$ is equal to $BD$,
therefore $BD$ is commensurable in length with $BC$
And as $BD$ is to $BC$, so is $DA$ to $AC$ [vi 1]
Therefore $DA$ is commensurable with $AC$ [x 11]
But $DA$ is rational
therefore $AC$ is also rational [x Def 4]
Therefore etc Q E D

PROPOSITION 20

*If a rational area be applied to a rational straight line it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied*

For let the rational area $AC$ be applied to $AB$, a straight line once more rational in any of the aforesaid ways producing $BC$ as breadth,

I say that $BC$ is rational and commensurable in length with $BA$



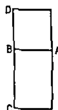

For on $AB$ let the square $AD$ be described
therefore $AD$ is rational [x Def 4]
But $AC$ is also rational
therefore $DA$ is commensurable with $AC$
And as $DA$ is to $AC$ so is $DB$ to $BC$ [vi 1]
Therefore $DB$ is also commensurable with $BC$, [x 11]
and $DB$ is equal to $BA$,
therefore $AB$ is also commensurable with $BC$
But $AB$ is rational
therefore $BC$ is al o rational and commensurable in length with $AB$
Therefore etc Q E D

PROPOSITION 21

*The rectangle contained by rational straight lines commensurable in square only is irrational and the side of the square equal to it is irrational Let the latter be called* medial

For let the rectangle $AC$ be contained by the rational straight lines $AB$, $BC$ commensurable in square only,

I say that $AC$ is irrational, and the side of the square equal to it is irrational, and let the latter be called *medial*




For on $AB$ let the square $AD$ be described,

therefore $AD$ is rational [x Def 4]

And since $AB$ is incommensurable in length with $BC$

for by hypothesis they are commensurable in square only,

while $AB$ is equal to $BD$

therefore $DB$ is also incommensurable in length with $BC$

And, as $DB$ is to $BC$, so is $AD$ to $AC$ [vi 1]

therefore $DA$ is incommensurable with $AC$ [x 11]

But $DA$ is rational,

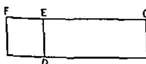
therefore $AC$ is irrational

so that the side of the square equal to $AC$ is also irrational [x Def 4]

And let the latter be called *medial* Q E D

## Lemma

If there be two straight lines then, as the first is to the second so is the square on the first to the rectangle contained by the two straight lines




Let $FE$, $EG$ be two straight lines

I say that, as $FE$ is to $EG$ so is the square on $FE$ to the rectangle $FE$, $EG$

For on $FE$ let the square $DF$ be described and let $GD$ be completed

Since then, as $FE$ is to $EG$ so is $FD$ to $DG$, [vi 1]

and $FD$ is the square on $FE$

and $DG$ the rectangle $DE$ $EG$ that is the rectangle $FE$ $EG$

therefore as $FE$ is to $EG$ so is the square on $FE$ to the rectangle $FE$, $EG$

Similarly also as the rectangle $GE$, $EF$ is to the square on $EF$ that is as $GD$ is to $FD$, so is $GE$ to $EF$ Q E D

## Proposition 22

*The square on a medial straight line if applied to a rational straight line produces as breadth a straight line rational and incommensurable in length with that to which it is applied*







Let $A$ be medial and $CB$ rational and let a rectangular area $BD$ equal to the square on $A$ be applied to $BC$ producing $CD$ as breadth

I say that $CD$ is rational and incommensurable in length with $CB$

For since $A$ is medial the square on it is equal to a rectangular area contained by rational straight lines commensurable in square only [x 21]

Let the square on it be equal to $GF$

But the square on it is also equal to $BD$

therefore $BD$ is equal to $GF$

But it is also equiangular with it,
and in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional, [VI 14]

therefore, proportionally, as $BC$ is to $EG$, so is $EF$ to $CD$

Therefore also as the square on $BC$ is to the square on $EG$, so is the square on $EF$ to the square on $CD$ [VI 22]

But the square on $CB$ is commensurable with the square on $EG$, for each of these straight lines is rational

*therefore the square on $EF$ is also commensurable with the square on $CD$* [X 11]

But the square on $EF$ is rational,

therefore the square on $CD$ is also rational, [X Def 4]

therefore $CD$ is rational

And since $EF$ is incommensurable in length with $EG$,

for they are commensurable in square only,

and as $EF$ is to $EG$ so is the square on $EF$ to the rectangle $FE$, $EG$, [Lemma]
therefore the square on $EF$ is incommensurable with the rectangle $FE$, $EG$ [X 11]

But the square on $CD$ is commensurable with the square on $EF$, for the *straight lines are rational in square*
and the rectangle $DC$, $CB$ is commensurable with the rectangle $FE$, $EG$ for they are equal to the square on $A$,
therefore the square on $CD$ is also incommensurable with the rectangle $DC$, $CB$ [X 13]

But as the square on $CD$ is to the rectangle $DC$, $CB$, so is $DC$ to $CB$, [Lemma]

therefore $DC$ is incommensurable in length with $CB$ [X 11]

Therefore $CD$ is rational and incommensurable in length with $CB$

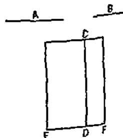
Q E D

PROPOSITION 23

*A straight line commensurable with a medial straight line is medial*

Let $A$ be medial and let $B$ be commensurable with $A$,

I say that $B$ is also medial




For let a rational straight line $CD$ be set out and to $CD$ let the rectangular area $CE$ equal to the square on $A$ be applied producing $ED$ as breadth, therefore $ED$ is rational and incommensurable in length with $CD$ [X 22]

And let the rectangular area $CF$ equal to the square on $B$ be applied to $CD$ producing $DF$ as breadth

Since, then $A$ is commensurable with $B$ the square on $A$ is also commensurable with the square on $B$

But $EC$ is equal to the square on $A$

and $CF$ is equal to the square on $B$,

therefore $EC$ is commensurable with $CF$

And as $EC$ is to $CF$ so is $ED$ to $DF$ [VI 1]

therefore $ED$ is commensurable in length with $DF$ [X 11]

But $ED$ is rational and incommensurable in length with $DC$
therefore $DF$ is also rational [x Def 3] and incommensurable in length with $DC$ [x 13]

Therefore $CD$, $DF$ are rational and commensurable in square only

But the straight line the square on which is equal to the rectangle contained by rational straight lines commensurable in square only is medial, [x 21]
therefore the side of the square equal to the rectangle $CD$, $DF$ is medial

And $B$ is the side of the square equal to the rectangle $CD$, $DF$,

therefore $B$ is medial Q E D

PORISM From this it is manifest that an area commensurable with a medial area is medial

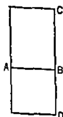
[And in the same way as was explained in the case of rationals [Lemma following x 18] it follows as regards medials, that a straight line commensurable in length with a medial straight line is called *medial* and commensurable with it not only in length but in square also since in general, straight lines commensurable in length are always commensurable in square also

But if any straight line be commensurable in square with a medial straight line then, if it is also commensurable in length with it, the straight lines are called in this case too medial and commensurable in length and in square but if in square only, they are called medial straight lines commensurable in square only ]

## PROPOSITION 24

*The rectangle contained by medial straight lines commensurable in length is medial*

For let the rectangle $AC$ be contained by the medial straight lines $AB$, $BC$ which are *commensurable in length*




I say that $AC$ is medial

For on $AB$ let the square $AD$ be described

therefore $AD$ is medial

And, since $AB$ is commensurable in length with $BC$
while $AB$ is equal to $BD$,
therefore $DB$ is also commensurable in length with $BC$,
so that $DA$ is also commensurable with $AC$ [vi 1 x 11]

But $DA$ is medial

therefore $AC$ is also medial [x 23 Por ]

Q E D

## PROPOSITION 25

*The rectangle contained by medial straight lines commensurable in square only is either rational or medial*

For let the rectangle $AC$ be contained by the medial straight lines $AB$, $BC$ which are commensurable in square only

I say that $AC$ is either rational or medial

For on $AB$ $BC$ let the squares $AD$ $BE$ be described

therefore each of the squares $AD$ $BE$ is medial

Let a rational straight line $FG$ be set out

to $FG$ let there be applied the rectangular parallelogram $GH$ equal to $AD$ producing $FH$ as breadth

to $HM$ let there be applied the rectangular parallelogram $MK$ equal to $AC$, producing $HK$ as breadth

and further to $KN$ let there be similarly applied $NL$ equal to $BE$ producing $KL$ as breadth,

therefore $FH$, $HK$, $KL$ are in a straight line

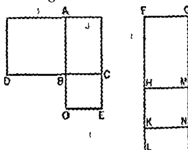
Since then each of the squares $AD$, $BE$ is medial

and $AD$ is equal to $GH$, and $BE$ to $NL$

therefore each of the rectangles $GH$, $NL$ is also medial

And they are applied to the rational straight line $FG$

therefore each of the straight lines $FH$, $KL$ is rational and incommensurable in length with $FG$ [x 22]




And since $AD$ is commensurable with $BE$

therefore $GH$ is also commensurable with $NL$

And as $GH$ is to $NL$ so is $FH$ to $KL$ [vi 1]

therefore $FH$ is commensurable in length with $KL$ [x 11]

Therefore $FH$, $KL$ are rational straight lines commensurable in length,

therefore the rectangle $FH$, $KL$ is rational [x 19]

And since $DB$ is equal to $BA$ and $OB$ to $BC$,

therefore as $DB$ is to $BC$ so is $AB$ to $BO$

But as $DB$ is to $BC$ so is $DA$ to $AC$ [vi 1]

and as $AB$ is to $BO$ so is $AC$ to $CO$, [id]

therefore as $DA$ is to $AC$ so is $AC$ to $CO$

But $AD$ is equal to $GH$, $AC$ to $MK$ and $CO$ to $NL$,

therefore as $GH$ is to $MK$, so is $MK$ to $NL$,

therefore also as $FH$ is to $HK$ so is $HK$ to $KL$ [vi 1, v 11]

therefore the rectangle $FH$, $KL$ is equal to the square on $HK$ [vi 17]

But the rectangle $FH$, $KL$ is rational

therefore the square on $HK$ is also rational

Therefore $HK$ is rational

And if it is commensurable in length with $FG$

$HN$ is rational [x 19]

but if it is incommensurable in length with $FG$

$KH$, $HM$ are rational straight lines commensurable in square only and therefore $HN$ is medial [x 21]

Therefore $HN$ is either rational or medial

But $HN$ is equal to $AC$

therefore $AC$ is either rational or medial

Therefore etc Q E D

## Proposition 26

*A medial area does not exceed a medial area by a rational area*

For if possible let the medial area $AB$ exceed the medial area $AC$ by the rational area $DB$

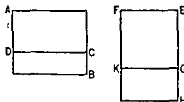
and let a rational straight line $EF$ be set out

to $EF$ let there be applied the rectangular parallelogram $FH$ equal to $AB$ producing $EH$ as breadth

and let the rectangle $FG$ equal to $AC$ be subtracted

therefore the remainder $BD$ is equal to the remainder $KH$

But $DB$ is rational




therefore $KH$ is also rational

Since then each of the rectangles $AB$, $AC$ is medial

and $AB$ is equal to $FH$ and $AC$ to $FG$

therefore each of the rectangles $FH$, $FG$ is also medial

And they are applied to the rational straight line $EF$,

therefore each of the straight lines $HE$ $EG$ is rational and incommensurable in length with $EF$ [x 22]

And, since [$DB$ is rational and is equal to $KH$,

therefore] $KH$ is [also] rational,

and it is applied to the rational straight line $EF$

therefore $GH$ is rational and commensurable in length with $EF$ [x 20]

But $EG$ is also rational and is incommensurable in length with $EF$

therefore $EG$ is incommensurable in length with $GH$ [x 13]

And as $EG$ is to $GH$ so is the square on $EG$ to the rectangle $EG$, $GH$,

therefore the square on $EG$ is incommensurable with the rectangle $EG$ $GH$ [x 11]

But the squares on $EG$, $GH$ are commensurable with the square on $EG$ for both are rational

and twice the rectangle $EG$, $GH$ is commensurable with the rectangle $EG$, $GH$ for it is double of it, [x 6]

therefore the squares on $EG$, $GH$ are incommensurable with twice the rectangle $EG$ $GH$ [x 13]

therefore also the sum of the squares on $EG$, $GH$ and twice the rectangle $EG$ $GH$, that is the square on $EH$ [II 4] is incommensurable with the squares on $EG$ $GH$ [x 16]

But the squares on $EG$ $GH$ are rational

therefore the square on $EH$ is irrational [x Def 4]

Therefore $EH$ is irrational

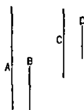
But it is also rational

which is impossible

Therefore etc Q E D

## Proposition 27

*To find medial straight lines commensurable in square only which contain a rational rectangle*




Let two rational straight lines $A$, $B$ commensurable in square only be set out,

let $C$ be taken a mean proportional between $A$, $B$ [vi 13]

and let it be contrived that

as $A$ is to $B$, so is $C$ to $D$ [vi 12]

Then since $A$ $B$ are rational and commensurable in square only

the rectangle $A$, $B$ that is the square on $C$ [vi 17] is medial [x 21]

Therefore $C$ is medial [x 21]

And since as $A$ is to $B$, so is $C$ to $D$,

and $A$, $B$ are commensurable in square only

therefore $C$, $D$ are also commensurable in square only [x 11]

And $C$ is medial,

therefore $D$ is also medial [x 23, addition]

Therefore $C$, $D$ are medial and commensurable in square only

I say that they also contain a rational rectangle

For since as $A$ is to $B$ so is $C$ to $D$

therefore alternately as $A$ is to $C$ so is $B$ to $D$ [v 16]

But as $A$ is to $C$ so is $C$ to $B$,

therefore also as $C$ is to $B$ so is $B$ to $D$

therefore the rectangle $C$ $D$ is equal to the square on $B$

But the square on $B$ is rational

therefore the rectangle $C$ $D$ is also rational

*Therefore medial straight lines commensurable in square only have been found which contain a rational rectangle* Q E D

## Proposition 28

*To find medial straight lines commensurable in square only which contain a medial rectangle*

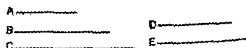
Let the rational straight lines $A$, $B$, $C$ commensurable in square only be set out

let $D$ be taken a mean proportional between $A$, $B$, [vi 13]

and let it be contrived that

as $B$ is to $C$ so is $D$ to $E$ [vi 12]

Since $A$ $B$ are rational straight lines commensurable in square only therefore the rectangle $A$ $B$ that is the square on $D$ [vi 17] is medial [x 21]




Therefore $D$ is medial [x 21]

And since $B$ $C$ are commensurable in square only

and as $B$ is to $C$ so is $D$ to $E$

therefore $D$ $E$ are also commensurable in square only [x 11]

But $D$ is medial

therefore $E$ is also medial [x 23 addition]

Therefore $D$ $E$ are medial straight lines commensurable in square only

I say next that they also contain a medial rectangle

For since as $B$ is to $C$ so is $D$ to $E$

therefore alternately as $B$ is to $D$ so is $C$ to $E$ [v 16]

But as $B$ is to $D$ so is $D$ to $A$

therefore also as $D$ is to $A$ so is $C$ to $E$

therefore the rectangle $A$ $C$ is equal to the rectangle $D$ $E$ [vi 16]

But the rectangle $A$ $C$ is medial [x 21]

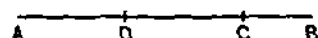
therefore the rectangle $D$ $E$ is also medial

Therefore medial straight lines commensurable in square only have been found which contain a medial rectangle Q E D

LEMMA 1

*To find two square numbers such that their sum is also square*

Let two numbers $AB$, $BC$ be set out, and let them be either both even or both odd




Then since whether an even number is subtracted from an even number, or an odd number from an odd number the remainder is even [IX 24 26]

therefore the remainder $AC$ is even

Let $AC$ be bisected at $D$

Let $AB$, $BC$ also be either similar plane numbers or square numbers which are themselves also similar plane numbers

Now the product of $AB$, $BC$ together with the square on $CD$ is equal to the square on $BD$ [II 6]

And the product of $AB$, $BC$ is square inasmuch as it was proved that if two similar plane numbers by multiplying one another make some number the product is square [IX 1]

Therefore two square numbers, the product of $AB$ $BC$, and the square on $CD$ have been found which when added together make the square on $BD$

And it is manifest that two square numbers, the square on $BD$ and the square on $CD$, have again been found such that their difference, the product of $AB$, $BC$ is a square, whenever $AB$ $BC$ are similar plane numbers

But when they are not similar plane numbers two square numbers the square on $BD$ and the square on $DC$, have been found such that their difference the product of $AB$ $BC$ is not square Q E D

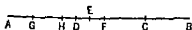
LEMMA 2

*To find two square numbers such that their sum is not square*

For let the product of $AB$, $BC$ as we said be square

and $CA$ even

and let $CA$ be bisected by $D$




It is then manifest that the square product of $AB$, $BC$ together with the square on $CD$ is equal to the square on $BD$

[See Lemma 1]

Let the unit $DE$ be subtracted,

therefore the product of $AB$ $BC$ together with the square on $CE$ is less than the square on $BD$

I say then that the square product of $AB$ $BC$ together with the square on $CE$ will not be square

For if it is square it is either equal to the square on $BE$ or less than the square on $BE$ but cannot any more be greater lest the unit be divided

First if possible, let the product of $AB$, $BC$ together with the square on $CE$ be equal to the square on $BE$

and let $GA$ be double of the unit $DE$

Since then the whole $AC$ is double of the whole $CD$

and in them $AG$ is double of $DE$

therefore the remainder $GC$ is also double of the remainder $EC$,

therefore $GC$ is bisected by $E$

Therefore the product of $GB$, $BC$ together with the square on $CE$ is equal to the square on $BE$ [II 6]

But the product of $AB$, $BC$ together with the square on $CE$ is also, by hypothesis equal to the square on $BE$,

therefore the product of $GB$ $BC$ together with the square on $CE$ is equal to the product of $AB$, $BC$ together with the square on $CE$

And, if the common square on $CE$ be subtracted,

it follows that $AB$ is equal to $GB$

which is absurd

Therefore the product of $AB$, $BC$ together with the square on $CE$ is not equal to the square on $BE$

I say next that neither is it less than the square on $BE$

For, if possible, let it be equal to the square on $BF$

and let $HA$ be double of $DF$

Now it will again follow that $HC$ is double of $CF$

so that $CH$ has also been bisected at $F$

and for this reason the product of $HB$ $BC$ together with the square on $FC$ is equal to the square on $BF$ [II 6]

But, by hypothesis the product of $AB$ $BC$ together with the square on $CE$ is also equal to the square on $BF$

Thus the product of $HB$ $BC$ together with the square on $CF$ will also be equal to the product of $AB$, $BC$ together with the square on $CE$

which is absurd

Therefore the product of $AB$, $BC$ together with the square on $CE$ is not less than the square on $BE$

And it was proved that neither is it equal to the square on $BE$

Therefore the product of $AB$, $BC$ together with the square on $CE$ is not square Q E D

PROPOSITION 29

*To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater*

For let there be set out any rational straight line $AB$ and two square numbers $CD$, $DE$ such that their difference $CE$ is not square, [Lemma 1]

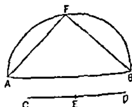
let there be described on $AB$ the semicircle $AFB$

and let it be contrived that

as $DC$ is to $CE$ so is the square on $BA$ to the square on $AF$ [x 6 Por]

Let $FB$ be joined

Since as the square on $BA$ is to the square on $AF$, so is $DC$ to $CE$




therefore the square on $BA$ has to the square on $AF$ the ratio which the number $DC$ has to the number $CE$,

therefore the square on $BA$ is commensurable with the square on $AF$ [x 6]

But the square on $AB$ is rational [x Def 4]

therefore the square on $AF$ is also rational [id]

therefore $AF$ is also rational

And since $DC$ has not to $CE$ the ratio which a square number has to a square number

neither has the square on $BA$ to the square on $AF$ the ratio which a square number has to a square number

therefore $AB$ is incommensurable in length with $AF$ [x 9]

Therefore $BA$, $AF$ are rational straight lines commensurable in square only

And since as $DC$ is to $CE$, so is the square on $BA$ to the square on $AF$,

therefore, *convertendo* as $CD$ is to $DE$, so is the square on $AB$ to the square on $BF$ [v 19 Por, III 31 I 47]

But $CD$ has to $DE$ the ratio which a square number has to a square number

therefore also the square on $AB$ has to the square on $BF$ the ratio which a square number has to a square number,

therefore $AB$ is commensurable in length with $BF$ [x 9]

And the square on $AB$ is equal to the squares on $AF$, $FB$

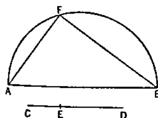
therefore the square on $AB$ is greater than the square on $AF$ by the square on $BF$ commensurable with $AB$

Therefore there have been found two rational straight lines $BA$, $AF$ commensurable in square only and such that the square on the greater $AB$ is greater than the square on the less $AF$ by the square on $BF$ commensurable in length with $AB$ Q E D

PROPOSITION 30

*To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater*

Let there be set out a rational straight line $AB$,




and two square numbers $CE$, $ED$ such that their sum $CD$ is not square [Lemma 2]

let there be described on $AB$ the semicircle $AFB$,

let it be contrived that

as $DC$ is to $CE$, so is the square on $BA$ to the square on $AF$ [x 6, Por]

and let $FB$ be joined

Then in a similar manner to the preceding we can prove that $BA$, $AF$ are rational straight lines commensurable in square only

And since as $DC$ is to $CE$, so is the square on $BA$ to the square on $AF$

therefore *convertendo* as $CD$ is to $DE$ so is the square on $AB$ to the square on $BF$ [v 19 Por III 31 I 47]

But $CD$ has not to $DE$ the ratio which a square number has to a square number

therefore neither has the square on $AB$ to the square on $BF$ the ratio which a square number has to a square number,

therefore $AB$ is incommensurable in length with $BF$ [x 9]

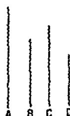
And the square on $AB$ is greater than the square on $AF$ by the square on $FB$ incommensurable with $AB$

Therefore $AB$ $AF$ are rational straight lines commensurable in square only and the square on $AB$ is greater than the square on $AF$ by the square on $FB$ incommensurable in length with $AB$ Q E D

PROPOSITION 31

*To find two medial straight lines commensurable in square only, containing a rational rectangle and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater*

Let there be set out two rational straight lines $A$ $B$ commensurable in square only and such that the square on $A$ being the greater is greater than the square on $B$ the less by the square on a straight line commensurable in length with $A$ [x 29]




And let the square on $C$ be equal to the rectangle $A$ $B$

Now the rectangle $A$, $B$ is medial [x 21]

therefore the square on $C$ is also medial,

therefore $C$ is also medial [x 21]

Let the rectangle $C$ $D$ be equal to the square on $B$

Now the square on $B$ is rational

therefore the rectangle $C$ $D$ is also rational

And since as $A$ is to $B$ so is the rectangle $A$ $B$ to the square on $B$

while the square on $C$ is equal to the rectangle $A$, $B$

and the rectangle $C$ $D$ is equal to the square on $B$,

therefore as $A$ is to $B$ so is the square on $C$ to the rectangle $C$, $D$

But as the square on $C$ is to the rectangle $C$ $D$ so is $C$ to $D$,

therefore also as $A$ is to $B$ so is $C$ to $D$

But $A$ is commensurable with $B$ in square only

therefore $C$ is also commensurable with $D$ in square only [x 11]

And $C$ is medial

therefore $D$ is also medial [x 23 addition]

And since as $A$ is to $B$ so is $C$ to $D$

and the square on $A$ is greater than the square on $B$ by the square on a straight line commensurable with $A$,

therefore also the square on $C$ is greater than the square on $D$ by the square on a straight line commensurable with $C$ [x 14]

Therefore two medial straight lines $C$ $D$ commensurable in square only and containing a rational rectangle have been found and the square on $C$ is greater than the square on $D$ by the square on a straight line commensurable in length with $C$

Similarly also it can be proved that the square on $C$ exceeds the square on $D$ by the square on a straight line incommensurable with $C$ when the square on $A$ is greater than the square on $B$ by the square on a straight line incommensurable with $A$ [x 30]

PROPOSITION 32

*To find two medial straight lines commensurable in square only containing a medial rectangle and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater*

Let there be set out three rational straight lines $A$ $B$ $C$ commensurable in square only and such that the square on $A$ is greater than the square on $C$ by the square on a straight line commensurable with $A$ [x 29]

and let the square on $D$ be equal to the rectangle $A$ $B$

Therefore the square on $D$ is medial

therefore $D$ is also medial [x. 21]

Let the rectangle $D$, $E$ be equal to the rectangle $B$, $C$.

Then since, as the rectangle $A$, $B$ is to the rectangle $B$, $C$, so is $A$ to $C$,

while the square on $D$ is equal to the rectangle $A$, $B$,

and the rectangle $D$, $E$ is equal to the rectangle $B$, $C$,

therefore, as $A$ is to $C$, so is the square on $D$ to the rectangle $D$, $E$.

But, as the square on $D$ is to the rectangle $D$, $E$, so is $D$ to $E$;

therefore also, as $A$ is to $C$, so is $D$ to $E$.

But $A$ is commensurable with $C$ in square only;

therefore $D$ is also commensurable with $E$ in square only. [x. 11]

But $D$ is medial;

therefore $E$ is also medial. [x. 23, addition]

And, since, as $A$ is to $C$, so is $D$ to $E$,

while the square on $A$ is greater than the square on $C$ by the square on a straight line commensurable with $A$,

therefore also the square on $D$ will be greater than the square on $E$ by the square on a straight line commensurable with $D$. [x. 14]

I say next that the rectangle $D$, $E$ is also medial.

For, since the rectangle $B$, $C$ is equal to the rectangle $D$, $E$, while the rectangle $B$, $C$ is medial, [x. 21]

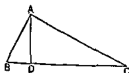
therefore the rectangle $D$, $E$ is also medial.

Therefore two medial straight lines $D$, $E$, commensurable in square only and containing a medial rectangle, have been found such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Similarly again it can be proved that the square on $D$ is greater than the square on $E$ by the square on a straight line incommensurable with $D$, when the square on $A$ is greater than the square on $C$ by the square on a straight line incommensurable with $A$. [x. 30]

### Lemma

Let $ABC$ be a right-angled triangle having the angle $A$ right, and let the perpendicular $AD$ be drawn;




I say that the rectangle $CB$, $BD$ is equal to the square on $BA$,

the rectangle $BC$, $CD$ equal to the square on $CA$,

the rectangle $BD$, $DC$ equal to the square on $AD$,

and, further, the rectangle $BC$, $AD$ equal to the rectangle $BA$, $AC$.

And first that the rectangle $CB$, $BD$ is equal to the square on $BA$.

For, since in a right-angled triangle $AD$ has been drawn from the right angle perpendicular to the base,

therefore the triangles $ABD$, $ADC$ are similar both to the whole $ABC$ and to one another. [vi. 8]

And since the triangle $ABC$ is similar to the triangle $ABD$, therefore, as $CB$ is to $BA$, so is $BA$ to $BD$; [vi. 4]

therefore the rectangle *CB* *BD* is equal to the square on *AB* [VI 17]

For the same reason the rectangle *BC*, *CD* is also equal to the square on *AC*

And since if in a right angled triangle a perpendicular be drawn from the right angle to the base the perpendicular so drawn is a mean proportional between the segments of the base [VI 8 Por]

therefore as *BD* is to *DA*, so is *AD* to *DC*,

therefore the rectangle *BD*, *DC* is equal to the square on *AD* [VI 17]

I say that the rectangle *BC* *AD* is also equal to the rectangle *BA* *AC*

For since as we said *ABC* is similar to *ABD*,

therefore as *BC* is to *CA* so is *BA* to *AD* [VI 4]

Therefore the rectangle *BC*, *AD* is equal to the rectangle *BA* *AC* [VI 16]

Q E D

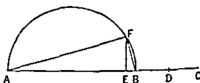
## Proposition 33

*To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial*

Let there be set out two rational straight lines *AB* *BC* commensurable in square only and such that the square on the greater *AB* is greater than the square on the less *BC* by the square on a straight line incommensurable with *AB* [X 30]

let *BC* be bisected at *D*

let there be applied to *AB* a parallelogram equal to the square on either of the straight lines *BD* *DC* and deficient by a square figure and let it be the rectangle *AE*, *EB*, [VI 28]




let the semicircle *AFB* be described on *AB*,

let *EF* be drawn at right angles to *AB*,

and let *AF* *FB* be joined

Then since *AB* *BC* are unequal straight lines and the square on *AB* is greater than the square on *BC* by the square on a straight line incommensurable with *AB*

while there has been applied to *AB* a parallelogram equal to the fourth part of the square on *BC* that is to the square on half of it and deficient by a square figure making the rectangle *AE* *EB*,

therefore *AE* is incommensurable with *EB* [X 18]

And as *AE* is to *EB* so is the rectangle *BA* *AE* to the rectangle *AB* *BE*

while the rectangle *BA* *AE* is equal to the square on *AF*

and the rectangle *AB* *BE* to the square on *BF*

therefore the square on *AF* is incommensurable with the square on *FB*,

therefore *AF* *FB* are incommensurable in square

And since *AB* is rational

therefore the square on *AB* is also rational,

so that the sum of the squares on *AF* *FB* is also rational [I 47]

And since again the rectangle *AE* *EB* is equal to the square on *EF*

and by hypothesis the rectangle *AE* *EB* is also equal to the square on *BD*

therefore *FE* is equal to *BD*

therefore *BC* is double of *FE*

so that the rectangle *AB* *BC* is also commensurable with the rectangle *AB* *EF*

But the rectangle $AB$ $BC$ is medial, [x 21]

therefore the rectangle $AB$, $EF$ is also medial [x 23 Por]

But the rectangle $AB$, $EF$ is equal to the rectangle $AF$ $FB$, [Lemma]

therefore the rectangle $AF$ $FB$ is also medial

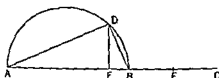
But it was also proved that the sum of the squares on these straight lines is rational

Therefore two straight lines $AF$, $FB$ incommensurable in square have been found which make the sum of the squares on them rational but the rectangle contained by them medial Q E D

## Proposition 34

*To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational*

Let there be set out two medial straight lines $AB$ $BC$, commensurable in square only such that the rectangle which they contain is rational and the square on $AB$ is greater than the square on $BC$ by the square on a straight line incommensurable with $AB$, [x 31, *ad fin*]




let the semicircle $ADB$ be described on $AB$,

let $BC$ be bisected at $E$

let there be applied to $AB$ a parallelogram equal to the square on $BE$ and deficient by a square figure, namely the rectangle $AF$ $FB$ [vi 28]

therefore $AF$ is incommensurable in length with $FB$ [x 18]

Let $FD$ be drawn from $F$ at right angles to $AB$,

and let $AD$ $DB$ be joined

Since $AF$ is incommensurable in length with $FB$

therefore the rectangle $BA$, $AF$ is also incommensurable with the rectangle $AB$, $BF$ [x 11]

But the rectangle $BA$ $AF$ is equal to the square on $AD$ and the rectangle $AB$, $BF$ to the square on $DB$,

therefore the square on $AD$ is also incommensurable with the square on $DB$

And since the square on $AB$ is medial

therefore the sum of the squares on $AD$ $DB$ is also medial [iii 31 i 47]

And since $BC$ is double of $DF$,

therefore the rectangle $AB$ $BC$ is also double of the rectangle $AB$ $FD$

But the rectangle $AB$ $BC$ is rational

therefore the rectangle $AB$ $FD$ is also rational [x 6]

But the rectangle $AB$ $FD$ is equal to the rectangle $AD$ $DB$ [Lemma]

so that the rectangle $AD$ $DB$ is also rational

Therefore two straight lines $AD$ $DB$ incommensurable in square have been found which make the sum of the squares on them medial but the rectangle contained by them rational Q E D

## Proposition 35

*To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them*

Let there be set out two medial straight lines $AB$, $BC$ commensurable in square only containing a medial rectangle and such that the square on $AB$ is greater than the square on $BC$ by the square on a straight line incommensurable with $AB$, [x 32, ad fin]

let the semicircle $ADB$ be described on $AB$,

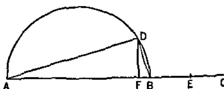
and let the rest of the construction be as above

Then, since $AF$ is incommensurable in length with $FB$ [x 18]

$AD$ is also incommensurable in square with $DB$ [x 11]

And since the square on $AB$ is medial

therefore the sum of the squares on $AD$ $DB$ is also medial [III 31 I 47]




And since the rectangle $AF$, $FB$ is equal to the square on each of the straight lines $BE$ $DF$,

therefore $BE$ is equal to $DF$,

therefore $BC$ is double of $FD$

so that the rectangle $AB$, $BC$ is also double of the rectangle $AB$, $FD$

But the rectangle $AB$ $BC$ is medial,

therefore the rectangle $AB$ $FD$ is also medial [x 32 Por]

And it is equal to the rectangle $AD$ $DB$, [Lemma after x 32]

therefore the rectangle $AD$ $DB$ is also medial

And since $AB$ is incommensurable in length with $BC$,

while $CB$ is commensurable with $BE$,

therefore $AB$ is also incommensurable in length with $BE$ [x 13]

so that the square on $AB$ is also incommensurable with the rectangle $AB$ $BE$ [x 11]

But the squares on $AD$ $DB$ are equal to the square on $AB$, [I 47]

and the rectangle $AB$, $FD$, that is the rectangle $AD$, $DB$, is equal to the rectangle $AB$ $BE$

therefore the sum of the squares on $AD$, $DB$ is incommensurable with the rectangle $AD$ $DB$

Therefore two straight lines $AD$, $DB$ incommensurable in square have been found which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them Q E D

## Proposition 36

*If two rational straight lines commensurable in square only be added together, the whole is irrational and let it be called* binomial

For let two rational straight lines $AB$, $BC$ commensurable in square only be added together

I say that the whole $AC$ is irrational

For since $AB$ is incommensurable in length with $BC$—for they are commensurable in square only—

A B C

and as $AB$ is to $BC$, so is the rectangle $AB$, $BC$ to the square on $BC$,

therefore the rectangle $AB$, $BC$ is incommensurable with the square on $BC$ [x 11]

But twice the rectangle $AB$, $BC$ is commensurable with the rectangle $AB$, $BC$ [x 6], and the squares on $AB$ $BC$ are commensurable with the square on $BC$—for $AB$, $BC$ are rational straight lines commensurable in square only— [x 15]

therefore twice the rectangle $AB$, $BC$ is incommensurable with the squares on $AB$, $BC$ [x 13]

And, *componendo* twice the rectangle $AB$, $BC$ together with the squares on $AB$, $BC$, that is, the square on $AC$ [II 4] is incommensurable with the sum of the squares on $AB$, $BC$ [x 16]

But the sum of the squares on $AB$ $BC$ is rational

therefore the square on $AC$ is irrational

so that $AC$ is also irrational [x *Def* 4]

And let it be called *binomial* Q E D

## Proposition 37

*If two medial straight lines commensurable in square only and containing a rational rectangle be added together the whole is irrational, and let it be called a* first bimedial *straight line*

For let two medial straight lines $AB$ $BC$ commensurable in square only and containing a rational rectangle be added together

I say that the whole $AC$ is irrational

For since $AB$ is incommensurable in length with $BC$

therefore the squares on $AB$ $BC$ are also incommensurable with twice the rectangle $AB$ $BC$, [cf x 36, ll 9–20]

and *componendo* the squares on $AB$, $BC$ together with twice the rectangle $AB$, $BC$, that is the square on $AC$ [II 4] is incommensurable with the rectangle $AB$, $BC$ [x 16]

But the rectangle $AB$ $BC$ is rational for by hypothesis $AB$, $BC$ are straight lines containing a rational rectangle

therefore the square on $AC$ is irrational,

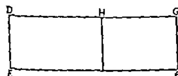
therefore $AC$ is irrational [x Def 4]

And let it be called a *first bimedial* straight line Q E D

## Proposition 38

*If two medial straight lines commensurable in square only and containing a medial rectangle be added together the whole is irrational and let it be called a* second bimedial *straight line*

For let two medial straight lines $AB$ $BC$ commensurable in square only and containing a medial rectangle be added together




I say that $AC$ is irrational

For let a rational straight line $DE$ be set out and let the parallelogram $DF$ equal to the square on $AC$ be applied to $DE$ producing $DG$ as breadth [I 44]

Then since the square on $AC$ is equal to the squares on $AB$ $BC$ and twice the rectangle $AB$ $BC$ [II 4]

let $EH$ equal to the squares on $AB$ $BC$ be applied to $DE$

therefore the remainder $HF$ is equal to twice the rectangle $AB$, $BC$

And, since each of the straight lines $AB$, $BC$ is medial

therefore the squares on $AB$, $BC$ are also medial

But by hypothesis, twice the rectangle $AB$, $BC$ is also medial

And $EH$ is equal to the squares on $AB$, $BC$,

while $FH$ is equal to twice the rectangle $AB$, $BC$,

therefore each of the rectangles $EH$, $HF$ is medial

And they are applied to the rational straight line $DE$,

therefore each of the straight lines $DH$, $HG$ is rational and incommensurable in length with $DE$ [x 22]

Since then $AB$ is incommensurable in length with $BC$,

and as $AB$ is to $BC$ so is the square on $AB$ to the rectangle $AB$ $BC$,

therefore the square on $AB$ is incommensurable with the rectangle $AB$ $BC$ [x 11]

But the sum of the squares on $AB$, $BC$ is commensurable with the square on $AB$ [x 15]

and twice the rectangle $AB$, $BC$ is commensurable with the rectangle $AB$ $BC$ [x 6]

Therefore the sum of the squares on $AB$, $BC$ is incommensurable with twice the rectangle $AB$ $BC$ [x 13]

But $EH$ is equal to the squares on $AB$ $BC$,

and $HF$ is equal to twice the rectangle $AB$, $BC$

Therefore $EH$ is incommensurable with $HF$

so that $DH$ is also incommensurable in length with $HG$ [vi 1, x 11]

Therefore $DH$, $HG$ are rational straight lines commensurable in square only,

so that $DG$ is irrational [x 36]

But $DE$ is rational,

and the rectangle contained by an irrational and a rational straight line is irrational [cf x 20]

therefore the area $DF$ is irrational

and the side of the square equal to it is irrational [x Def 4]

But $AC$ is the side of the square equal to $DF$

therefore $AC$ is irrational

And let it be called a *second bimedial straight line* Q E D

## Proposition 39

*If two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial, be added together, the whole straight line is irrational and let it be called* major

For let two straight lines $AB$ $BC$ incommensurable in square and fulfilling the given conditions [x 33] be added together

I say that $AC$ is irrational




For, since the rectangle $AB$ $BC$ is medial

twice the rectangle $AB$ $BC$ is also medial [x 6 and 23 Por]

But the sum of the squares on $AB$ $BC$ is rational

therefore twice the rectangle $AB$ $BC$ is incommensurable with the sum of the squares on $AB$ $BC$

so that the squares on $AB$ $BC$ together with twice the rectangle $AB$ $BC$ that

is, the square on $AC$, is also incommensurable with the sum of the squares on $AB$, $BC$, [x 16]

therefore the square on $AC$ is irrational

so that $AC$ is also irrational [x Def 4]

And let it be called *major* Q E D

## Proposition 40

*If two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational be added together, the whole straight line is irrational and let it be called the* side of a rational plus a medial area

For let two straight lines $AB$, $BC$ incommensurable in square and fulfilling the given conditions [x 34] be added together,

A B C

I say that $AC$ is irrational

For, since the sum of the squares on $AB$, $BC$ is medial while twice the rectangle $AB$ $BC$ is rational

therefore the sum of the squares on $AB$, $BC$ is incommensurable with twice the rectangle $AB$, $BC$,

so that the square on $AC$ is also incommensurable with twice the rectangle $AB$, $BC$ [x 16]

But twice the rectangle $AB$, $BC$ is rational,

therefore the square on $AC$ is irrational

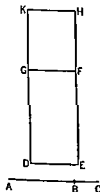
Therefore $AC$ is irrational [x Def 4]

And let it be called the *side of a rational plus a medial area* Q E D

## Proposition 41

*If two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and also incommensurable with the sum of the squares on them be added together, the whole straight line is irrational and let it be called the* side of the sum of two medial areas

For let two straight lines $AB$, $BC$ incommensurable in square and satisfying the given conditions [x 35] be added together




I say that $AC$ is irrational

Let a rational straight line $DE$ be set out and let there be applied to $DE$ the rectangle $DF$ equal to the squares on $AB$, $BC$, and the rectangle $GH$ equal to twice the rectangle $AB$ $BC$

therefore the whole $DH$ is equal to the square on $AC$ [II 4]

Now since the sum of the squares on $AB$, $BC$ is medial, and is equal to $DF$,

therefore $DF$ is also medial

And it is applied to the rational straight line $DE$

therefore $DG$ is rational and incommensurable in length with $DE$ [x 22]

For the same reason $GK$ is also rational and incommensurable in length with $GF$ that is $DE$

And since the squares on $AB$ $BC$ are incommensurable with twice the rectangle $AB$, $BC$

$DF$ is incommensurable with $GH$

so that $DG$ is also incommensurable with $GK$ [VI 1, X 11]

And they are rational

therefore $DG$ $GK$ are rational straight lines commensurable in square only,

therefore $DK$ is irrational and what is called binomial [X 36]

But $DE$ is rational

therefore $DH$ is irrational and the side of the square which is equal to it is irrational [X Def 4]

But $AC$ is the side of the square equal to $HD$,

therefore $AC$ is irrational

And let it be called the *side of the sum of two medial areas* Q E D

### Lemma

And that the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question we will now prove after premising the following lemma

Let the straight line $AB$ be set out let the whole be cut into unequal parts at each of the points $C$ $D$

and let $AC$ be supposed greater than $DB$

I say that the squares on $AC$ $CB$ are greater than the squares on $AD$ $DB$

For let $AB$ be bisected at $E$

Then since $AC$ is greater than $DB$,

let $DC$ be subtracted from each

therefore the remainder $AD$ is greater than the remainder $CB$

But $AE$ is equal to $EB$

therefore $DE$ is less than $EC$

therefore the points $C$ $D$ are not equidistant from the point of bisection

And since the rectangle $AC$, $CB$ together with the square on $EC$ is equal to the square on $EB$ [II 5]

and further the rectangle $AD$ $DB$ together with the square on $DE$ is equal to the square on $EB$ [id]

therefore the rectangle $AC$ $CB$ together with the square on $EC$ is equal to the rectangle $AD$ $DB$ together with the square on $DE$

And of these the square on $DE$ is less than the square on $EC$,

therefore the remainder the rectangle $AC$ $CB$ is also less than the rectangle $AD$ $DB$

so that twice the rectangle $AC$ $CB$ is also less than twice the rectangle $AD$, $DB$

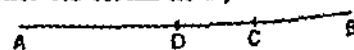
Therefore also the remainder the sum of the squares on $AC$, $CB$, is greater than the sum of the squares on $AD$ $DB$ Q E D

### Proposition 42

*A binomial straight line is divided into its terms at one point only*

Let $AB$ be a binomial straight line divided into its terms at $C$,

therefore $AC$ $CB$ are rational straight lines commensurable in square only




I say that $AB$ is not divided at another point into two rational straight lines commensurable in square only

For if possible let it be divided at $D$ also so that $AD$ $DB$ are also rational straight lines commensurable in square only

It is then manifest that $AC$ is not the same with $DB$

For, if possible let it be so

Then $AD$ will also be the same as $CB$,

and as $AC$ is to $CB$, so will $BD$ be to $DA$,

thus $AB$ will be divided at $D$ also in the same way as by the division at $C$

which is contrary to the hypothesis

Therefore $AC$ is not the same with $DB$

For this reason also the points $C$, $D$ are not equidistant from the point of bisection

Therefore that by which the squares on $AC$, $CB$ differ from the squares on $AD$ $DB$ is also that by which twice the rectangle $AD$ $DB$ differs from twice the rectangle $AC$, $CB$,

because both the squares on $AC$ $CB$ together with twice the rectangle $AC$, $CB$ and the squares on $AD$ $DB$ together with twice the rectangle $AD$, $DB$ are equal to the square on $AB$ [II 4]

But the squares on $AC$, $CB$ differ from the squares on $AD$, $DB$ by a rational area

for both are rational

therefore twice the rectangle $AD$, $DB$ also differs from twice the rectangle $AC$, $CB$ by a rational area though they are medial [X 21]

which is absurd for a medial area does not exceed a medial by a rational area [X 26]

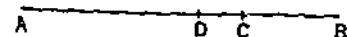
Therefore a binomial straight line is not divided at different points

therefore it is divided at one point only Q E D

PROPOSITION 43

*A first bimedial straight line is divided at one point only*

Let $AB$ be a first bimedial straight line divided at $C$, so that $AC$ $CB$ are medial straight lines commensurable in square only and containing a rational rectangle; [X 37]




I say that $AB$ is not so divided at another point

For if possible let it be divided at $D$ also, so that $AD$, $DB$ are also medial straight lines commensurable in square only and containing a rational rectangle

Since then that by which twice the rectangle $AD$ $DB$ differs from twice the rectangle $AC$ $CB$ is that by which the squares on $AC$ $CB$ differ from the squares on $AD$ $DB$

while twice the rectangle $AD$ $DB$ differs from twice the rectangle $AC$, $CB$ by a rational area—for both are rational—

therefore the squares on $AC$ $CB$ also differ from the squares on $AD$ $DB$ by a rational area though they are medial

which is absurd [X 26]

Therefore a first bimedial straight line is not divided into its terms at different points

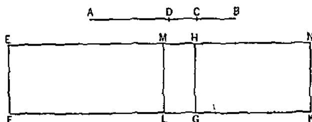
therefore it is so divided at one point only Q E D

## Proposition 44

*A second bimedial straight line is divided at one point only*

Let $AB$ be a second bimedial straight line divided at $C$, so that $AC$, $CB$ are medial straight lines commensurable in square only and containing a medial rectangle, [x 38]
it is then manifest that $C$ is not at the point of bisection, because the segments are not commensurable in length

I say that $AB$ is not so divided at another point




For if possible let it be divided at $D$ also so that $AC$ is not the same with $DB$, but $AC$ is supposed greater
it is then clear that the squares on $AD$ $DB$ are also as we proved above [Lemma] less than the squares on $AC$ $CB$
and suppose that $AD$ $DB$ are medial straight lines commensurable in square only and containing a medial rectangle

Now let a rational straight line $EF$ be set out
let there be applied to $EF$ the rectangular parallelogram $EK$ equal to the square on $AB$,

and let $EG$, equal to the squares on $AC$ $CB$, be subtracted
therefore the remainder $HK$ is equal to twice the rectangle $AC$ $CB$ [II 4]

Again let there be subtracted $EL$ equal to the squares on $AD$ $DB$, which were proved less than the squares on $AC$, $CB$ [Lemma]
therefore the remainder $MK$ is also equal to twice the rectangle $AD$ $DB$

Now since the squares on $AC$ $CB$ are medial
therefore $EG$ is medial

And it is applied to the rational straight line $EF$,
therefore $EH$ is rational and incommensurable in length with $EF$ [x 22]

For the same reason
$HN$ is also rational and incommensurable in length with $EF$

And since $AC$ $CB$ are medial straight lines commensurable in square only
therefore $AC$ is incommensurable in length with $CB$

But as $AC$ is to $CB$ so is the square on $AC$ to the rectangle $AC$ $CB$,
therefore the square on $AC$ is incommensurable with the rectangle $AC$ $CB$ [x 11]

But the squares on $AC$ $CB$ are commensurable with the square on $AC$, for $AC$ $CB$ are commensurable in square [x 15]

And twice the rectangle $AC$ $CB$ is commensurable with the rectangle $AC$, $CB$ [x 6]

Therefore the squares on $AC$ $CB$ are also incommensurable with twice the rectangle $AC$ $CB$ [x 13]

But $EG$ is equal to the squares on $AC$ $CB$,

and $HK$ is equal to twice the rectangle $AC$, $CB$,

therefore $EG$ is incommensurable with $HK$,

so that $EH$ is also incommensurable in length with $HN$ [VI 1, X 11]

And they are rational

therefore $EH$ $HN$ are rational straight lines commensurable in square only

But if two rational straight lines commensurable in square only be added together the whole is the irrational which is called binomial [X 36]

Therefore $EN$ is a binomial straight line divided at $H$

In the same way $EM$, $MN$ will also be proved to be rational straight lines commensurable in square only,

and $EN$ will be a binomial straight line divided at different points $H$ and $M$

And $EH$ is not the same with $MN$

For the squares on $AC$ $CB$ are greater than the squares on $AD$, $DB$

But the squares on $AD$, $DB$ are greater than twice the rectangle $AD$ $DB$

therefore also the squares on $AC$ $CB$, that is, $EG$ are much greater than twice the rectangle $AD$, $DB$ that is $MK$

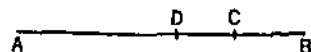
so that $EH$ is also greater than $MN$

Therefore $EH$ is not the same with $MN$ Q E D

PROPOSITION 45

*A major straight line is divided at one and the same point only*

Let $AB$ be a major straight line divided at $C$, so that $AC$ $CB$ are incommensurable in square and make the sum of the squares on $AC$ $CB$ rational but the rectangle $AC$ $CB$ medial,




I say that $AB$ is not so divided at another point

For if possible let it be divided at $D$ also so that $AD$ $DB$ are also incommensurable in square and make the sum of the squares on $AD$ $DB$ rational but the rectangle contained by them medial

Then since that by which the squares on $AC$ $CB$ differ from the squares on $AD$ $DB$ is also that by which twice the rectangle $AD$ $DB$ differs from twice the rectangle $AC$ $CB$

while the squares on $AC$ $CB$ exceed the squares on $AD$ $DB$ by a rational area —for both are rational—

therefore twice the rectangle $AD$ $DB$ also exceeds twice the rectangle $AC$ $CB$ by a rational area though they are medial

which is impossible [X 26]

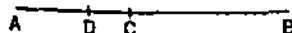
Therefore a major straight line is not divided at different points

therefore it is only divided at one and the same point Q E D

PROPOSITION 46

*The side of a rational plus a medial area is divided at one point only*

Let $AB$ be the side of a rational plus a medial area divided at $C$ so that $AC$, $CB$ are incommensurable in square and make the sum of the squares on $AC$ $CB$ medial but twice the rectangle $AC$ $CB$ rational [X 40]




I say that $AB$ is not so divided at another point

For if possible let it be divided at $D$ also so that $AD$ $DB$ are also incom

mensurable in square and make the sum of the squares on $AD$, $DB$ medial but twice the rectangle $AD$, $DB$ rational.

Since then that by which twice the rectangle $AC$, $CB$ differs from twice the rectangle $AD$, $DB$ is also that by which the squares on $AD$, $DB$ differ from the squares on $AC$, $CB$,

while twice the rectangle $AC$, $CB$ exceeds twice the rectangle $AD$, $DB$ by a rational area,

therefore the squares on $AD$, $DB$ also exceed the squares on $AC$, $CB$ by a rational area though they are medial

which is impossible. [x 26]

Therefore the side of a rational plus a medial area is not divided at different points,

therefore it is divided at one point only. Q E D

PROPOSITION 47

*The side of the sum of two medial areas is divided at one point only.*

Let $AB$ be divided at $C$ so that $AC$, $CB$ are incommensurable in square and make the sum of the squares on $AC$, $CB$ medial and the rectangle $AC$, $CB$ medial and also incommensurable with the sum of the squares on them.

I say that $AB$ is not divided at another point so as to fulfil the given conditions.

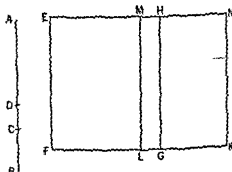
For if possible let it be divided at $D$, so that again $AC$ is of course not the same as $BD$, but $AC$ is supposed greater;

let a rational straight line $EF$ be set out,

and let there be applied to $EF$ the rectangle $EG$ equal to the squares on $AC$, $CB$,

and the rectangle $HK$ equal to twice the rectangle $AC$, $CB$;

therefore the whole $EK$ is equal to the square on $AB$. [II 4]




Again let $EL$, equal to the squares on $AD$, $DB$, be applied to $EF$;

therefore the remainder, twice the rectangle $AD$, $DB$, is equal to the remainder $MK$.

And since, by hypothesis, the sum of the squares on $AC$, $CB$ is medial,

therefore $EG$ is also medial.

And it is applied to the rational straight line $EF$;

therefore $HE$ is rational and incommensurable in length with $EF$. [x 22]

For the same reason

$HN$ is also rational and incommensurable in length with $EF$.

And, since the sum of the squares on $AC$, $CB$ is incommensurable with twice the rectangle $AC$, $CB$,

therefore $EG$ is also incommensurable with $GN$,

so that $EH$ is also incommensurable with $HN$. [vi 1, x 11]

And they are rational;

therefore $EH$, $HN$ are rational straight lines commensurable in square only;

therefore $EN$ is a binomial straight line divided at $H$. [x 36]

Similarly we can prove that it is also divided at $M$

And $EH$ is not the same with $MN$,

therefore a binomial has been divided at different points which is absurd [x 42]

Therefore a side of the sum of two medial areas is not divided at different points

therefore it is divided at one point only Q E D

## DEFINITIONS II

1 Given a rational straight line and a binomial divided into its terms such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then if the greater term be commensurable in length with the rational straight line set out, let the whole be called a *first binomial* straight line

2 but if the lesser term be commensurable in length with the rational straight line set out let the whole be called a *second binomial*

3 and if neither of the terms be commensurable in length with the rational straight line set out let the whole be called a *third binomial*

4 Again if the square on the greater term be greater than the square on the lesser by the square on a straight line incommensurable in length with the greater then if the greater term be commensurable in length with the rational straight line set out let the whole be called a *fourth binomial*,

5 if the lesser a *fifth binomial*,

6 and if neither a *sixth binomial*

### PROPOSITION 48

*To find the first binomial straight line*

D———— H————

E——F——G

A——C—B

Let two numbers $AC$ $CB$ be set out such that the sum of them $AB$ has to $BC$ the ratio which a square number has to a square number but has not to $CA$ the ratio which a square number has to a square number, [Lemma 1 after x 28]

let any rational straight line $D$ be set out and let $EF$ be commensurable in length with $D$

Therefore $EF$ is also rational

Let it be contrived that

as the number $BA$ is to $AC$ so is the square on $EF$ to the square on $FG$ [x 6 Por]

But $AB$ has to $AC$ the ratio which a number has to a number

therefore the square on $EF$ also has to the square on $FG$ the ratio which a number has to a number

so that the square on $EF$ is commensurable with the square on $FG$ [x 6]

And $EF$ is rational

therefore $FG$ is also rational

And since $BA$ has not to $AC$ the ratio which a square number has to a square number

neither therefore has the square on $EF$ to the square on $FG$ the ratio which a square number has to a square number

therefore $EF$ is incommensurable in length with $FG$ [x 9]

Therefore $EF$, $FG$ are rational straight lines commensurable in square only,

therefore $EG$ is binomial [x 36]

I say that it is also a first binomial straight line

For since as the number $BA$ is to $AC$, so is the square on $EF$ to the square on $FG$,

while $BA$ is greater than $AC$,

therefore the square on $EF$ is also greater than the square on $FG$

Let then the squares on $FG$ $H$ be equal to the square on $EF$

Now since, as $BA$ is to $AC$, so is the square on $EF$ to the square on $FG$,

therefore, *convertendo*

as $AB$ is to $BC$, so is the square on $EF$ to the square on $H$ [v 19 Por]

But $AB$ has to $BC$ the ratio which a square number has to a square number therefore the square on $EF$ also has to the square on $H$ the ratio which a square number has to a square number

Therefore $EF$ is commensurable in length with $H$ [x 9]

therefore the square on $EF$ is greater than the square on $FG$ by the square on a straight line commensurable with $EF$

And $EF$ $FG$ are rational and $EF$ is commensurable in length with $D$

Therefore $EF$ is a first binomial straight line Q E D

PROPOSITION 49

*To find the second binomial straight line*

Let two numbers $AC$ $CB$ be set out such that the sum of them $AB$ has to $BC$ the ratio which a square number has to a square number but has not to $AC$ the ratio which a square number has to a square number,

let a rational straight line $D$ be set out and let $EF$ be commensurable in length with $D$

therefore $EF$ is rational

Let it be contrived then that

as the number $CA$ is to $AB$ so also is the square on $EF$ to the square on $FG$ [x 6 Por]

therefore the square on $EF$ is commensurable with the square on $FG$ [x 6]

Therefore $FG$ is also rational

Now since the number $CA$ has not to $AB$ the ratio which a square number has to a square number neither has the square on $EF$ to the square on $FG$ the ratio which a square number has to a square number

Therefore $EF$ is incommensurable in length with $FG$, [x 9]

therefore $EF$ $FG$ are rational straight lines commensurable in square only

therefore $EG$ is binomial [x 36]

It is next to be proved that it is also a second binomial straight line

For since inversely as the number $BA$ is to $AC$, so is the square on $GF$ to the square on $FE$,

while $BA$ is greater than $AC$

therefore the square on $GF$ is greater than the square on $FE$

Let the squares on $EF$, $H$ be equal to the square on $GF$,

therefore *convertendo* as $AB$ is to $BC$ so is the square on $FG$ to the square on $H$ [v 19 Por]

But $AB$ has to $BC$ the ratio which a square number has to a square number,

therefore the square on $FG$ also has to the square on $H$ the ratio which a square number has to a square number

Therefore $FG$ is commensurable in length with $H$, [x 9]
so that the square on $FG$ is greater than the square on $FE$ by the square on a straight line commensurable with $FG$

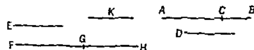
And $FG$ $FE$ are rational straight lines commensurable in square only and $EF$, the lesser term is commensurable in length with the rational straight line $D$ set out

Therefore $EG$ is a second binomial straight line Q E D

PROPOSITION 50

*To find the third binomial straight line*

Let two numbers $AC$, $CB$ be set out such that the sum of them $AB$ has to $BC$ the ratio which a square number has to a square number but has not to $AC$ the ratio which a square number has to a square number




Let any other number $D$, not square be set out also, and let it not have to either of the numbers $BA$, $AC$ the ratio which a square number has to a square number

Let any rational straight line $E$ be set out and let it be contrived that as $D$ is to $AB$ so is the square on $E$ to the square on $FG$, [x 6, Por]
therefore the square on $E$ is commensurable with the square on $FG$ [x 6]

And $E$ is rational,

therefore $FG$ is also rational

And since $D$ has not to $AB$ the ratio which a square number has to a square number
neither has the square on $E$ to the square on $FG$ the ratio which a square number has to a square number

therefore $E$ is incommensurable in length with $FG$ [x 9]

Next let it be contrived that as the number $BA$ is to $AC$ so is the square on $FG$ to the square on $GH$ [x 6 Por]
therefore the square on $FG$ is commensurable with the square on $GH$ [x 6]

But $FG$ is rational

therefore $GH$ is also rational

And since $BA$ has not to $AC$ the ratio which a square number has to a square number
neither has the square on $FG$ to the square on $HG$ the ratio which a square number has to a square number

therefore $FG$ is incommensurable in length with $GH$ [x 9]

Therefore $FG$ $GH$ are rational straight lines commensurable in square only

therefore $FH$ is binomial [x 36]

I say next that it is also a third binomial straight line

For since as $D$ is to $AB$ so is the square on $E$ to the square on $FG$
and as $BA$ is to $AC$ so is the square on $FG$ to the square on $GH$
therefore *ex aequali*, as $D$ is to $AC$ so is the square on $E$ to the square on $GH$ [v 22]

But $D$ has not to $AC$ the ratio which a square number has to a square number,

therefore neither has the square on $E$ to the square on $GH$ the ratio which a square number has to a square number,

therefore $E$ is incommensurable in length with $GH$ [x 9]

And since, as $BA$ is to $AC$, so is the square on $FG$ to the square on $GH$,

therefore the square on $FG$ is greater than the square on $GH$

Let then the squares on $GH$, $K$ be equal to the square on $FG$,

therefore *convertendo* as $AB$ is to $BC$, so is the square on $FG$ to the square on $K$ [v 19, Por]

But $AB$ has to $BC$ the ratio which a square number has to a square number,

therefore the square on $FG$ also has to the square on $K$ the ratio which a square number has to a square number,

therefore $FG$ is commensurable in length with $K$ [x 9]

Therefore the square on $FG$ is greater than the square on $GH$ by the square on a straight line commensurable with $FG$

And $FG$ $GH$ are rational straight lines commensurable in square only, and neither of them is commensurable in length with $E$

Therefore $FH$ is a third binomial straight line Q E D

## Proposition 51

*To find the fourth binomial straight line*

Let two numbers $AC$, $CB$ be set out such that $AB$ neither has to $BC$, nor yet to $AC$ the ratio which a square number has to a square number

Let a rational straight line $D$ be set out

and let $EF$ be commensurable in length with $D$,

therefore $EF$ is also rational




Let it be contrived that, as the number $BA$ is to $AC$ so is the square on $EF$ to the square on $FG$ [x 6, Por]

therefore the square on $EF$ is commensurable with the square on $FG$, [x 6]

therefore $FG$ is also rational

Now since $BA$ has not to $AC$ the ratio which a square number has to a square number

neither has the square on $EF$ to the square on $FG$ the ratio which a square number has to a square number

therefore $EF$ is incommensurable in length with $FG$ [x 9]

Therefore $EF$, $FG$ are rational straight lines commensurable in square only

so that $EG$ is binomial

I say next that it is also a fourth binomial straight line

For since as $BA$ is to $AC$ so is the square on $EF$ to the square on $FG$

therefore the square on $EF$ is greater than the square on $FG$

Let then the squares on $FG$ $H$ be equal to the square on $EF$,

therefore *convertendo* as the number $AB$ is to $BC$, so is the square on $EF$ to the square on $H$ [v 19 Por]

But $AB$ has not to $BC$ the ratio which a square number has to a square number

therefore neither has the square on $EF$ to the square on $H$ the ratio which a square number has to a square number

Therefore $EF$ is incommensurable in length with $H$, [x 9]
therefore the square on $EF$ is greater than the square on $GF$ by the square on a straight line incommensurable with $EF$

And $EF$, $FG$ are rational straight lines commensurable in square only, and $EF$ is commensurable in length with $D$

Therefore $EG$ is a fourth binomial straight line Q E D

### Proposition 52

*To find the fifth binomial straight line*

Let two numbers $AC$, $CB$ be set out such that $AB$ has not to either of them the ratio which a square number has to a square number,

let any rational straight line $D$ be set out and let $EF$ be commensurable with $D$,

therefore $EF$ is rational

Let it be contrived that, as $CA$ is to $AB$, so is the square on $EF$ to the square on $FG$ [x 6, Por ]

But $CA$ has not to $AB$ the ratio which a square number has to a square number,

therefore neither has the square on $EF$ to the square on $FG$ the ratio which a square number has to a square number

Therefore $EF$, $FG$ are rational straight lines commensurable in square only, [x 9]

therefore $EG$ is binomial [x 36]

I say next that it is also a fifth binomial straight line

For since, as $CA$ is to $AB$ so is the square on $EF$ to the square on $FG$ inversely, as $BA$ is to $AC$ so is the square on $FG$ to the square on $FE$,

therefore the square on $GF$ is greater than the square on $FE$

Let then the squares on $EF$, $H$ be equal to the square on $GF$,

therefore, *convertendo* as the number $AB$ is to $BC$, so is the square on $GF$ to the square on $H$ [v 19 Por ]

But $AB$ has not to $BC$ the ratio which a square number has to a square number

therefore neither has the square on $FG$ to the square on $H$ the ratio which a square number has to a square number

Therefore $FG$ is incommensurable in length with $H$ [x 9]

so that the square on $FG$ is greater than the square on $FE$ by the square on a straight line incommensurable with $FG$

And $GF$, $FE$ are rational straight lines commensurable in square only and the lesser term $EF$ is commensurable in length with the rational straight line $D$ set out

Therefore $EG$ is a fifth binomial straight line Q E D

### Proposition 53

*To find the sixth binomial straight line*

Let two numbers $AC$, $CB$ be set out such that $AB$ has not to either of them the ratio which a square number has to a square number

and let there also be another number $D$ which is not square and which has not to either of the numbers $BA$, $AC$ the ratio which a square number has to a square number

Let any rational straight line $E$ be set out, and let it be contrived that, as $D$ is to $AB$ so is the square on $E$ to the square on $FG$ [x 6 Por]

therefore the square on $E$ is commensurable with the square on $FG$ [x 6]



And $E$ is rational

therefore $FG$ is also rational

Now since $D$ has not to $AB$ the ratio which a square number has to a square number neither has the square on $E$ to the square on $FG$ the ratio which a square number has to a square number,

therefore $E$ is incommensurable in length with $FG$ [x 9]

Again let it be contrived that as $BA$ is to $AC$, so is the square on $FG$ to the square on $GH$ [x 6, Por]

Therefore the square on $FG$ is commensurable with the square on $HG$ [x 6]

Therefore the square on $HG$ is rational

therefore $HG$ is rational

And since $BA$ has not to $AC$ the ratio which a square number has to a square number,

neither has the square on $FG$ to the square on $GH$ the ratio which a square number has to a square number,

therefore $FG$ is incommensurable in length with $GH$ [x 9]

Therefore $FG$, $GH$ are rational straight lines commensurable in square only

therefore $FH$ is binomial [x 36]

It is next to be proved that it is also a sixth binomial straight line

For since as $D$ is to $AB$ so is the square on $E$ to the square on $FG$ and also as $BA$ is to $AC$, so is the square on $FG$ to the square on $GH$ therefore *ex aequali* as $D$ is to $AC$ so is the square on $E$ to the square on $GH$ [v 22]

But $D$ has not to $AC$ the ratio which a square number has to a square number

therefore neither has the square on $E$ to the square on $GH$ the ratio which a square number has to a square number

therefore $E$ is incommensurable in length with $GH$ [x 9]

But it was also proved incommensurable with $FG$

therefore each of the straight lines $FG$ $GH$ is incommensurable in length with $E$

And since as $BA$ is to $AC$ so is the square on $FG$ to the square on $GH$

therefore the square on $FG$ is greater than the square on $GH$

Let then the squares on $GH$ $K$ be equal to the square on $FG$

therefore *convertendo* as $AB$ is to $BC$, so is the square on $FG$ to the square on $K$ [v 19 Por]

But $AB$ has not to $BC$ the ratio which a square number has to a square number

so that neither has the square on $FG$ to the square on $K$ the ratio which a square number has to a square number

Therefore $FG$ is incommensurable in length with $K$ [x 9]

therefore the square on $FG$ is greater than the square on $GH$ by the square on a straight line incommensurable with $FG$

And $FG$, $GH$ are rational straight lines commensurable in square only, and neither of them is commensurable in length with the rational straight line $E$ set out.

Therefore $FH$ is a sixth binomial straight line. Q. E. D.

LEMMA

Let there be two squares $AB$, $BC$, and let them be placed so that $DB$ is in a straight line with $BE$;

therefore $FB$ is also in a straight line with $BG$.

Let the parallelogram $AC$ be completed.

I say that $AC$ is a square, that $DG$ is a mean proportional between $AB$, $BC$, and further that $DC$ is a mean proportional between $AC$, $CB$.

For, since $DB$ is equal to $BF$, and $BE$ to $BG$,

therefore the whole $DE$ is equal to the whole $FG$.

But $DE$ is equal to each of the straight lines $AH$, $KC$, and $FG$ is equal to each of the straight lines $AK$, $HC$; [I. 34]

therefore each of the straight lines $AH$, $KC$ is also equal to each of the straight lines $AK$, $HC$.

Therefore the parallelogram $AC$ is equilateral.

And it is also rectangular;

therefore $AC$ is a square.

And since, as $FB$ is to $BG$, so is $DB$ to $BE$,

while, as $FB$ is to $BG$, so is $AB$ to $DG$,

and, as $DB$ is to $BE$, so is $DG$ to $BC$, [VI. 1]

therefore also, as $AB$ is to $DG$, so is $DG$ to $BC$. [V. 11]

Therefore $DG$ is a mean proportional between $AB$, $BC$.

I say next that $DC$ is also a mean proportional between $AC$, $CB$.

For since, as $AD$ is to $DK$, so is $KG$ to $GC$—

for they are equal respectively—

and, *componendo*, as $AK$ is to $KD$, so is $KC$ to $CG$, [V. 18]

while, as $AK$ is to $KD$, so is $AC$ to $CD$,

and, as $KC$ is to $CG$, so is $DC$ to $CB$, [VI. 1]

therefore also, as $AC$ is to $DC$, so is $DC$ to $BC$. [V. 11]

Therefore $DC$ is a mean proportional between $AC$, $CB$.

Being what it was proposed to prove.

PROPOSITION 54.

*If an area be contained by a rational straight line and the first binomial, the "side" of the area is the irrational straight line which is called binomial.*

For let the area $AC$ be contained by the rational straight line $AB$ and the first binomial $AD$;

I say that the "side" of the area $AC$ is the irrational straight line which is called binomial.

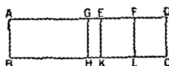
For, since $AD$ is a first binomial straight line, let it be divided into its terms at $E$,

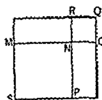
and let $AE$ be the greater term.

It is then manifest that $AE$, $ED$ are rational straight lines commensurable in square only,

the square on $AE$ is greater than the square on $ED$ by the square on a straight line commensurable with $AE$,
and $AE$ is commensurable in length with the rational straight line $AB$ set out. [x. Deff. ii. 1]

Let $ED$ be bisected at the point $F$.




Then since the square on $AE$ is greater than the square on $ED$ by the square on a straight line commensurable with $AE$,
therefore, if there be applied to the greater $AE$ a parallelogram equal to the fourth part of the square on the less, that is, to the square on $EF$, and deficient by a square figure, it divides it into commensurable parts. [x. 17]

Let then the rectangle $AG$, $GE$ equal to the square on $EF$ be applied to $AE$;
therefore $AG$ is commensurable in length with $EG$.

Let $GH$, $EK$, $FL$ be drawn from $G$, $E$, $F$ parallel to either of the straight lines $AB$, $CD$;
let the square $SN$ be constructed equal to the parallelogram $AH$, and the square $NQ$ equal to $GK$, [ii. 14]
and let them be placed so that $MN$ is in a straight line with $NO$;
therefore $RN$ is also in a straight line with $NP$.
And let the parallelogram $SQ$ be completed;
therefore $SQ$ is a square. [Lemma]

Now, since the rectangle $AG$, $GE$ is equal to the square on $EF$,
therefore, as $AG$ is to $EF$, so is $FE$ to $EG$; [vi. 17]
therefore also, as $AH$ is to $EL$, so is $EL$ to $KG$; [vi. 1]
therefore $EL$ is a mean proportional between $AH$, $GK$.
But $AH$ is equal to $SN$, and $GK$ to $NQ$;
therefore $EL$ is a mean proportional between $SN$, $NQ$.
But $MR$ is also a mean proportional between the same $SN$, $NQ$; [Lemma]
therefore $EL$ is equal to $MR$,
so that it is also equal to $PO$.
But $AH$, $GK$ are also equal to $SN$, $NQ$;
therefore the whole $AC$ is equal to the whole $SQ$, that is, to the square on $MO$;
therefore $MO$ is the "side" of $AC$.

I say next that $MO$ is binomial.
For, since $AG$ is commensurable with $GE$,
therefore $AE$ is also commensurable with each of the straight lines $AG$, $GE$. [x. 15]
But $AE$ is also by hypothesis commensurable with $AB$;
therefore $AG$, $GE$ are also commensurable with $AB$. [x. 12]
And $AB$ is rational;
therefore each of the straight lines $AG$, $GE$ is also rational;
therefore each of the rectangles $AH$, $GK$ is rational, [x. 19]

and $AH$ is commensurable with $GK$

But $AH$ is equal to $SN$, and $GK$ to $NQ$,

therefore $SN$, $NQ$ that is, the squares on $MN$, $NO$ are rational and commensurable

And since $AE$ is incommensurable in length with $ED$

while $AE$ is commensurable with $AG$, and $DE$ is commensurable with $EF$

therefore $AG$ is also incommensurable with $EF$ [X 13]

so that $AH$ is also incommensurable with $EL$ [VI 1, X 11]

But $AH$ is equal to $SN$, and $EL$ to $MR$,

therefore $SN$ is also incommensurable with $MR$

But, as $SN$ is to $MR$, so is $PN$ to $NR$, [VI 1]

therefore $PN$ is incommensurable with $NR$ [X 11]

But $PN$ is equal to $MN$, and $NR$ to $NO$,

therefore $MN$ is incommensurable with $NO$

And the square on $MN$ is commensurable with the square on $NO$

and each is rational,

therefore $MN$, $NO$ are rational straight lines commensurable in square only

Therefore $MO$ is binomial [X 36] and the 'side' of $AC$ Q E D

PROPOSITION 55

*If an area be contained by a rational straight line and the second binomial the 'side' of the area is the irrational straight line which is called a first bimedial*

For let the area $ABCD$ be contained by the rational straight line $AB$ and the second binomial $AD$,

I say that the 'side' of the area $AC$ is a first bimedial straight line

For, since $AD$ is a second binomial straight line let it be divided into its terms at $E$ so that $AE$ is the greater term

therefore $AE$, $ED$ are rational straight lines commensurable in square only, the square on $AE$ is greater than the square on $ED$ by the square on a straight line commensurable with $AE$

and the lesser term $ED$ is commensurable in length with $AB$ [X Deff II 2]

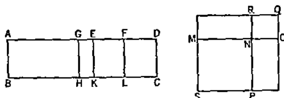
Let $ED$ be bisected at $F$

and let there be applied to $AE$ the rectangle $AG$, $GE$ equal to the square on $EF$ and deficient by a square figure

therefore $AG$ is commensurable in length with $GE$ [X 17]

Through $G$, $E$, $F$ let $GH$, $EK$, $FL$ be drawn parallel to $AB$, $CD$

let the square $SN$ be constructed equal to the parallelogram $AH$, and the square $NQ$ equal to $GK$




and let them be placed so that $MN$ is in a straight line with $NO$

therefore $RN$ is also in a straight line with $NP$

Let the square $SQ$ be completed

It is then manifest from what was proved before that $MR$ is a mean proportional between $SN$, $NQ$ and is equal to $EL$, and that $MO$ is the "side" of the area $AC$

It is now to be proved that $MO$ is a first bimedial straight line

Since $AE$ is commensurable in length with $ED$

while $ED$ is commensurable with $AB$

therefore $AE$ is incommensurable with $AB$ [x 13]

And since $AG$ is commensurable with $EG$,

$AE$ *is also commensurable with each of the straight lines* $AG$ $GE$ [x 15]

But $AE$ is incommensurable in length with $AB$,

therefore $AG$ $GE$ are also incommensurable with $AB$ [x 13]

Therefore $BA$ $AG$ and $BA$, $GE$ are pairs of rational straight lines commensurable in square only

so that each of the rectangles $AH$, $GK$ is medial [x 21]

Hence each of the squares $SN$ $NQ$ is medial

Therefore $MN$ $NO$ are also medial

And since $AG$ is commensurable in length with $GE$

$AH$ is also commensurable with $GK$, [vi 1, x 11]

that is $SN$ is commensurable with $NQ$

*that is the square on* $MN$ *with the square on* $NO$

And since $AE$ is incommensurable in length with $ED$,

while $AE$ is commensurable with $AG$,

and $ED$ is commensurable with $EF$

therefore $AG$ is incommensurable with $EF$, [x 13]

so that $AH$ is also incommensurable with $EL$,

that is $SN$ is incommensurable with $MR$,

that is $PN$ with $NR$, [vi 1, x 11]

that is $MN$ is incommensurable in length with $NO$

But $MN$ $NO$ were proved to be both medial and commensurable in square

therefore $MN$ $NO$ are medial straight lines commensurable in square only

I say next that they also contain a rational rectangle

For since $DE$ is by hypothesis commensurable with each of the straight lines $AB$ $EF$

therefore $EF$ is also commensurable with $EK$ [x 12]

And each of them is rational,

therefore $EL$ that is $MR$ is rational [x 19]

and $MR$ is the rectangle $MN$ $NO$

But if two medial straight lines commensurable in square only and containing a rational rectangle be added together the whole is irrational and is called a first bimedial straight line [x 37]

Therefore $MO$ is a first bimedial straight line Q E D

## Proposition 56

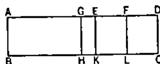
*If an area be contained by a rational straight line and the third binomial the 'side' of the area is the irrational straight line called a second bimedial*

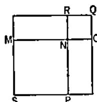
For let the area $ABCD$ be contained by the rational straight line $AB$ and the third binomial $AD$ divided into its terms at $E$ of which terms $AE$ is the greater

I say that the side of *the area* $AC$ *is the irrational straight line* called a second bimedial

For let the same construction be made as before

Now, since $AD$ is a third binomial straight line,




therefore $AE$, $ED$ are rational straight lines commensurable in square only, the square on $AE$ is greater than the square on $ED$ by the square on a straight line commensurable with $AE$,

and neither of the terms $AE$, $ED$ is commensurable in length with $AB$ [x Deff II 3]

Then in manner similar to the foregoing, we shall prove that $MO$ is the 'side' of the area $AC$

and $MN$ $NO$ are medial straight lines commensurable in square only,

so that $MO$ is bimedial

It is next to be proved that it is also a second bimedial straight line

Since $DE$ is incommensurable in length with $AB$ that is with $EK$

and $DE$ is commensurable with $EF$

therefore $EF$ is incommensurable in length with $EK$ [x 13]

And they are rational,

therefore $FE$ $EK$ are rational straight lines commensurable in square only

Therefore $EL$ that is $MR$ is medial [x 21]

And it is contained by $MN$, $NO$

therefore the rectangle $MN$ $NO$ is medial

Therefore $MO$ is a second bimedial straight line [x 38]

Q E D

## Proposition 57

*If an area be contained by a rational straight line and the fourth binomial the 'side' of the area is the irrational straight line called major*

For let the area $AC$ be contained by the rational straight line $AB$ and the fourth binomial $AD$ divided into its terms at $E$, of which terms let $AE$ be the greater

I say that the side of the area $AC$ is the irrational straight line called major

For since $AD$ is a fourth binomial straight line

therefore $AE$ $ED$ are rational straight lines commensurable in square only the square on $AE$ is greater than the square on $ED$ by the square on a straight line incommensurable with $AE$

and $AE$ is commensurable in length with $AB$ [x Deff II 4]

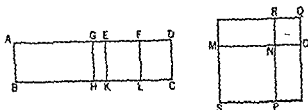
Let $DE$ be bisected at $F$

and let there be applied to $AE$ a parallelogram the rectangle $AG$, $GE$ equal to the square on $EF$

therefore $AG$ is incommensurable in length with $GE$ [x 18]

Let $GH$ $EK$ $FL$ be drawn parallel to $AB$

and let the rest of the construction be as before
it is then manifest that $MO$ is the "side" of the area $AC$



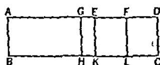

It is next to be proved that $MO$ is the irrational straight line called major
Since $AG$ is incommensurable with $EG$
$AH$ is also incommensurable with $GK$ that is $SN$ with $NQ$ [vi 1 x 11]
therefore $MN$ $NO$ are incommensurable in square
And since $AE$ is commensurable with $AB$
$AK$ is rational [x 19]
and it is equal to the squares on $MN$ $NO$
therefore the sum of the squares on $MN$ $NO$ is also rational
And, since $DE$ is incommensurable in length with $AB$ that is with $EK$
while $DE$ is commensurable with $EF$,
therefore $EF$ is incommensurable in length with $EK$ [x 13]
Therefore $EK$ $EF$ are rational straight lines commensurable in square only
therefore $LE$ that is, $MR$, is medial [x 21]
And it is contained by $MN$ $NO$,
therefore the rectangle $MN$ $NO$ is medial
And the [sum] of the squares on $MN$ $NO$ is rational
and $MN$ $NO$ are incommensurable in square
But if two straight lines incommensurable in square and making the sum of the squares on them rational but the rectangle contained by them medial be added together the whole is irrational and is called major [x 39]
Therefore $MO$ is the irrational straight line called major and is the "side" of the area $AC$ Q E D

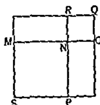
## Proposition 58

*If an area be contained by a rational straight line and the fifth binomial the "side" of the area is the irrational straight line called the side of a rational plus a medial area*

For let the area $AC$ be contained by the rational straight line $AB$ and the fifth binomial $AD$ divided into its terms at $E$ so that $AE$ is the greater term,
I say that the "side" of the area $AC$ is the irrational straight line called the side of a rational plus a medial area
For let the same construction be made as before shown,
it is then manifest that $MO$ is the "side" of the area $AC$
It is then to be proved that $MO$ is the side of a rational plus a medial area
For since $AG$ is incommensurable with $GE$ [x 18]
therefore $AH$ is also commensurable with $HE$ [vi 1, x 11]
that is the square on $MN$ with the square on $NO$,
therefore $MN$ $NO$ are incommensurable in square

And since $AD$ is a fifth binomial straight line and $ED$ the lesser segment therefore $ED$ is commensurable in length with $AB$ [X. Deff II 5]




But $AE$ is incommensurable with $ED$,

therefore $AB$ is also incommensurable in length with $AE$ [X 13]

Therefore $AK$ that is the sum of the squares on $MN$ $NO$, is medial [X 21]

And, since $DE$ is commensurable in length with $AB$, that is, with $FK$,

while $DE$ is commensurable with $EF$,

therefore $EF$ is also commensurable with $EK$ [X 12]

And $EK$ is rational,

therefore $EL$ that is, $MR$, that is the rectangle $MN$ $NO$, is also rational [X 19]

Therefore $MN$, $NO$ are straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational

Therefore $MO$ is the side of a rational plus a medial area [X 40] and is the "side" of the area $AC$ Q E D

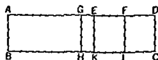
## Proposition 59

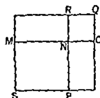
*If an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the irrational straight line called the side of the sum of two medial areas*

For let the area $ABCD$ be contained by the rational straight line $AB$ and the sixth binomial $AD$, divided into its terms at $E$ so that $AE$ is the greater term

I say that the 'side' of $AC$ is the side of the sum of two medial areas

Let the same construction be made as before shown




It is then manifest that $MO$ is the 'side' of $AC$ and that $MN$ is incommensurable in square with $NO$

Now since $EA$ is incommensurable in length with $AB$

therefore $EA$ $AB$ are rational straight lines commensurable in square only

therefore $AK$ that is the sum of the squares on $MN$ $NO$ is medial [X 21]

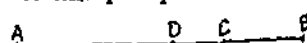
Again since $ED$ is incommensurable in length with $AB$,

therefore $FE$ is also incommensurable with $EK$, [x 13]
therefore $FE$, $EK$ are rational straight lines commensurable in square only,
therefore $EL$ that is $MR$, that is the rectangle $MN$, $NO$ is medial [x 21]
And since $AE$ is incommensurable with $EF$
$AK$ is also incommensurable with $EL$ [vi 1 x 11]
But $AK$ is the sum of the squares on $MN$, $NO$
and $EL$ is the rectangle $MN$, $NO$
therefore the sum of the squares on $MN$, $NO$ is incommensurable with the rectangle $MN$ $NO$
And each of them is medial, and $MN$ $NO$ are incommensurable in square
Therefore $MO$ is the side of the sum of two medial areas [x 41] and is the side ' of $AC$
Q E D

## Lemma

If a straight line be cut into unequal parts, the squares on the unequal parts are greater than twice the rectangle contained by the unequal parts

Let $AB$ be a straight line and let it be cut into unequal parts at $C$ and let $AC$ be the greater




I say that the squares on $AC$ $CB$ are greater than twice the rectangle $AC$, $CB$
For let $AB$ be bisected at $D$
Since then, a straight line has been cut into equal parts at $D$, and into unequal parts at $C$
therefore the rectangle $AC$, $CB$ together with the square on $CD$ is equal to the square on $AD$ [ii 5]
so that the rectangle $AC$ $CB$ is less than the square on $AD$,
therefore twice the rectangle $AC$, $CB$ is less than double of the square on $AD$
But the squares on $AC$, $CB$ are double of the squares on $AD$ $DC$, [ii 9]
therefore the squares on $AC$, $CB$ are greater than twice the rectangle $AC$ $CB$
Q E D

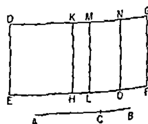
## Proposition 60

*The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial*

Let $AB$ be a binomial straight line divided into its terms at $C$ so that $AC$ is the greater term

let a rational straight line $DF$ be set out
and let $DEFG$ equal to the square on $AB$ be applied to $DE$ producing $DG$ as its breadth

I say that $DG$ is a first binomial straight line




For let there be applied to $DE$ the rectangle $DH$ equal to the square on $AC$ and $KL$ equal to the square on $BC$
therefore the remainder twice the rectangle $AC$ $CB$ is equal to $MF$
Let $MG$ be bisected at $N$ and let $NO$ be drawn parallel [to $ML$ or $GF$]
Therefore each of the rectangles $MO$ $NF$ is equal to once the rectangle $AC$ $CB$
Now, since $AB$ is a binomial divided into its terms at $C$

therefore $AC$, $CB$ are rational straight lines commensurable in square only, [x 36]

therefore the squares on $AC$, $CB$ are rational and commensurable with one another

so that the sum of the squares on $AC$, $CB$ is also rational [x 15]

And it is equal to $DL$,

therefore $DL$ is rational

And it is applied to the rational straight line $DE$,

therefore $DM$ is rational and commensurable in length with $DE$ [x 20]

Again since $AC$ $CB$ are rational straight lines commensurable in square only

therefore twice the rectangle $AC$, $CB$ that is $MF$ is medial [x 21]

And it is applied to the rational straight line $ML$,

therefore $MG$ is also rational and incommensurable in length with $ML$, that is $DE$ [x 22]

But $MD$ is also rational and is commensurable in length with $DE$,

therefore $DM$ is incommensurable in length with $MG$ [x 13]

And they are rational

therefore $DM$ $MG$ are rational straight lines commensurable in square only

therefore $DG$ is binomial [x 36]

It is next to be proved that it is also a first binomial straight line

Since the rectangle $AC$, $CB$ is a mean proportional between the squares on $AC$ $CB$ [cf Lemma after x 53]

therefore $MO$ is also a mean proportional between $DH$ $KL$

Therefore, as $DH$ is to $MO$ so is $MO$ to $KL$

that is as $DK$ is to $MN$ so is $MK$ to $MK$ [vi 1]

therefore the rectangle $DK$ $KM$ is equal to the square on $MK$ [vi 17]

And since the square on $AC$ is commensurable with the square on $CB$

$DH$ is also commensurable with $KL$

so that $DK$ is also commensurable with $KM$ [vi 1 x 11]

And, since the squares on $AC$, $CB$ are greater than twice the rectangle $AC$ $CB$, [Lemma]

therefore $DL$ is also greater than $MF$,

so that $DM$ is also greater than $MG$ [vi 1]

And the rectangle $DK$, $KM$ is equal to the square on $MN$ that is to the fourth part of the square on $MG$

and $DK$ is commensurable with $KM$

But if there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure and if it divide it into commensurable parts the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater [x 17]

therefore the square on $DM$ is greater than the square on $MG$ by the square on a straight line commensurable with $DM$

And $DM$ $MG$ are rational

and $DM$, which is the greater term is commensurable in length with the rational straight line $DE$ set out

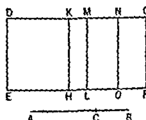
Therefore $DG$ is a first binomial straight line [x Deff ii 1]

Q E D

PROPOSITION 61

*The square on the first bimedial straight line applied to a rational straight line produces as breadth the second binomial*

Let $AB$ be a first bimedial straight line divided into its medials at $C$, of which medials $AC$ is the greater,




let a rational straight line $DE$ be set out, and let there be applied to $DE$ the parallelogram $DF$ equal to the square on $AB$, producing $DG$ as its breadth

I say that $DG$ is a second binomial straight line

For let the same construction as before be made

Then, since $AB$ is a first bimedial divided at $C$, therefore $AC$ $CB$ are medial straight lines commensurable in square only and containing a rational rectangle, [x 37]

so that the squares on $AC$, $CB$ are also medial [x 21]

Therefore $DL$ is medial [x 15 and 13, Por]

And it has been applied to the rational straight line $DE$,

therefore $MD$ is rational and incommensurable in length with $DE$ [x 22]

Again since twice the rectangle $AC$ $CB$ is rational $MF$ is also rational

And it is applied to the rational straight line $ML$

therefore $MG$ is also rational and commensurable in length with $ML$ that is $DE$ [x 20]

therefore $DM$ is incommensurable in length with $MG$ [x 13]

And they are rational

therefore $DM$ $MG$ are rational straight lines commensurable in square only

therefore $DG$ is binomial [x 36]

It is next to be proved that it is also a second binomial straight line

For since the squares on $AC$ $CB$ are greater than twice the rectangle $AC$, $CB$

therefore $DL$ is also greater than $MF$

so that $DM$ is also greater than $MG$ [vi 1]

And since the square on $AC$ is commensurable with the square on $CB$,

$DH$ is also commensurable with $KL$

so that $DK$ is also commensurable with $KM$ [vi 1 x 11]

And the rectangle $DK$ $KM$ is equal to the square on $MN$,

therefore the square on $DM$ is greater than the square on $MG$ by the square on a straight line commensurable with $DM$ [x 17]

And $MG$ is commensurable in length with $DE$

Therefore $DG$ is a second binomial straight line [x Deff ii 2]

PROPOSITION 62

*The square on the second bimedial straight line applied to a rational straight line produces as breadth the third binomial*

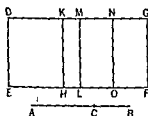
Let $AB$ be a second bimedial straight line divided into its medials at $C$, so that $AC$ is the greater segment

let $DE$ be any rational straight line

and to $DE$ let there be applied the parallelogram $DF$ equal to the square on $AB$ and producing $DG$ as its breadth,

I say that $DG$ is a third binomial straight line

Let the same construction be made as before shown




Then, since $AB$ is a second bimedial divided at $C$,

therefore $AC$, $CB$ are medial straight lines commensurable in square only and containing a medial rectangle [x 35]

so that the sum of the squares on $AC$, $CB$ is also medial [x 15 and 23 Por]

And it is equal to $DL$

therefore $DL$ is also medial

And it is applied to the rational straight line $DE$

therefore $MD$ is also rational and incommensurable in length with $DE$ [x 22]

For the same reason

$MG$ is also rational and incommensurable in length with $ML$ that is, with $DE$,

therefore each of the straight lines $DM$ $MG$ is rational and incommensurable in length with $DE$

*And since $AC$ is incommensurable in length with $CB$,*

and as $AC$ is to $CB$ so is the square on $AC$ to the rectangle $AC$ $CB$,

therefore the square on $AC$ is also incommensurable with the rectangle $AC$, $CB$ [x 11]

Hence the sum of the squares on $AC$, $CB$ is incommensurable with twice the rectangle $AC$, $CB$, [x 12 13]

that is, $DL$ is incommensurable with $MF$,

so that $DM$ is also incommensurable with $MG$ [vi 1, x 11]

And they are rational,

therefore $DG$ is binomial [x 36]

It is to be proved that it is also a third binomial straight line

In manner similar to the foregoing we may conclude that $DM$ is greater than $MG$

and that $DK$ is commensurable with $KM$

And the rectangle $DK$ $KM$ is equal to the square on $MN$,

therefore the square on $DM$ is greater than the square on $MG$ by the square on a straight line commensurable with $DM$

And neither of the straight lines $DM$ $MG$ is commensurable in length with $DE$

Therefore $DG$ is a third binomial straight line [x Def ii 3]

Q E D

## Proposition 63

*The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial*

Let $AB$ be a major straight line divided at $C$ so that $AC$ is greater than $CB$

let $DE$ be a rational straight line

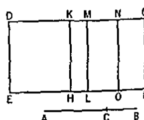
and to $DE$ let there be applied the parallelogram $DF$ equal to the square on $AB$ and producing $DG$ as its breadth

I say that $DG$ is a fourth binomial straight line

Let the same construction be made as before shown

Then since $AB$ is a major straight line divided at $C$,

$AC$, $CB$ are straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial [x 39]




Since, then the sum of the squares on $AC$, $CB$ is rational therefore $DL$ is rational therefore $DM$ is also rational and commensurable in length with $DE$ [x 20]

Again since twice the rectangle $AC$, $CB$, that is $MF$ is medial, and it is applied to the rational straight line $ML$ therefore $MG$ is also rational and incommensurable in length with $DE$, [x 22]

therefore $DM$ is also incommensurable in length with $MG$ [x 13]

Therefore $DM$ $MG$ are rational straight lines commensurable in square only

therefore $DG$ is binomial [x 36]

It is to be proved that it is also a fourth binomial straight line

*In manner similar to the foregoing we can prove that* $DM$ *is greater than* $MG$

and that the rectangle $DK$ $KM$ is equal to the square on $MN$

Since then the square on $AC$ is incommensurable with the square on $CB$,

therefore $DH$ is also incommensurable with $KL$,

so that $DK$ is also incommensurable with $KM$ [vi 1 x 11]

But if there be two unequal straight lines and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure and if it divide it into incommensurable parts then the square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater [x 18]

therefore the square on $DM$ is greater than the square on $MG$ by the square on a straight line incommensurable with $DM$

And $DM$ $MG$ are rational straight lines commensurable in square only and $DM$ is commensurable with the rational straight line $DE$ set out

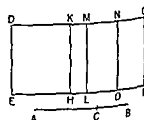
Therefore $DG$ is a fourth binomial straight line [x Deff II 4]

Q E D

## Proposition 64

*The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial*

Let $AB$ be the side of a rational plus a medial area divided into its straight lines at $C$ so that $AC$ is the greater,




let a rational straight line $DE$ be set out and let there be applied to $DE$ the parallelogram $DF$ equal to the square on $AB$ producing $DG$ as its breadth,

I say that $DG$ is a fifth binomial straight line

Let the same construction as before be made

Since then $AB$ is the side of a rational plus a medial area divided at $C$

therefore $AC$ $CB$ are straight lines incommensurable in square which make

the sum of the squares on them medial but the rectangle contained by them rational [x 40]

Since, then the sum of the squares on $AC$ $CB$ is medial

therefore $DL$ is medial

so that $DM$ is rational and incommensurable in length with $DE$ [x 22]

Again, since twice the rectangle $AC$, $CB$, that is $MF$ is rational

therefore $MG$ is rational and commensurable with $DE$ [x 20]

Therefore $DM$ is incommensurable with $MG$ [x 13]

therefore $DM$, $MG$ are rational straight lines commensurable in square only

therefore $DG$ is binomial [x 36]

I say next that it is also a fifth binomial straight line

For it can be proved similarly that the rectangle $DK$ $KM$ is equal to the square on $MN$

*and that $DK$ is incommensurable in length with $KM$,*

therefore the square on $DM$ is greater than the square on $MG$ by the square on a straight line incommensurable with $DM$ [x 18]

And $DM$, $MG$ are commensurable in square only, and the less $MG$, is commensurable in length with $DE$

Therefore $DG$ is a fifth binomial Q E D

## Proposition 65

*The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial*

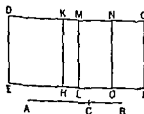
Let $AB$ be the side of the sum of two medial areas, divided at $C$

let $DE$ be a rational straight line

and let there be applied to $DE$ the parallelogram $DF$ equal to the square on $AB$ producing $DG$ as its breadth,

I say that $DG$ is a sixth binomial straight line

*For let the same construction be made as before*




Then since $AB$ is the side of the sum of two medial areas divided at $C$

therefore $AC$ $CB$ are straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and moreover the sum of the squares on them incommensurable with the rectangle contained by them [x 41]

so that in accordance with what was before proved each of the rectangles $DL$ $MF$ is medial

And they are applied to the rational straight line $DE$

therefore each of the straight lines $DM$ $MG$ is rational and incommensurable in length with $DE$ [x 22]

And since the sum of the squares on $AC$, $CB$ is incommensurable with twice the rectangle $AC$ $CB$

therefore $DL$ is incommensurable with $MF$

Therefore $DM$ is also incommensurable with $MG$ [vi 1 x 11]

therefore $DM$ $MG$ are rational straight lines commensurable in square only

*therefore $DG$ is binomial* [x 36]

I say next that it is also a sixth binomial straight line

Similarly again we can prove that the rectangle $DK$, $KM$ is equal to the square on $MN$,

and that $DK$ is incommensurable in length with $KM$,

and for the same reason the square on $DM$ is greater than the square on $MG$ by the square on a straight line incommensurable in length with $DM$

And neither of the straight lines $DM$ $MG$ is commensurable in length with the rational straight line $DE$ set out

Therefore $DG$ is a sixth binomial straight line Q E D

PROPOSITION 66

*A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order*

Let $AB$ be binomial and let $CD$ be commensurable in length with $AB$,

I say that $CD$ is binomial and the same in order with $AB$

For since $AB$ is binomial

let it be divided into its terms at $E$

and let $AE$ be the greater term,

therefore $AE$ $EB$ are rational straight lines commensurable in square only [x 36]

Let it be contrived that

as $AB$ is to $CD$ so is $AE$ to $CF$, [vi 12]

therefore also the remainder $EB$ is to the remainder $FD$ as $AB$ is to $CD$ [v 19]

But $AB$ is commensurable in length with $CD$

therefore $AE$ is also commensurable with $CF$, and $EB$ with $FD$ [x 11]

And $AE$ $EB$ are rational

therefore $CF$ $FD$ are also rational

And as $AE$ is to $CF$ so is $EB$ to $FD$ [v 11]

Therefore alternately as $AE$ is to $EB$ so is $CF$ to $FD$ [v 16]

But $AE$ $EB$ are commensurable in square only

therefore $CF$ $FD$ are also commensurable in square only [x 11]

And they are rational

therefore $CD$ is binomial [x 36]

I say next that it is the same in order with $AB$

For the square on $AE$ is greater than the square on $EB$ either by the square on a straight line commensurable with $AE$ or by the square on a straight line incommensurable with it

If then the square on $AE$ is greater than the square on $EB$ by the square on a straight line commensurable with $AE$

the square on $CF$ will also be greater than the square on $FD$ by the square on a straight line commensurable with $CF$ [x 14]

And if $AE$ is commensurable with the rational straight line set out, $CF$ will also be commensurable with it [x 12]

and for this reason each of the straight lines $AB$ $CD$ is a first binomial that is the same in order [x Deff II 1]

But if $EB$ is commensurable with the rational straight line set out $FD$ is also commensurable with it [x 12]

and for this reason again $CD$ will be the same in order with $AB$

for each of them will be a second binomial [x Deff II 2]

But, if neither of the straight lines $AE$, $EB$ is commensurable with the rational straight line set out neither of the straight lines $CF$, $FD$ will be commensurable with it [x 13]
and each of the straight lines $AB$, $CD$ is a third binomial [x Deff II 3]

But, if the square on $AE$ is greater than the square on $EB$ by the square on a straight line incommensurable with $AE$,
the square on $CF$ is also greater than the square on $FD$ by the square on a straight line incommensurable with $CF$ [x 14]

And if $AE$ is commensurable with the rational straight line set out, $CF$ is also commensurable with it,
and each of the straight lines $AB$, $CD$ is a fourth binomial [x Deff II 4]

But if $EB$ is so commensurable, so is $FD$ also
and each of the straight lines $AB$ $CD$ will be a fifth binomial [x Deff II 5]

But if neither of the straight lines $AE$ $EB$ is so commensurable, neither of the straight lines $CF$, $FD$ is commensurable with the rational straight line set out
and each of the straight lines $AB$ $CD$ will be a sixth binomial [x Deff II 6]

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order Q E D

PROPOSITION 67

*A straight line commensurable in length with a bimedial straight line is itself also bimedial and the same in order*

Let $AB$ be bimedial, and let $CD$ be commensurable in length with $AB$,

A E B
C F D

I say that $CD$ is bimedial and the same in order with $AB$

For, since $AB$ is bimedial,
let it be divided into its medials at $E$,
therefore $AE$ $EB$ are medial straight lines commensurable in square only [x 37 38]

And let it be contrived that
as $AB$ is to $CD$ so is $AE$ to $CF$,
therefore also the remainder $EB$ is to the remainder $FD$ as $AB$ is to $CD$ [v 19]

But $AB$ is commensurable in length with $CD$,
therefore $AE$ $EB$ are also commensurable with $CF$ $FD$ respectively [x 11]

But $AE$, $EB$ are medial,
therefore $CF$ $FD$ are also medial [x 23]

And since, as $AE$ is to $EB$ so is $CF$ to $FD$ [v 11]
and $AE$, $EB$ are commensurable in square only,
$CF$ $FD$ are also commensurable in square only [x 11]

But *they were also proved medial*
therefore $CD$ is bimedial

I say next that it is also the same in order with $AB$

For since as $AE$ is to $EB$ so is $CF$ to $FD$
therefore also as the square on $AE$ is to the rectangle $AE$, $EB$, so is the square on $CF$ to the rectangle $CF$ $FD$,
therefore alternately
as the square on $AE$ is to the square on $CF$ so is the rectangle $AE$ $EB$ to the rectangle $CF$ $FD$ [v 16]

But the square on $AE$ is commensurable with the square on $CF$,

therefore the rectangle $AE$, $EB$ is also commensurable with the rectangle $CF$, $FD$

If therefore the rectangle $AE$ $EB$ is rational the rectangle $CF$, $FD$ is also rational

[and for this reason $CD$ is a first bimedial], [x 37]

but if medial medial [x 23, Por]

and each of the straight lines $AB$, $CD$ is a second bimedial [x 38]

And for this reason $CD$ will be the same in order with $AB$ Q E D

PROPOSITION 68

*A straight line commensurable with a major straight line is itself also major*

Let $AB$ be major and let $CD$ be commensurable with $AB$,

I say that $CD$ is major

Let $AB$ be divided at $E$,

therefore $AE$, $EB$ are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial [x 39]

Let the same construction be made as before

Then since as $AB$ is to $CD$ so is $AE$ to $CF$ and $EB$ to $FD$,

therefore also as $AE$ is to $CF$, so is $EB$ to $FD$ [v 11]

But $AB$ is commensurable with $CD$,

therefore $AE$, $EB$ are also commensurable with $CF$, $FD$ respectively [x 11]

And since as $AE$ is to $CF$ so is $EB$ to $FD$,

alternately also

as $AE$ is to $EB$, so is $CF$ to $FD$, [v 16]

therefore also *componendo*

as $AB$ is to $BE$, so is $CD$ to $DF$, [v 18]

therefore also as the square on $AB$ is to the square on $BE$, so is the square on $CD$ to the square on $DF$ [vi 20]

Similarly we can prove that as the square on $AB$ is to the square on $AE$ so also is the square on $CD$ to the square on $CF$

Therefore also as the square on $AB$ is to the squares on $AE$ $EB$ so is the square on $CD$ to the squares on $CF$ $FD$,

therefore also alternately

as the square on $AB$ is to the square on $CD$ so are the squares on $AE$, $EB$ to the squares on $CF$ $FD$ [v 16]

But the square on $AB$ is commensurable with the square on $CD$,

therefore the squares on $AE$ $EB$ are also commensurable with the squares on $CF$ $FD$

And the squares on $AE$ $EB$ together are rational,

therefore the squares on $CF$ $FD$ together are rational

Similarly also twice the rectangle $AE$, $EB$ is commensurable with twice the rectangle $CF$ $FD$

And twice the rectangle $AE$ $EB$ is medial,

therefore twice the rectangle $CF$ $FD$ is also medial [x 23 Por]

Therefore $CF$ $FD$ are straight lines incommensurable in square which make at the same time the sum of the squares on them rational but the rectangle contained by them medial therefore the whole $CD$ is the irrational straight line called major [x 39]

Therefore a straight line commensurable with the major straight line is major Q E D

## Proposition 69

*A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area*

Let $AB$ be the side of a rational plus a medial area and let $CD$ be commensurable with $AB$

it is to be proved that $CD$ is also the side of a rational plus a medial area

Let $AB$ be divided into its straight lines at $E$,

therefore $AE$ $EB$ are straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational [X 40]

Let the same construction be made as before

We can then prove similarly that

$CF$ $FD$ are incommensurable in square

and the sum of the squares on $AE$, $EB$ is commensurable with the sum of the squares on $CF$ $FD$

and the rectangle $AE$, $EB$ with the rectangle $CF$, $FD$,

so that the sum of the squares on $CF$ $FD$ is also medial, and the rectangle $CF$ $FD$ rational

Therefore $CD$ is the side of a rational plus a medial area Q E D

## Proposition 70

*A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas*

Let $AB$ be the side of the sum of two medial areas and $CD$ commensurable with $AB$

it is to be proved that $CD$ is also the side of the sum of two medial areas

For since $AB$ is the side of the sum of two medial areas let it be divided into its straight lines at $E$,

therefore $AE$ $EB$ are straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and furthermore the sum of the squares on $AE$, $EB$ incommensurable with the rectangle $AE$ $EB$ [X 41]

Let the same construction be made as before

We can then prove similarly that

$CF$ $FD$ are also incommensurable in square

the sum of the squares on $AE$ $EB$ is commensurable with the sum of the squares on $CF$ $FD$

and the rectangle $AE$ $EB$ with the rectangle $CF$ $FD$

so that the sum of the squares on $CF$ $FD$ is also medial

the rectangle $CF$ $FD$ is medial

and moreover the sum of the squares on $CF$ $FD$ is incommensurable with the rectangle $CF$ $FD$

Therefore $CD$ is the side of the sum of two medial areas Q E D

PROPOSITION 71

*If a rational and a medial area be added together, four irrational straight lines arise namely a binomial or a first bimedial or a major or a side of a rational plus a medial area*

Let $AB$ be rational and $CD$ medial

I say that the 'side' of the area $AD$ is a binomial or a first bimedial or a major or a side of a rational plus a medial area

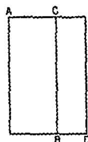
For $AB$ is either greater or less than $CD$

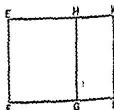
First let it be greater

let a rational straight line $EF$ be set out

let there be applied to $EF$ the rectangle $EG$ equal to $AB$, producing $EH$ as breadth

and let $HI$ equal to $DC$ be applied to $EF$ producing $HK$ as breadth




Then since $AB$ is rational and is equal to $EG$

therefore $EG$ is also rational

And it has been applied to $EF$ producing $EH$ as breadth

therefore $EH$ is rational and commensurable in length with $EF$ [x 20]

Again since $CD$ is medial and is equal to $HI$

therefore $HI$ is also medial

And it is applied to the rational straight line $EF$ producing $HK$ as breadth,

therefore $HK$ is rational and incommensurable in length with $EF$ [x [illegible]]

And since $CD$ is medial

while $AB$ is rational

therefore $AB$ is incommensurable with $CD$,

so that $EG$ is also incommensurable with $HI$

But as $EG$ is to $HI$ so is $EH$ to $HK$ [vi 1]

therefore $EH$ is also incommensurable in length with $HK$ [x 11]

And both are rational

therefore $EH$ $HK$ are rational straight lines commensurable in square only

therefore $EK$ is a binomial straight line divided at $H$ [x 36]

*And since $AB$ is greater than $CD$*

while $AB$ is equal to $EG$ and $CD$ to $HI$

therefore $EG$ is also greater than $HI$

therefore $EH$ is also greater than $HK$

The square then on $EH$ is greater than the square on $HK$ either by the square on a straight line commensurable in length with $EH$ or by the square on a straight line incommensurable with it

First let the square on it be greater by the square on a straight line commensurable with itself

Now the greater straight line $HE$ is commensurable in length with the rational straight line $EF$ set out

therefore $EK$ is a first binomial [x Deff ii 1]

But $EF$ is rational

and if an area be contained by a rational straight line and the first binomial the side of the square equal to the area is binomial [x 54]

Therefore the "side" of $EI$ is binomial

so that the "side" of $AD$ is also binomial

Next let the square on $EH$ be greater than the square on $HK$ by the square on a straight line incommensurable with $EH$

Now the greater straight line $EH$ is commensurable in length with the rational straight line $EF$ set out

therefore $EK$ is a fourth binomial [x Deff II 4]

But $EF$ is rational

and if an area be contained by a rational straight line and the fourth binomial, the 'side' of the area is the irrational straight line called major [x 57]

Therefore the 'side' of the area $EI$ is major,

so that the "side' of the area $AD$ is also major

Next, let $AB$ be less than $CD$,

therefore $EG$ is also less than $HI$,

so that $EH$ is also less than $HK$

Now the square on $HK$ is greater than the square on $EH$ either by the square on a straight line commensurable with $HK$ or by the square on a straight line incommensurable with it

First let the square on it be greater by the square on a straight line commensurable in length with itself

Now the lesser straight line $EH$ is commensurable in length with the rational straight line $EF$ set out,

therefore $EK$ is a second binomial [x Deff II 2]

But $EF$ is rational

and if an area be contained by a rational straight line and the second binomial the side of the square equal to it is a first bimedial, [x 55]

therefore the 'side of the area $EI$ is a first bimedial

so that the 'side of $AD$ is also a first bimedial

Next let the square on $HK$ be greater than the square on $HE$ by the square on a straight line incommensurable with $HK$

Now the lesser straight line $EH$ is commensurable with the rational straight line $EF$ set out,

therefore $EK$ is a fifth binomial [x Deff II 5]

But $EF$ is rational

and, if an area be contained by a rational straight line and the fifth binomial the side of the square equal to the area is a side of a rational plus a medial area [x 58]

Therefore the 'side of the area $EI$ is a side of a rational plus a medial area so that the 'side of the area $AD$ is also a side of a rational plus a medial area

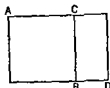
*Therefore etc* Q E D

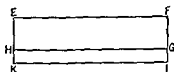
## Proposition 72

*If two medial areas incommensurable with one another be added together, the remaining two irrational straight lines arise namely either a second bimedial or a side of the sum of two medial areas*

For let two medial areas $AB$, $CD$ incommensurable with one another be added together,

I say that the "side" of the area $AD$ is either a second bimedial or a side of the sum of two medial areas




For $AB$ is either greater or less than $CD$

First if it so chance let $AB$ be greater than $CD$

Let the rational straight line $EF$ be set out

and to $EF$ let there be applied the rectangle $EG$ equal to $AB$ and producing $EH$ as breadth and the rectangle $HI$ equal to $CD$ and producing $HK$ as breadth.

Now since each of the areas $AB$, $CD$ is medial

therefore each of the areas $EG$ $HI$ is also medial

And they are applied to the rational straight line $FE$ producing $EH$, $HK$ as breadth

therefore each of the straight lines $EH$, $HK$ is rational and incommensurable in length with $EF$ [x 22]

And, since $AB$ is incommensurable with $CD$,

and $AB$ is equal to $EG$, and $CD$ to $HI$,

therefore $EG$ is also incommensurable with $HI$

But as $EG$ is to $HI$, so is $EH$ to $HK$ [vi 1]

therefore $EH$ is incommensurable in length with $HK$ [x 11]

Therefore $EH$, $HK$ are rational straight lines commensurable in square only,

therefore $EK$ is binomial [x 36]

But the square on $EH$ is greater than the square on $HK$ either by the square on a straight line commensurable with $EH$ or by the square on a straight line incommensurable with it

First let the square on it be greater by the square on a straight line commensurable in length with itself

Now neither of the straight lines $EH$ $HK$ is commensurable in length with the rational straight line $EF$ set out

therefore $EK$ is a third binomial [x Deff II 3]

But $FF$ is rational,

and if an area be contained by a rational straight line and the third binomial the "side" of the area is a second bimedial [x 56]

therefore the "side" of $EI$ that is of $AD$ is a second bimedial

Next let the square on $EH$ be greater than the square on $HK$ by the square on a straight line incommensurable in length with $EH$

Now each of the straight lines $EH$, $HK$ is incommensurable in length with $EF$,

therefore $EK$ is a sixth binomial [x Deff II 6]

But, if an area be contained by a rational straight line and the sixth bi

nomial the 'side' of the area is the side of the sum of two medial areas [X. 59] so that the "side" of the area $AD$ is also the side of the sum of two medial areas

Therefore etc. Q. E. D.

The binomial straight line and the irrational straight lines after it are neither the same with the medial nor with one another

For the square on a medial if applied to a rational straight line produces as breadth a straight line rational and incommensurable in length with that to which it is applied [X. 22]

But the square on the binomial if applied to a rational straight line produces as breadth the first binomial [X. 60]

The square on the first bimedial if applied to a rational straight line produces as breadth the second binomial [X. 61]

The square on the second bimedial if applied to a rational straight line produces as breadth the third binomial [X. 62]

The square on the major if applied to a rational straight line, produces as breadth the fourth binomial [X. 63]

The square on the side of a rational plus a medial area, if applied to a rational straight line produces as breadth the fifth binomial [X. 64]

The square on the side of the sum of two medial areas if applied to a rational straight line produces as breadth the sixth binomial [X. 65]

And the said breadths differ both from the first and from one another from the first because it is rational and from one another because they are not the same in order

so that the irrational straight lines themselves also differ from one another

## PROPOSITION 73

*If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only the remainder is irrational and let it be called an* apotome

For from the rational straight line $AB$ let the rational straight line $BC$ commensurable with the whole in square only be subtracted,

A C B

I say that the remainder $AC$ is the irrational straight line called *apotome*

For, since $AB$ is incommensurable in length with $BC$,

and as $AB$ is to $BC$ so is the square on $AB$ to the rectangle $AB$ $BC$,

therefore the square on $AB$ is incommensurable with the rectangle $AB$ $BC$ [X. 11]

But the squares on $AB$ $BC$ are commensurable with the square on $AB$ [X. 15]

and twice the rectangle $AB$ $BC$ is commensurable with the rectangle $AB$ $BC$ [X. 6]

And inasmuch as the squares on $AB$, $BC$ are equal to twice the rectangle $AB$, $BC$ together with the square on $CA$ [II. 7]

therefore the squares on $AB$ $BC$ are also incommensurable with the remainder the square on $AC$ [X. 13 16]

But the squares on $AB$ $BC$ are rational

therefore $AC$ is irrational [X. Def 4]

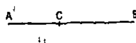
And let it be called an *apotome* Q. E. D.

PROPOSITION 74

*If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle the remainder is irrational And let it be called a* first apotome of a medial *straight line*

For from the medial straight line $AB$ let there be subtracted the medial straight line $BC$ which is commensurable with $AB$ in square only and with $AB$ makes the rectangle $AB$ $BC$ rational,

I say that the remainder $AC$ is irrational and let it be called a *first apotome of a medial* straight line



For since $AB$ $BC$ are medial

the squares on $AB$, $BC$ are also medial

But twice the rectangle $AB$, $BC$ is rational,

therefore the squares on $AB$, $BC$ are incommensurable with twice the rectangle $AB$, $BC$,

therefore twice the rectangle $AB$, $BC$ is also incommensurable with the remainder the square on $AC$, [cf II 7]

since if the whole is incommensurable with one of the magnitudes, the original magnitudes will also be incommensurable [x 16]

But twice the rectangle $AB$, $BC$ is rational

therefore the square on $AC$ is irrational,

therefore $AC$ is irrational [x Def 4]

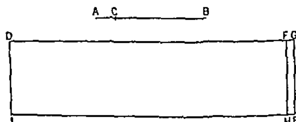
And let it be called a *first apotome of a medial* straight line Q E D

PROPOSITION 75

*If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only and which contains with the whole a medial rectangle the remainder is irrational and let it be called a* second apotome of a medial *straight line*

For from the medial straight line $AB$ let there be subtracted the medial straight line $CB$ which is commensurable with the whole $AB$ in square only and such that the rectangle $AB$, $BC$ which it contains with the whole $AB$, is medial [x 28]

I say that the remainder $AC$ is irrational and let it be called a *second apotome of a medial* straight line




For let a rational straight line $DI$ be set out,
let $DE$, equal to the squares on $AB$, $BC$, be applied to $DI$, producing $DG$ as breadth,

and let $DH$ equal to twice the rectangle $AB$ $BC$ be applied to $DI$ producing $DF$ as breadth,

therefore the remainder $FE$ is equal to the square on $AC$ [II 7]

Now, since the squares on $AB$, $BC$ are medial and commensurable

therefore $DE$ is also medial [X 15 and 23, Por]

And it is applied to the rational straight line $DI$ producing $DG$ as breadth, therefore $DG$ is rational and incommensurable in length with $DI$ [X 22]

Again since the rectangle $AB$, $BC$ is medial,

therefore twice the rectangle $AB$, $BC$ is also medial [X 23, Por]

And it is equal to $DH$,

therefore $DH$ is also medial

*And it has been applied to the rational straight line $DI$, producing $DF$ as* breadth

therefore $DF$ is rational and incommensurable in length with $DI$ [X 22]

And, since $AB$, $BC$ are commensurable in square only,

therefore $AB$ is incommensurable in length with $BC$

therefore the square on $AB$ is also incommensurable with the rectangle $AB$ $BC$ [X 11]

But the squares on $AB$, $BC$ are commensurable with the square on $AB$, [X 15]

and twice the rectangle $AB$ $BC$ is commensurable with the rectangle $AB$ $BC$, [X 6]

therefore twice the rectangle $AB$, $BC$ is incommensurable with the squares on $AB$, $BC$ [X 13]

But $DE$ is equal to the squares on $AB$, $BC$,

and $DH$ to twice the rectangle $AB$, $BC$,

therefore $DE$ is incommensurable with $DH$

But as $DE$ is to $DH$ so is $GD$ to $DF$, [VI 1]

therefore $GD$ is incommensurable with $DF$ [X 11]

And both are rational,

therefore *$GD$ $DF$ are rational straight lines commensurable in square only*

therefore $FG$ is an apotome [X 73]

But $DI$ is rational

and the rectangle contained by a rational and an irrational straight line is irrational, [deduction from X 20]

and its 'side' is irrational

And $AC$ is the 'side' of $FE$

therefore $AC$ is irrational

And let it be called a *second apotome of a medial* straight line Q E D

## Proposition 76

*If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the squares on them added together rational but the rectangle contained by them medial the remainder is irrational and let it be called* minor

For from the straight line $AB$ let there be subtracted the straight line $BC$ which is incommensurable in square with the whole and fulfils the given conditions [X 33]

A C B

I say that the remainder $AC$ is the irrational straight line called *minor*

For, since the sum of the squares on $AB$, $BC$ is rational while twice the rectangle $AB$, $BC$ is medial,

therefore the squares on $AB$, $BC$ are incommensurable with twice the rectangle $AB$, $BC$;

and, *convertendo*, the squares on $AB$, $BC$ are incommensurable with the remainder, the square on $AC$. [II. 7, X. 16]

But the squares on $AB$, $BC$ are rational;

therefore the square on $AC$ is irrational;

therefore $AC$ is irrational.

And let it be called *minor*. Q. E. D.

PROPOSITION 77.

*If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, but twice the rectangle contained by them rational, the remainder is irrational; and let it be called* that which produces with a rational area a medial whole.

For from the straight line $AB$ let there be subtracted the straight line $BC$ which is incommensurable in square with $AB$ and fulfils the given conditions; [X. 34]

I say that the remainder $AC$ is the irrational straight line aforesaid.

For, since the sum of the squares on $AB$, $BC$ is medial,

while twice the rectangle $AB$, $BC$ is rational,

therefore the squares on $AB$, $BC$ are incommensurable with twice the rectangle $AB$, $BC$;

therefore the remainder also, the square on $AC$, is incommensurable with twice the rectangle $AB$, $BC$. [II. 7, X. 16]

And twice the rectangle $AB$, $BC$ is rational;

therefore the square on $AC$ is irrational;

therefore $AC$ is irrational.

And let it be called *that which produces with a rational area a medial whole*. Q. E. D.

PROPOSITION 78.

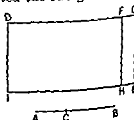
*If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, twice the rectangle contained by them medial, and further, the squares on them incommensurable with twice the rectangle contained by them, the remainder is irrational; and let it be called* that which produces with a medial area a medial whole.

For from the straight line $AB$ let there be subtracted the straight line $BC$ incommensurable in square with $AB$ and fulfilling the given conditions; [X. 35]

I say that the remainder $AC$ is the irrational straight line called *that which produces with a medial area a medial whole*.

For let a rational straight line $DI$ be set out,

to $DI$ let there be applied $DE$ equal to the squares on $AB$, $BC$, producing $DG$ as breadth,



and let $DH$ equal to twice the rectangle $AB$, $BC$ be subtracted

Therefore the remainder $FE$ is equal to the square on $AC$ [II 7]

so that $AC$ is the 'side' of $FE$

Now since the sum of the squares on $AB$, $BC$ is medial and is equal to $DE$,

therefore $DE$ is medial

And it is applied to the rational straight line $DI$ producing $DG$ as breadth, therefore $DG$ is rational and incommensurable in length with $DI$ [X 22]

Again since twice the rectangle $AB$ $BC$ is medial and is equal to $DH$

therefore $DH$ is medial

And it is applied to the rational straight line $DI$, producing $DF$ as breadth therefore $DF$ is also rational and incommensurable in length with $DI$ [X 22]

And since the squares on $AB$, $BC$ are incommensurable with twice the rectangle $AB$, $BC$,

therefore $DE$ is also incommensurable with $DH$

But as $DE$ is to $DH$, so also is $DG$ to $DF$ [VI 1]

therefore $DG$ is incommensurable with $DF$ [X 11]

And both are rational,

therefore $GD$ $DF$ are rational straight lines commensurable in square only

Therefore $FG$ is an apotome [X 73]

And $FH$ is rational,

but the rectangle contained by a rational straight line and an apotome is irrational [deduction from X 20]

and its 'side' is irrational

And $AC$ is the 'side' of $FE$

therefore $AC$ is irrational

And let it be called *that which produces with a medial area a medial whole*

Q E D

PROPOSITION 79

*To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only*

Let $AB$ be an apotome and $BC$ an annex to it,

A B C D

therefore $AC$, $CB$ are rational straight lines commensurable in square only [X 73]

I say that no other rational straight line can be annexed to $AB$ which is commensurable with the whole in square only

For if possible let $BD$ be so annexed

therefore $AD$, $DB$ are also rational straight lines commensurable in square only [X 73]

Now since the excess of the squares on $AD$ $DB$ over twice the rectangle $AD$ $DB$ *is also the excess of the squares on $AC$ $CB$ over twice the rectangle* $AC$, $CB$,

for both exceed by the same the square on $AB$ [II 7]

therefore alternately the excess of the squares on $AD$ $DB$ over the squares on $AC$, $CB$ is the excess of twice the rectangle $AD$, $DB$ over twice the rectangle $AC$, $CB$

But the squares on $AD$, $DB$ exceed the squares on $AC$ $CB$ by a rational area

for both are rational

therefore twice the rectangle $AD$ $DB$ also exceeds twice the rectangle $AC$, $CB$ by a rational area

which is impossible

for both are medial [x 21] and a medial area does not exceed a medial by a rational area [x 26]

Therefore no other rational straight line can be annexed to $AB$ which is commensurable with the whole in square only

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only Q E D

PROPOSITION 80

*To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle*

For let $AB$ be a first apotome of a medial straight line and let $BC$ be an annex to $AB$,

A B C D

therefore $AC$ $CB$ are medial straight lines commensurable in square only and such that the rectangle $AC$ $CB$ which they contain is rational, [x 74]

I say that no other medial straight line can be annexed to $AB$ which is commensurable with the whole in square only and which contains with the whole a rational area

For if possible let $DB$ also be so annexed

therefore $AD$ $DB$ are medial straight lines commensurable in square only and such that the rectangle $AD$ $DB$ which they contain is rational [x 74]

Now since the excess of the squares on $AD$ $DB$ over twice the rectangle $AD$ $DB$ is also the excess of the squares on $AC$ $CB$ over twice the rectangle $AC$ $CB$,

for they exceed by the same the square on $AB$, [II 7]

therefore alternately the excess of the squares on $AD$ $DB$ over the squares on $AC$ $CB$ is also the excess of twice the rectangle $AD$ $DB$ over twice the rectangle $AC$ $CB$

But twice the rectangle $AD$ $DB$ exceeds twice the rectangle $AC$, $CB$ by a rational area

for both are rational

Therefore the squares on $AD$ $DB$ also exceed the squares on $AC$, $CB$ by a rational area

which is impossible

for both are medial [x 15 and 23 Por ] and a medial area does not exceed a medial by a rational area [x 26]

*Therefore etc* Q E D

PROPOSITION 81

*To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle*

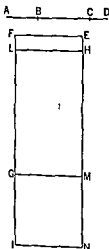
Let $AB$ be a second apotome of a medial straight line and $BC$ an annex to $AB$,

therefore $AC$ $CB$ are medial straight lines commensurable in square only and such that the rectangle $AC$ $CB$ which they contain is medial [x 75]

I say that no other medial straight line can be annexed to $AB$ which is com

mensurable with the whole in square only and which contains with the whole a medial rectangle

For, if possible let $BD$ also be so annexed,




therefore $AD$, $DB$ are also medial straight lines commensurable in square only and such that the rectangle $AD$, $DB$ which they contain is medial [x 75]

Let a rational straight line $EF$ be set out let $EG$ equal to the squares on $AC$, $CB$ be applied to $EF$, producing $EM$ as breadth and let $HG$ equal to twice the rectangle $AC$, $CB$ be subtracted producing $HM$ as breadth

therefore the remainder $EL$ is equal to the square on $AB$, [II 7]

so that $AB$ is the "side" of $EL$

Again let $EI$ equal to the squares on $AD$ $DB$ be applied to $EF$, producing $EN$ as breadth

But $EL$ is also equal to the square on $AB$, therefore the remainder $HI$ is equal to twice the rectangle $AD$ $DB$ [II 7]

Now since $AC$, $CB$ are medial straight lines therefore the squares on $AC$, $CB$ are also medial

And they are equal to $EG$

therefore $EG$ is also medial [x 15 and 23 Por]

And it is applied to the rational straight line $EF$, producing $EM$ as breadth therefore $EM$ is rational and incommensurable in length with $EF$ [x 22]

Again since the rectangle $AC$ $CB$ is medial

twice the rectangle $AC$, $CB$ is also medial [x 23 Por]

And it is equal to $HG$,

therefore $HG$ is also medial

And it is applied to the rational straight line $EF$ producing $HM$ as breadth, therefore $HM$ is also rational and incommensurable in length with $EF$ [x 22]

And since $AC$ $CB$ are commensurable in square only

therefore $AC$ is incommensurable in length with $CB$

But as $AC$ is to $CB$, so is the square on $AC$ to the rectangle $AC$ $CB$, therefore the square on $AC$ is incommensurable with the rectangle $AC$, $CB$ [x 11]

But the squares on $AC$, $CB$ are commensurable with the square on $AC$, while twice the rectangle $AC$, $CB$ is commensurable with the rectangle $AC$, $CB$ [x 6]

therefore the squares on $AC$ $CB$ are incommensurable with twice the rectangle $AC$, $CB$ [x 13]

And $EG$ is equal to the squares on $AC$ $CB$,

while $GH$ is equal to twice the rectangle $AC$ $CB$

therefore $EG$ is incommensurable with $HG$

But, as $EG$ is to $HG$ so is $EM$ to $HM$ [vi 1]

therefore $EM$ is incommensurable in length with $MH$ [x 11]

And both are rational

therefore $EM$ $MH$ are rational straight lines commensurable in square only

therefore $EH$ is an apotome and $HM$ an annex to it [x 73]

Similarly we can prove that $HN$ is also an annex to it,
therefore to an apotome different straight lines are annexed which are commensurable with the wholes in square only

which is impossible [x 79]

Therefore etc Q E D

PROPOSITION 82

*To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes with the whole, the sum of the squares on them rational but twice the rectangle contained by them medial*

Let $AB$ be the minor straight line and let $BC$ be an annex to $AB$, therefore $AC$, $CB$ are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial [x 76]

A B C D

I say that no other straight line can be annexed to $AB$ fulfilling the same conditions

For, if possible let $BD$ be so annexed,
therefore $AD$, $DB$ are also straight lines incommensurable in square which fulfil the aforesaid conditions [x 76]

Now since the excess of the squares on $AD$ $DB$ over the squares on $AC$ $CB$ is also the excess of twice the rectangle $AD$, $DB$ over twice the rectangle $AC$, $CB$,
while the squares on $AD$, $DB$ exceed the squares on $AC$, $CB$ by a rational area, for both are rational
therefore twice the rectangle $AD$ $DB$ also exceeds twice the rectangle $AC$ $CB$ by a rational area

which is impossible for both are medial [x 26]

Therefore to a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes the squares on them added together rational but twice the rectangle contained by them medial Q E D

PROPOSITION 83

*To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial but twice the rectangle contained by them rational*

Let $AB$ be the straight line which produces with a rational area a medial whole
and let $BC$ be an annex to $AB$,
therefore $AC$ $CB$ are straight lines incommensurable in square which fulfil the given conditions [x 77]

A B C D

I say that no other straight line can be annexed to $AB$ which fulfils the same conditions

For if possible let $BD$ be so annexed
therefore $AD$ $DB$ are also straight lines incommensurable in square which fulfil the given conditions [x 77]

Since then as in the preceding cases
the excess of the squares on $AD$, $DB$ over the squares on $AC$, $CB$ is also the

excess of twice the rectangle $AD$, $DB$ over twice the rectangle $AC$, $CB$, while twice the rectangle $AD$, $DB$ exceeds twice the rectangle $AC$, $CB$ by a rational area

for both are rational

therefore the squares on $AD$, $DB$ also exceed the squares on $AC$, $CB$ by a rational area

which is impossible for both are medial [x 26]

Therefore no other straight line can be annexed to $AB$ which is incommensurable in square with the whole and which with the whole fulfils the aforesaid conditions,

therefore only one straight line can be so annexed Q E D

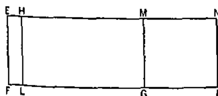
## Proposition 84

*To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them*

Let $AB$ be the straight line which produces with a medial area a medial whole

and $BC$ an annex to it,

therefore $AC$, $CB$ are straight lines incommensurable in square which fulfil the aforesaid conditions [x 78]




I say that no other straight line can be annexed to $AB$ which fulfils the aforesaid conditions

For if possible let $BD$ be so annexed

so that $AD$, $DB$ are also straight lines incommensurable in square which make the squares on $AD$, $DB$ added together medial twice the rectangle $AD$, $DB$ medial and also the squares on $AD$, $DB$ incommensurable with twice the rectangle $AD$, $DB$ [x 78]

Let a rational straight line $EF$ be set out

let $EG$ equal to the squares on $AC$, $CB$ be applied to $EF$ producing $EM$ as breadth

and let $HG$ equal to twice the rectangle $AC$, $CB$ be applied to $EF$ producing $HM$ as breadth

therefore the remainder the square on $AB$ [II 7] is equal to $EL$

therefore $AB$ is the side of $EL$

Again let $EI$ equal to the squares on $AD$, $DB$ be applied to $EF$ producing $EN$ as breadth

But the square on $AB$ is also equal to $EL$,

therefore the remainder twice the rectangle $AD$, $DB$ [II 7] is equal to $HI$

Now since the sum of the squares on $AC$, $CB$ is medial and is equal to $EG$

therefore $EG$ is also medial

And it is applied to the rational straight line $EF$ producing $EM$ as breadth

therefore $EM$ is rational and incommensurable in length with $EF$ [x 22]

Again, since twice the rectangle $AC$, $CB$ is medial and is equal to $HG$
therefore $HG$ is also medial

And it is applied to the rational straight line $EF$ producing $HM$ as breadth
therefore $HM$ is rational and incommensurable in length with $EF$ [x 22]

And, since the squares on $AC$, $CB$ are incommensurable with twice the rectangle $AC$ $CB$

$EG$ is also incommensurable with $HG$,
therefore $EM$ is also incommensurable in length with $MH$ [vi 1 x 11]

And both are rational,
therefore $EM$, $MH$ are rational straight lines commensurable in square only
therefore $EH$ is an apotome and $HM$ an annex to it [x 73]

Similarly we can prove that $EH$ is again an apotome and $HN$ an annex to it

Therefore to an apotome different rational straight lines are annexed which are commensurable with the wholes in square only
which was proved impossible [x 79]

Therefore no other straight line can be so annexed to $AB$

Therefore to $AB$ only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial twice the rectangle contained by them medial, and also the squares on them incommensurable with twice the rectangle contained by them Q E D

## DEFINITIONS III

1 Given a rational straight line and an apotome if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole and the whole be commensurable in length with the rational straight line set out let the apotome be called a *first apotome*

2 But if the annex be commensurable in length with the rational straight line set out and the square on the whole be greater than that on the annex by the square on a straight line commensurable with the whole let the apotome be called a *second apotome*

3 But if neither be commensurable in length with the rational straight line set out and the square on the whole be greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a *third apotome*

4 Again if the square on the whole be greater than the square on the annex by the square on a straight line incommensurable with the whole then if the whole be commensurable in length with the rational straight line set out let the apotome be called a *fourth apotome*

5 if the annex be so commensurable a *fifth*

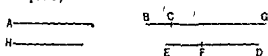
6 and if neither a *sixth*

### PROPOSITION 85

*To find the first apotome*

Let a rational straight line $A$ be set out
and let $BG$ be commensurable in length with $A$,
therefore $BG$ is also rational

Let two square numbers $DE$, $EF$ be set out, and let their difference $FD$ not be square,




therefore neither has $ED$ to $DF$ the ratio which a square number has to a square number

Let it be contrived that

as $ED$ is to $DF$, so is the square on $BG$ to the square on $GC$, [X 6 Por ]

therefore the square on $BG$ is commensurable with the square on $GC$ [X 6]

But the square on $BG$ is rational,

therefore the square on $GC$ is also rational,

therefore $GC$ is also rational

And since $ED$ has not to $DF$ the ratio which a square number has to a square number,

therefore neither has the square on $BG$ to the square on $GC$ the ratio which a square number has to a square number,

therefore $BG$ is incommensurable in length with $GC$ [X 9]

And both are rational,

therefore $BG$, $GC$ are rational straight lines commensurable in square only

therefore $BC$ is an apotome [X 73]

I say next that it is also a first apotome

For let the square on $H$ be that by which the square on $BG$ is greater than the square on $GC$

Now since as $ED$ is to $FD$ so is the square on $BG$ to the square on $GC$,

therefore also, *convertendo* [V 19, Por ]

as $DE$ is to $EF$, so is the square on $GB$ to the square on $H$

But $DE$ has to $EF$ the ratio which a square number has to a square number,

for each is square,

therefore the square on $GB$ also has to the square on $H$ the ratio which a square number has to a square number

therefore $BG$ is commensurable in length with $H$ [X 9]

And the square on $BG$ is greater than the square on $GC$ by the square on $H$,

therefore the square on $BG$ is greater than the square on $GC$ by the square on a straight line commensurable in length with $BG$

And the whole $BG$ is commensurable in length with the rational straight line $A$ set out

Therefore $BC$ is a first apotome [X Deff III 1]

Therefore the first apotome $BC$ has been found

(Being) that which it was required to find Q E D

PROPOSITION 86

*To find the second apotome*

Let a rational straight line $A$ be set out and $GC$ commensurable in [illegible] with $A$,

therefore $GC$ is rational

Let two square numbers $DE$ $EF$ be set out and let their dif[illegible] be square

Now let it be contrived that as $FD$ is to $DE$ so is the squ[illegible] square on $GB$

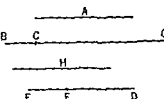
Therefore the square on $CG$ is commensurable with the [illegible]

But the square on $CG$ is rational,

therefore the square on $GB$ is also rational,

therefore $BG$ is rational

And, since the square on $GC$ has not to the square on $GB$ the ratio which a square number has to a square number $CG$ is incommensurable in length with $GB$ [x 9]




And both are rational, therefore $CG$, $GB$ are rational straight lines commensurable in square only,

therefore $BC$ is an apotome [x 73]

I say next that it is also a second apotome

For let the square on $H$ be that by which the square on $BG$ is greater than the square on $GC$

Since then as the square on $BG$ is to the square on $GC$, so is the number $ED$ to the number $DF$

therefore *convertendo*

as the square on $BG$ is to the square on $H$, so is $DE$ to $EF$ [v 19 Por]

And each of the numbers $DE$, $EF$ is square

therefore the square on $BG$ has to the square on $H$ the ratio which a square number has to a square number

therefore $BG$ is commensurable in length with $H$ [x 9]

And the square on $BG$ is greater than the square on $GC$ by the square on $H$, therefore the square on $BG$ is greater than the square on $GC$ by the square on a straight line commensurable in length with $BG$

And $CG$ the annex is commensurable with the rational straight line $A$ set out

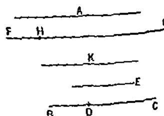
Therefore $BC$ is a second apotome [x Deff III 2]

Therefore the second apotome $BC$ has been found Q E D

## Proposition 87

*To find the third apotome*

Let a rational straight line $A$ be set out let three numbers $E$ $BC$ $CD$ be set out which have not to one another the ratio which a square number has to a square number but let $CB$ have to $BD$ the ratio which a square number has to a square number




Let it be contrived that as $E$ is to $BC$ so is the square on $A$ to the square on $FG$ and as $BC$ is to $CD$, so is the square on $FG$ to the square on $GH$ [x 6 Por]

Since then, as $E$ is to $BC$ so is the square on $A$ to the square on $FG$, therefore the square on $A$ is commensurable with the square on $FG$ [x 6]

But the square on $A$ is rational

therefore the square on $FG$ is also rational

therefore $FG$ is rational

And since $E$ has not to $BC$ the ratio which a square number has to a square number,
therefore neither has the square on $A$ to the square on $FG$ the ratio which a square number has to a square number,

therefore $A$ is incommensurable in length with $FG$ [x 9]

Again, since as $BC$ is to $CD$ so is the square on $FG$ to the square on $GH$
therefore the square on $FG$ is commensurable with the square on $GH$ [x 6]

But the square on $FG$ is rational,

therefore the square on $GH$ is also rational,

therefore $GH$ is rational

And since $BC$ has not to $CD$ the ratio which a square number has to a square number,
therefore neither has the square on $FG$ to the square on $GH$ the ratio which a square number has to a square number

therefore $FG$ is incommensurable in length with $GH$ [x 9]

And both are rational
therefore $FG$, $GH$ are rational straight lines commensurable in square only,

therefore $FH$ is an apotome [x 73]

I say next that it is also a third apotome

For since, as $E$ is to $BC$ so is the square on $A$ to the square on $FG$
and, as $BC$ is to $CD$ so is the square on $FG$ to the square on $HG$
therefore *ex aequali*, as $E$ is to $CD$ so is the square on $A$ to the square on $HG$ [v 22]

But $E$ has not to $CD$ the ratio which a square number has to a square number
therefore neither has the square on $A$ to the square on $GH$ the ratio which a square number has to a square number,

therefore $A$ is incommensurable in length with $GH$ [x 9]

Therefore neither of the straight lines $FG$ $GH$ is commensurable in length with the rational straight line $A$ set out

Now let the square on $K$ be that by which the square on $FG$ is greater than the square on $GH$

Since then as $BC$ is to $CD$ so is the square on $FG$ to the square on $GH$
therefore *convertendo* as $BC$ is to $BD$ so is the square on $FG$ to the square on $K$ [v 19 Por]

But $BC$ has to $BD$ the ratio which a square number has to a square number,
therefore the square on $FG$ also has to the square on $K$ the ratio which a square number has to a square number

Therefore $FG$ is commensurable in length with $K$ [x 9]
and the square on $FG$ is greater than the square on $GH$ by the square on a straight line commensurable with $FG$

And neither of the straight lines $FG$ $GH$ is commensurable in length with the rational straight line $A$ set out

therefore $FH$ is a third apotome [x Deff III 3]

Therefore the third apotome $FH$ has been found Q E D

## Proposition 88

*To find the fourth apotome*

Let a rational straight line $A$ be set out and $BG$ commensurable in length with it

therefore $BG$ is also rational

Let two numbers $DF$, $FE$ be set out such that the whole $DE$ has not to either of the numbers $DF$, $EF$ the ratio which a square number has to a square number




Let it be contrived that as $DE$ is to $EF$, so is the square on $BG$ to the square on $GC$, [x 6 Por]

therefore the square on $BG$ is commensurable with the square on $GC$ [x 6]

But the square on $BG$ is rational

therefore the square on $GC$ is also rational,

therefore $GC$ is rational

Now since $DE$ has not to $EF$ the ratio which a square number has to a square number,

therefore neither has the square on $BG$ to the square on $GC$ the ratio which a square number has to a square number,

therefore $BG$ is incommensurable in length with $GC$ [x 9]

And both are rational,

therefore $BG$, $GC$ are rational straight lines commensurable in square only,

therefore $BC$ is an apotome [x 73]

Now let the square on $H$ be that by which the square on $BG$ is greater than the square on $GC$

Since then as $DE$ is to $EF$ so is the square on $BG$ to the square on $GC$,

therefore also *convertendo* as $ED$ is to $DF$ so is the square on $GB$ to the square on $H$ [v 19 Por]

But $ED$ has not to $DF$ the ratio which a square number has to a square number

therefore neither has the square on $GB$ to the square on $H$ the ratio which a square number has to a square number

therefore $BG$ is incommensurable in length with $H$ [x 9]

And the square on $BG$ is greater than the square on $GC$ by the square on $H$,

therefore the square on $BG$ is greater than the square on $GC$ by the square on a straight line incommensurable with $BG$

And the whole $BG$ is commensurable in length with the rational straight line $A$ set out

Therefore $BC$ is a fourth apotome [x Deff III 4]

Therefore the fourth apotome has been found Q E D

PROPOSITION 89

*To find the fifth apotome*

Let a rational straight line $A$ be set out and let $CG$ be commensurable in length with $A$

therefore $CG$ is rational

Let two numbers $DF$, $FE$ be set out such that $DE$ again has not to either of the numbers $DF$, $FE$ the ratio which a square number has to a square number,

and let it be contrived that as $FE$ is to $ED$ so is the square on $CG$ to the square on $GB$

Therefore the square on $GB$ is also rational, [x 6]

therefore $BG$ is also rational

Now since as $DE$ is to $EF$, so is the square on $BG$ to the square on $GC$

while $DE$ has not to $EF$ the ratio which a square number has to a square number

therefore neither has the square on $BG$ to the square on $GC$ the ratio which a square number has to a square number

therefore $BG$ is incommensurable in length with $GC$ [x 9]

And both are rational,

therefore $BG$ $GC$ are rational straight lines commensurable in square only,

therefore $BC$ is an apotome [x 73]

I say next that it is also a fifth apotome

For let the square on $H$ be that by which the square on $BG$ is greater than the square on $GC$

Since then as the square on $BG$ is to the square on $GC$, so is $DE$ to $EF$, therefore, *convertendo* as $ED$ is to $DF$ so is the square on $BG$ to the square on $H$ [v 19 Por]

But $ED$ has not to $DF$ the ratio which a square number has to a square number,

therefore neither has the square on $BG$ to the square on $H$ the ratio which a square number has to a square number

therefore $BG$ is incommensurable in length with $H$ [x. 9]

And the square on $BG$ is greater than the square on $GC$ by the square on $H$, therefore the square on $GB$ is greater than the square on $GC$ by the square on a straight line incommensurable in length with $GB$

And the annex $CG$ is commensurable in length with the rational straight line $A$ set out

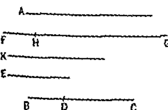
therefore $BC$ is a fifth apotome [x Deff III 5]

Therefore the fifth apotome $BC$ has been found Q E D

## Proposition 90

*To find the sixth apotome*

Let a rational straight line $A$ be set out and three numbers $E$ $BC$, $CD$ not having to one another the ratio which a square number has to a square number and further let $CB$ also not have to $BD$ the ratio which a square number has to a square number




Let it be contrived that as $E$ is to $BC$, so is the square on $A$ to the square on $FG$,

and as $BC$ is to $CD$ so is the square on $FG$ to the square on $GH$ [x 6 Por]

Now since as $E$ is to $BC$ so is the square on $A$ to the square on $FG$

therefore the square on $A$ is commensurable with the square on $FG$ [x 6]

But the square on $A$ is rational

therefore the square on *FG* is also rational

therefore *FG* is also rational

And, since *E* has not to *BC* the ratio which a square number has to a square number,

therefore neither has the square on *A* to the square on *FG* the ratio which a square number has to a square number

therefore *A* is incommensurable in length with *FG* [x 9]

Again since as *BC* is to *CD* so is the square on *FG* to the square on *GH* therefore the square on *FG* is commensurable with the square on *GH* [x 6]

But the square on *FG* is rational

therefore the square on *GH* is also rational,

therefore *GH* is also rational

And since *BC* has not to *CD* the ratio which a square number has to a square number,

therefore neither has the square on *FG* to the square on *GH* the ratio which a square number has to a square number

therefore *FG* is incommensurable in length with *GH* [x 9]

And both are rational

therefore *FG*, *GH* are rational straight lines commensurable in square only,

therefore *FH* is an apotome [x 73]

I say next that it is also a sixth apotome

For since as *E* is to *BC*, so is the square on *A* to the square on *FG*

and as *BC* is to *CD* so is the square on *FG* to the square on *GH*,

therefore *ex aequali* as *E* is to *CD*, so is the square on *A* to the square on *GH* [v 22]

But *E* has not to *CD* the ratio which a square number has to a square number

therefore neither has the square on *A* to the square on *GH* the ratio which a square number has to a square number,

therefore *A* is incommensurable in length with *GH*, [x 9]

therefore neither of the straight lines *FG* *GH* is commensurable in length with the rational straight line *A*

Now let the square on *K* be that by which the square on *FG* is greater than the square on *GH*

Since then as *BC* is to *CD* so is the square on *FG* to the square on *GH* therefore *convertendo* as *CB* is to *BD*, so is the square on *FG* to the square on *K* [v 19, Por]

But *CB* has not to *BD* the ratio which a square number has to a square number,

therefore neither has the square on *FG* to the square on *K* the ratio which a square number has to a square number

therefore *FG* is incommensurable in length with *K* [x 9]

And the square on *FG* is greater than the square on *GH* by the square on *K*,

therefore the square on *FG* is greater than the square on *GH* by the square on a straight line incommensurable in length with *FG*

And neither of the straight lines *FG* *GH* is commensurable with the rational straight line *A* set out

Therefore *FH* is a sixth apotome [x Deff III 6]

Therefore the sixth apotome *FH* has been found Q E D

PROPOSITION 91

*If an area be contained by a rational straight line and a first apotome the "side" of the area is an apotome*

For let the area $AB$ be contained by the rational straight line $AC$ and the first apotome $AD$

I say that the "side" of the area $AB$ is an apotome

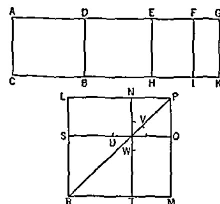
For since $AD$ is a first apotome let $DG$ be its annex,

therefore $AG$ $GD$ are rational straight lines commensurable in square only [x 73]

And the whole $AG$ is commensurable with the rational straight line $AC$ set out

and the square on $AG$ is greater than the square on $GD$ by the square on a straight line commensurable in length with $AG$ [x Deff III 1]

if therefore there be applied to $AG$ a parallelogram equal to the fourth part of the square on $DG$ and deficient by a square figure it divides it into commensurable parts [x 17]




Let $DG$ be bisected at $E$

let there be applied to $AG$ a parallelogram equal to the square on $EG$ and deficient by a square figure

and let it be the rectangle $AF$ $FG$

therefore $AF$ is commensurable with $FG$

And through the points $E$ $F$ $G$ let $EH$ $FI$ $GK$ be drawn parallel to $AC$

Now since $AF$ is commensurable in length with $FG$

therefore $AG$ is also commensurable in length with each of the straight lines $AF$ $FG$ [x 15]

But $AG$ is commensurable with $AC$,

therefore each of the straight lines $AF$ $FG$ is commensurable in length with $AC$ [x 12]

And $AC$ is rational

therefore each of the straight lines $AF$ $FG$ is also rational

so that each of the rectangles $AI$ $FK$ is also rational [x 19]

Now since $DE$ is commensurable in length with $EG$

therefore $DG$ is also commensurable in length with each of the straight lines $DE$ $EG$ [x 15]

But $DG$ is rational and incommensurable in length with $AC$

therefore each of the straight lines $DE$, $EG$ is also rational and incommensurable in length with $AC$ [x 13]

therefore each of the rectangles $DH$ $EK$ is medial [x 21]

Now let the square $LM$ be made equal to $AI$ and let there be subtracted the square $NO$ having a common angle with it the angle $LPM$, and equal to $FK$

therefore the squares $LM$ $NO$ are about the same diameter [vi 26]

Let $PR$ be their diameter and let the figure be drawn

Since then the rectangle contained by $AF$, $FG$ is equal to the square on $EG$
therefore, as $AF$ is to $EG$, so is $EG$ to $FG$ [VI 17]

But, as $AF$ is to $EG$ so is $AI$ to $EK$
and as $EG$ is to $FG$ so is $EK$ to $KF$, [VI 1]
*therefore $EK$ is a mean proportional between $AI$, $KF$* [V 11]

But $MN$ is also a mean proportional between $LM$, $NO$, as was before proved [Lemma after X 53]

and $AI$ is equal to the square $LM$, and $KF$ to $NO$,
therefore $MN$ is also equal to $EK$

But $EK$ is equal to $DH$ and $MN$ to $LO$,
therefore $DK$ is equal to the gnomon $UVW$ and $NO$

But $AK$ is also equal to the squares $LM$, $NO$
therefore the remainder $AB$ is equal to $ST$

But $ST$ is the square on $LN$,
therefore the square on $LN$ is equal to $AB$,
therefore $LN$ is the "side" of $AB$

I say next that $LN$ is an apotome

For since each of the rectangles $AI$, $FK$ is rational,
and they are equal to $LM$, $NO$
therefore each of the squares $LM$, $NO$ that is, the squares on $LP$, $PN$ respectively is also rational
therefore each of the straight lines $LP$, $PN$ is also rational

Again since $DH$ is medial and is equal to $LO$
therefore $LO$ is also medial

Since then $LO$ is medial
while $NO$ is rational
therefore $LO$ is incommensurable with $NO$

But as $LO$ is to $NO$ so is $LP$ to $PN$ [VI 1]
therefore $LP$ is incommensurable in length with $PN$ [X 11]

And both are rational
therefore $LP$, $PN$ are rational straight lines commensurable in square only
therefore $LN$ is an apotome [X 73]

And it is the "side" of the area $AB$
therefore the "side" of the area $AB$ is an apotome

Therefore etc Q E D

## Proposition 92

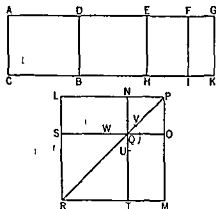
*If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line*

For let the area $AB$ be contained by the rational straight line $AC$ and the second apotome $AD$

I say that the "side" of the area $AB$ is a first apotome of a medial straight line

For let $DG$ be the annex to $AD$
therefore $AG$, $GD$ are rational straight lines commensurable in square only [X 73]
and the annex $DG$ is commensurable with the rational straight line $AC$ set out,
while the square on the whole $AG$ is greater than the square on the annex $GD$
by the square on a straight line commensurable in length with $AG$ [X Deff III 2]

Since, then the square on $AG$ is greater than the square on $GD$ by the square on a straight line commensurable with $AG$,




therefore, if there be applied to $AG$ a parallelogram equal to the fourth part of the square on $GD$ and deficient by a square figure it divides it into commensurable parts [x 17]

Let then $DG$ be bisected at $E$, let there be applied to $AG$ a parallelogram equal to the square on $EG$ and deficient by a square figure,

and let it be the rectangle $AF$, $FG$,

therefore $AF$ is commensurable in length with $FG$

Therefore $AG$ is also commensurable in length with each of the straight lines $AF$, $FG$ [x 15]

But $AG$ is rational and incommensurable in length with $AC$, therefore each of the straight lines $AF$, $FG$ is also rational and incommensurable in length with $AC$, [x 13]

therefore each of the rectangles $AI$, $FK$ is medial [x 21]

Again since $DE$ is commensurable with $EG$ therefore $DG$ is also commensurable with each of the straight lines $DE$, $EG$ [x 15]

But $DG$ is commensurable in length with $AC$

Therefore each of the rectangles $DH$, $EK$ is rational [x 19]

Let then the square $LM$ be constructed equal to $AI$, and let there be subtracted $NO$ equal to $FK$ and being about the same angle with $LM$ namely the angle $LPM$,

therefore the squares $LM$, $NO$ are about the same diameter [vi 26

Let $PR$ be their diameter and let the figure be drawn

Since then $AI$, $FK$ are medial and are equal to the squares on $LP$ $PN$,

the squares on $LP$ $PN$ are also medial

therefore $LP$ $PN$ are also medial straight lines commensurable in square only

And since the rectangle $AF$ $FG$ is equal to the square on $EG$,

therefore, as $AF$ is to $EG$ so is $EG$ to $FG$, [vi 17]

while, as $AF$ is to $EG$ so is $AI$ to $EK$

and as $EG$ is to $FG$ so is $EK$ to $FK$ [vi 1]

therefore $EK$ is a mean proportional between $AI$ $FK$ [v 11]

But $MN$ is also a mean proportional between the squares $LM$, $NO$

and $AI$ is equal to $LM$ and $FK$ to $NO$,

therefore $MN$ is also equal to $EK$

But $DH$ is equal to $EK$ and $LO$ equal to $MN$

therefore the whole $DK$ is equal to the gnomon $UVW$ and $NO$

Since then the whole $AK$ is equal to $LM$ $NO$

and, in these $DK$ is equal to the gnomon $UVW$ and $NO$

therefore the remainder $AB$ is equal to $TS$

But $TS$ is the square on $LN$

therefore the square on $LN$ is equal to the area $AB$,

therefore $LN$ is the "side" of the area $AB$

I say that $LN$ is a first apotome of a medial straight line

For, since $EK$ is rational and is equal to $LO$

therefore $LO$ that is the rectangle $LP$, $PN$, is rational

But $NO$ was proved medial,

therefore $LO$ is incommensurable with $NO$

But, as $LO$ is to $NO$, so is $LP$ to $PN$, [VI 1]

therefore $LP$, $PN$ are incommensurable in length [X 11]

Therefore $LP$ $PN$ are medial straight lines commensurable in square only which contain a rational rectangle,

therefore $LN$ is a first apotome of a medial straight line [X 74]

And it is the 'side' of the area $AB$

Therefore the "side of the area $AB$ is a first apotome of a medial straight line

Q E D

## Proposition 93

*If an area be contained by a rational straight line and a third apotome the 'side' of the area is a second apotome of a medial straight line*

For let the area $AB$ be contained by the rational straight line $AC$ and the third apotome $AD$

I say that the side of the area $AB$ is a second apotome of a medial straight line

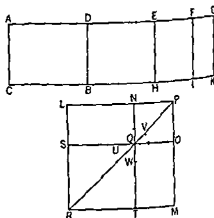
For let $DG$ be the annex to $AD$

therefore $AG$, $GD$ are rational straight lines commensurable in square only, and neither of the straight lines $AG$, $GD$ is commensurable in length with the rational straight line $AC$ set out

while the square on the whole $AG$ is greater than the square on the annex $DG$ by the square on a straight line commensurable with $AG$ [X Deff III 3]

Since, then, the square on $AG$ is greater than the square on $GD$ by the square on a straight line commensurable with $AG$,

therefore if there be applied to $AG$ a parallelogram equal to the fourth part of the square on $DG$ and deficient by a square figure it will divide it into commensurable parts [X 17]




Let then $DG$ be bisected at $E$

let there be applied to $AG$ a parallelogram equal to the square on $EG$ and deficient by a square figure,

and let it be the rectangle $AF$ $FG$

Let $EH$ $FI$ $GK$ be drawn through the points $E$ $F$ $G$ parallel to $AC$

Therefore $AF$ $FG$ are commensurable

therefore $AI$ is also commensurable with $FK$ [VI 1 X 11]

And, since $AF$ $FG$ are commensurable in length,

therefore $AG$ is also commensurable in length with each of the straight lines $AF$, $FG$ [X 15]

But $AG$ is rational and incommensurable in length with $AC$,
so that $AF$ $FG$ are so also [x 13]
Therefore each of the rectangles $AI$ $FK$ is medial [x 21]
Again since $DE$ is commensurable in length with $EG$
therefore $DG$ is also commensurable in length with each of the straight lines $DE$ $EG$ [x 15]
But $GD$ is rational and incommensurable in length with $AC$
therefore each of the straight lines $DE$, $EG$ is also rational and incommensurable in length with $AC$ [x 13]
therefore each of the rectangles $DH$ $EK$ is medial [x 21]
And since $AG$ $GD$ are commensurable in square only
therefore $AG$ is incommensurable in length with $GD$
But $AG$ is commensurable in length with $AF$ and $DG$ with $EG$
therefore $AF$ is incommensurable in length with $EG$ [x 13]
But as $AF$ is to $EG$ so is $AI$ to $EK$ [vi 1]
therefore $AI$ is incommensurable with $EK$ [x 11]
Now let the square $LM$ be constructed equal to $AI$
and let there be subtracted $NO$ equal to $FK$ and being about the same angle with $LM$
therefore $LM$ $NO$ are about the same diameter [vi 26]
Let $PR$ be their diameter and let the figure be drawn
Now since the rectangle $AF$ $FG$ is equal to the square on $EG$
therefore as $AF$ is to $EG$ so is $EG$ to $FG$ [vi 17]
But as $AF$ is to $EG$, so is $AI$ to $EK$
and as $EG$ is to $FG$ so is $EK$ to $FK$, [vi 1]
therefore also as $AI$ is to $EK$ so is $EK$ to $FK$ [v 11]
therefore $EK$ is a mean proportional between $AI$ $FK$
But $MN$ is also a mean proportional between the squares $LM$ $NO$
and $AI$ is equal to $LM$ and $FK$ to $NO$
therefore $EK$ is also equal to $MN$
But $MN$ is equal to $LO$ and $EK$ equal to $DH$
therefore the whole $DK$ is also equal to the gnomon $UVW$ and $NO$
But $AK$ is also equal to $LM$ $NO$
therefore the remainder $AB$ is equal to $ST$ that is to the square on $LN$
therefore $LN$ is the side of the area $AB$
I say that $LN$ is a second apotome of a medial straight line
For since $AI$ $FK$ were proved medial and are equal to the squares on $LP$ $PN$
therefore each of the squares on $LP$ $PN$ is also medial
therefore each of the straight lines $LP$ $PN$ is medial
And since $AI$ is commensurable with $FK$ [vi 1 x 11]
therefore the square on $LP$ is also commensurable with the square on $PN$
Again since $AI$ was proved incommensurable with $EK$
therefore $LM$ is also incommensurable with $MN$
that is the square on $LP$ with the rectangle $LP$ $PN$
so that $LP$ is also incommensurable in length with $PN$, [vi 1, x 11]
therefore $LP$ $PN$ are medial straight lines commensurable in square only
I say next that they also contain a medial rectangle
For since $EK$ was proved medial and is equal to the rectangle $LP$, $PN$

therefore the rectangle $LP$, $PN$ is also medial

so that $LP$, $PN$ are medial straight lines commensurable in square only which contain a medial rectangle

Therefore $LN$ is a second apotome of a medial straight line, [x 75]

and it is the "side" of the area $AB$

Therefore the "side" of the area $AB$ is a second apotome of a medial straight line

Q E D

## Proposition 94

*If an area be contained by a rational straight line and a fourth apotome, the "side" of the area is minor*

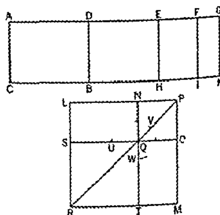
For let the area $AB$ be contained by the rational straight line $AC$ and the fourth apotome $AD$,

I say that the "side" of the area $AB$ is minor

For let $DG$ be the annex to $AD$

therefore $AG$, $GD$ are rational straight lines commensurable in square only, $AG$ is commensurable in length with the rational straight line $AC$ set out and the square on the whole $AG$ is greater than the square on the annex $DG$ by the square on a straight line incommensurable in length with $AG$

[x Deff III 4]




Since then the square on $AG$ is greater than the square on $GD$ by the square on a straight line incommensurable in length with $AG$

therefore if there be applied to $AG$ a parallelogram equal to the fourth part of the square on $DG$ and deficient by a square figure it will divide it into incommensurable parts [x 18]

Let then $DG$ be bisected at $E$

let there be applied to $AG$ a parallelogram equal to the square on $EG$ and deficient by a square figure

and let it be the rectangle $AF$, $FG$

therefore $AF$ is incommensurable in length with $FG$

Let $EH$, $FI$, $GK$ be drawn through $E$, $F$, $G$ parallel to $AC$, $BD$

Since then $AG$ is rational and commensurable in length with $AC$,

therefore the whole $AK$ is rational [x 19]

Again since $DG$ is incommensurable in length with $AC$ and both are rational

therefore $DK$ is medial [x 21]

Again since $AF$ is incommensurable in length with $FG$

therefore $AI$ is also incommensurable with $FK$ [vi 1, x 11]

Now let the square $LM$ be constructed equal to $AI$

and let there be subtracted $NO$ equal to $FK$ and about the same angle, the angle $LPM$

Therefore the squares $LM$, $NO$ are about the same diameter [vi 26]

Let $PR$ be their diameter and let the figure be drawn

Since then, the rectangle $AF$, $FG$ is equal to the square on $EG$,
therefore, proportionally, as $AF$ is to $EG$ so is $EG$ to $FG$ [VI 17]
But as $AF$ is to $EG$, so is $AI$ to $EK$,
and as $EG$ is to $FG$ so is $EK$ to $FK$, [VI 1]
therefore $EK$ is a mean proportional between $AI$, $FK$ [V 11]
But $MN$ is also a mean proportional between the squares $LM$, $NO$,
and $AI$ is equal to $LM$ and $FK$ to $NO$,
therefore $EK$ is also equal to $MN$
But $DH$ is equal to $EK$, and $LO$ is equal to $MN$,
therefore the whole $DK$ is equal to the gnomon $UVW$ and $NO$
Since then the whole $AK$ is equal to the squares $LM$, $NO$,
and in these $DK$ is equal to the gnomon $UVW$ and the square $NO$
therefore the remainder $AB$ is equal to $ST$, that is to the square on $LN$,
therefore $LN$ is the "side" of the area $AB$
I say that $LN$ is the irrational straight line called minor
For, since $AK$ is rational and is equal to the squares on $LP$, $PN$,
therefore the sum of the squares on $LP$, $PN$ is rational
Again since $DK$ is medial,
and $DK$ is equal to twice the rectangle $LP$, $PN$,
therefore twice the rectangle $LP$, $PN$ is medial
And since $AI$ was proved incommensurable with $FK$
therefore the square on $LP$ is also incommensurable with the square on $PN$
*Therefore $LP$, $PN$ are straight lines incommensurable in square which make the sum of the squares on them rational but twice the rectangle contained by them medial*
Therefore $LN$ is the irrational straight line called minor, [X 76]
and it is the "side" of the area $AB$
Therefore the "side" of the area $AB$ is minor Q E D

PROPOSITION 95

*If an area be contained by a rational straight line and a fifth apotome the "side" of the area is a straight line which produces with a rational area a medial whole*

For let the area $AB$ be contained by the rational straight line $AC$ and the fifth apotome $AD$

I say that the "side" of the area $AB$ is a straight line which produces with a rational area a medial whole

For let $DG$ be the annex to $AD$,
therefore $AG$, $GD$ are rational straight lines commensurable in square only,
the annex $GD$ is commensurable in length with the rational straight line $AC$ set out
and the square on the whole $AG$ is greater than the square on the annex $DG$ by the square on a straight line incommensurable with $AG$ [X Deff III 5]

Therefore, if there be applied to $AG$ a parallelogram equal to the fourth part of the square on $DG$ and deficient by a square figure it will divide it into incommensurable parts [X 18]

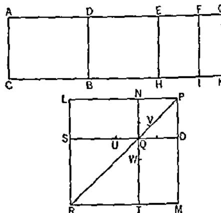
Let then $DG$ be bisected at the point $E$,
let there be applied to $AG$ a parallelogram equal to the square on $EG$ and deficient by a square figure and let it be the rectangle $AF$, $FG$,
therefore $AF$ is incommensurable in length with $FG$

Now since $AG$ is incommensurable in length with $CA$, and both are rational

therefore $AK$ is medial [x 21]

Again since $DG$ is rational and commensurable in length with $AC$

$DK$ is rational [x 19]




Now let the square $LM$ be constructed equal to $AI$ and let the square $NO$ equal to $FK$ and about the same angle the angle $LPM$ be subtracted

therefore the squares $LM$, $NO$ are about the same diameter [vi 26]

Let $PR$ be their diameter and let the figure be drawn

Similarly then we can prove that $LN$ is the 'side' of the area $AB$

I say that $LN$ is the straight line which produces with a rational area a medial whole

For since $AK$ was proved medial and is equal to the squares on $LP$, $PN$,

therefore the sum of the squares on $LP$, $PN$ is medial

Again, since $DK$ is rational and is equal to twice the rectangle $LP$, $PN$,

the latter is itself also rational

And since $AI$ is incommensurable with $FK$,

therefore the square on $LP$ is also incommensurable with the square on $PN$,

therefore $LP$, $PN$ are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational

Therefore the remainder $LN$ is the irrational straight line called that which produces with a rational area a medial whole [x 77]

and it is the 'side' of the area $AB$

Therefore the 'side' of the area $AB$ is a straight line which produces with a rational area a medial whole Q E D

PROPOSITION 96

*If an area be contained by a rational straight line and a sixth apotome the "side" of the area is a straight line which produces with a medial area a medial whole*

For let the area $AB$ be contained by the rational straight line $AC$ and the sixth apotome $AD$

I say that the 'side' of the area $AB$ is a straight line which produces with a medial area a medial whole

For let $DG$ be the annex to $AD$

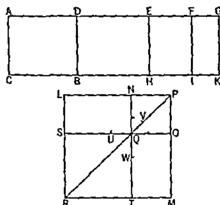
therefore $AG$, $GD$ are rational straight lines commensurable in square only neither of them is commensurable in length with the rational straight line $AC$ set out

and the square on the whole $AG$ is greater than the square on the annex $DG$ by the square on a straight line incommensurable in length with $AG$

[x Deff III 6]

Since then the square on $AG$ is greater than the square on $GD$ by the square on a straight line incommensurable in length with $AG$,

therefore if there be applied to $AG$ a parallelogram equal to the fourth part of the square on $DG$ and deficient by a square figure it will divide it into incommensurable parts [X 18]




Let then $DG$ be bisected at $E$ let there be applied to $AG$ a parallelogram equal to the square on $EG$ and deficient by a square figure, and let it be the rectangle $AF$, $FG$,

therefore $AF$ is incommensurable in length with $FG$

But as $AF$ is to $FG$ so is $AI$ to $FK$ [VI 1]

therefore $AI$ is incommensurable with $FK$ [X 11]

And since $AG$, $AC$ are rational straight lines commensurable in square only

$AK$ is medial [X 21]

Again since $AC$ $DG$ are rational straight lines and incommensurable in length

$DK$ is also medial [X 21]

Now, since $AG$ $GD$ are commensurable in square only

therefore $AG$ is incommensurable in length with $GD$

But as $AG$ is to $GD$ so is $AK$ to $KD$, [VI 1]

therefore $AK$ is incommensurable with $KD$ [X 11]

Now let the square $LM$ be constructed equal to $AI$,

and let $NO$ equal to $FK$ and about the same angle be subtracted,

therefore the squares $LM$ $NO$ are about the same diameter [VI 26]

Let $PR$ be their diameter and let the figure be drawn

Then in manner similar to the above we can prove that $LN$ is the 'side' of the area $AB$

I say that $LN$ is a straight line which produces with a medial area a medial whole

For since $AK$ was proved medial and is equal to the squares on $LP$ $PN$

therefore the sum of the squares on $LP$ $PN$ is medial

Again since $DK$ was proved medial and is equal to twice the rectangle $LP$ $PN$

twice the rectangle $LP$ $PN$ is also medial

And since $AK$ was proved incommensurable with $DK$

the squares on $LP$ $PN$ are also incommensurable with twice the rectangle $LP$ $PN$

And since $AI$ is incommensurable with $FK$

therefore the square on $LP$ is also incommensurable with the square on $PN$

therefore $LP$ $PN$ are straight lines incommensurable in square which make the sum of the squares on them medial twice the rectangle contained by them medial and further the squares on them incommensurable with twice the rectangle contained by them

Therefore $LN$ is the irrational straight line called that which produces with a medial area a medial whole [X 78]

and it is the "side" of the area $AB$

Therefore the "side" of the area is a straight line which produces with a medial area a medial whole. Q. E. D.

PROPOSITION 97

*The square on an apotome applied to a rational straight line produces as breadth a first apotome.*

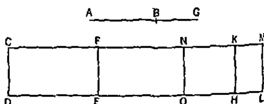
Let $AB$ be an apotome and $CD$ rational,

and to $CD$ let there be applied $CE$ equal to the square on $AB$ and producing $CF$ as breadth;

I say that $CF$ is a first apotome.

For let $BG$ be the annex to $AB$;

therefore $AG$, $GB$ are rational straight lines commensurable in square only. [x. 73]




To $CD$ let there be applied $CH$ equal to the square on $AG$, and $KL$ equal to the square on $BG$.

Therefore the whole $CL$ is equal to the squares on $AG$, $GB$, and, in these, $CE$ is equal to the square on $AB$;

therefore the remainder $FL$ is equal to twice the rectangle $AG$, $GB$. [II. 7]

Let $FM$ be bisected at the point $N$,

and let $NO$ be drawn through $N$ parallel to $CD$;

therefore each of the rectangles $FO$, $LN$ is equal to the rectangle $AG$, $GB$.

Now, since the squares on $AG$, $GB$ are rational,

and $DM$ is equal to the squares on $AG$, $GB$,

therefore $DM$ is rational.

And it has been applied to the rational straight line $CD$, producing $CM$ as breadth;

therefore $CM$ is rational and commensurable in length with $CD$. [x. 20]

Again, since twice the rectangle $AG$, $GB$ is medial, and $FL$ is equal to twice the rectangle $AG$, $GB$,

therefore $FL$ is medial.

And it is applied to the rational straight line $CD$, producing $FM$ as breadth;

therefore $FM$ is rational and incommensurable in length with $CD$. [x. 22]

And, since the squares on $AG$, $GB$ are rational,

while twice the rectangle $AG$, $GB$ is medial,

therefore the squares on $AG$, $GB$ are incommensurable with twice the rectangle $AG$, $GB$.

And $CL$ is equal to the squares on $AG$, $GB$,

and $FL$ to twice the rectangle $AG$, $GB$;

therefore $DM$ is incommensurable with $FL$.

But, as $DM$ is to $FL$, so is $CM$ to $FM$; [VI. 1]

therefore $CM$ is incommensurable in length with $FM$. [x. 11]

And both are rational;

therefore $CM$, $MF$ are rational straight lines commensurable in square only;

therefore $CF$ is an apotome. [x. 73]

I say next that it is also a first apotome

For since the rectangle $AG$, $GB$ is a mean proportional between the squares on $AG$ $GB$

and $CH$ is equal to the square on $AG$,
$KL$ equal to the square on $BG$,
and $NL$ equal to the rectangle $AG$, $GB$,
therefore $NL$ is also a mean proportional between $CH$, $KL$,
therefore as $CH$ is to $NL$, so is $NL$ to $KL$

But as $CH$ is to $NL$, so is $CK$ to $NM$,
and, as $NL$ is to $KL$ so is $NM$ to $KM$, [VI 1]

therefore the rectangle $CK$, $KM$ is equal to the square on $NM$ [VI 17] that is, to the fourth part of the square on $FM$

And since the square on $AG$ is commensurable with the square on $GB$,
$CH$ is also commensurable with $KL$

But, as $CH$ is to $KL$ so is $CK$ to $KM$ [VI 1]
therefore $CK$ is commensurable with $KM$ [X 11]

Since, then, $CM$ $MF$ are two unequal straight lines and to $CM$ there has been applied the rectangle $CK$ $KM$ equal to the fourth part of the square on $FM$ and deficient by a square figure
while $CK$ is commensurable with $KM$

therefore the square on $CM$ is greater than the square on $MF$ by the square on a straight line commensurable in length with $CM$ [X 17]

And $CM$ is commensurable in length with the rational straight line $CD$ set out,

therefore $CF$ is a first apotome [X Deff III 1]

Therefore etc Q E D

PROPOSITION 98

*The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome*

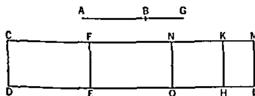
Let $AB$ be a first apotome of a medial straight line and $CD$ a rational straight line

and to $CD$ let there be applied $CE$ equal to the square on $AB$, producing $CF$ as breadth

I say that $CF$ is a second apotome

For let $BG$ be the annex to $AB$,

therefore $AG$, $GB$ are medial straight lines commensurable in square only which contain a rational rectangle [X 74]




To $CD$ let there be applied $CH$ equal to the square on $AG$, producing $CK$ as breadth, and $KL$ equal to the square on $GB$, producing $KM$ as breadth

therefore the whole $CL$ is equal to the squares on $AG$, $GB$,

therefore $CL$ is also medial [X 15 and 23 Por]

And it is applied to the rational straight line $CD$ producing $CM$ as breadth

therefore $CM$ is rational and incommensurable in length with $CD$ [X 22]

Now since $CL$ is equal to the squares on $AG$, $GB$,
and in these the square on $AB$ is equal to $CE$,
therefore the remainder, twice the rectangle $AG$, $GB$, is equal to $FL$. [II. 7]
But twice the rectangle $AG$, $GB$ is rational;
therefore $FL$ is rational.
And it is applied to the rational straight line $FE$, producing $FM$ as breadth;
therefore $FM$ is also rational and commensurable in length with $CD$. [X. 20]
Now, since the sum of the squares on $AG$, $GB$, that is, $CL$, is medial, while twice the rectangle $AG$, $GB$, that is, $FL$, is rational,
therefore $CL$ is incommensurable with $FL$.
But, as $CL$ is to $FL$, so is $CM$ to $FM$; [VI. 1]
therefore $CM$ is incommensurable in length with $FM$. [X. 11]
And both are rational;
therefore $CM$, $MF$ are rational straight lines commensurable in square only;
therefore $CF$ is an apotome. [X. 73]
I say next that it is also a second apotome.
For let $FM$ be bisected at $N$,
and let $NO$ be drawn through $N$ parallel to $CD$;
therefore each of the rectangles $FO$, $NL$ is equal to the rectangle $AG$, $GB$.
Now, since the rectangle $AG$, $GB$ is a mean proportional between the squares on $AG$, $GB$,
and the square on $AG$ is equal to $CH$,
the rectangle $AG$, $GB$ to $NL$,
and the square on $BG$ to $KL$,
therefore $NL$ is also a mean proportional between $CH$, $KL$;
therefore, as $CH$ is to $NL$, so is $NL$ to $KL$.
But, as $CH$ is to $NL$, so is $CK$ to $NM$,
and, as $NL$ is to $KL$, so is $NM$ to $MK$; [VI. 1]
therefore, as $CK$ is to $NM$, so is $NM$ to $KM$; [V. 11]
therefore the rectangle $CK$, $KM$ is equal to the square on $NM$ [VI. 17], that is, to the fourth part of the square on $FM$.
Since then $CM$, $MF$ are two unequal straight lines, and the rectangle $CK$, $KM$ equal to the fourth part of the square on $MF$ and deficient by a square figure has been applied to the greater, $CM$, and divides it into commensurable parts,
therefore the square on $CM$ is greater than the square on $MF$ by the square on a straight line commensurable in length with $CM$. [X. 17]
And the annex $FM$ is commensurable in length with the rational straight line $CD$ set out;
therefore $CF$ is a second apotome. [X. Deff. III. 2]
Therefore etc.
Q. E. D.

## Proposition 99

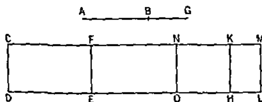
*The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.*

Let $AB$ be a second apotome of a medial straight line, and $CD$ rational, and to $CD$ let there be applied $CE$ equal to the square on $AB$, producing $CF$ as breadth;

I say that $CF$ is a third apotome.

For let $BG$ be the annex to $AB$,

therefore $AG$, $GB$ are medial straight lines commensurable in square only which contain a medial rectangle [X. 75]




Let $CH$ equal to the square on $AG$ be applied to $CD$, producing $CK$ as breadth, and let $KL$ equal to the square on $BG$ be applied to $KH$ producing $KM$ as breadth,

therefore the whole $CL$ is equal to the squares on $AG$, $GB$,

therefore $CL$ is also medial [X. 15 and 23, Por.]

And it is applied to the rational straight line $CD$ producing $CM$ as breadth,

therefore $CM$ is rational and incommensurable in length with $CD$ [X. 22]

Now, since the whole $CL$ is equal to the squares on $AG$, $GB$, and in these $CE$ is equal to the square on $AB$,

therefore the remainder $LF$ is equal to twice the rectangle $AG$, $GB$ [II. 7]

Let then $FM$ be bisected at the point $N$,

and let $NO$ be drawn parallel to $CD$,

therefore each of the rectangles $FO$, $NL$ is equal to the rectangle $AG$, $GB$

But the rectangle $AG$, $GB$ is medial,

therefore $FL$ is also medial

And it is applied to the rational straight line $EF$ producing $FM$ as breadth,

therefore $FM$ is also rational and incommensurable in length with $CD$ [X. 22]

And since $AG$, $GB$ are commensurable in square only,

therefore $AG$ is incommensurable in length with $GB$,

therefore the square on $AG$ is also incommensurable with the rectangle $AG$, $GB$ [VI. 1, X. 11]

But the squares on $AG$, $GB$ are commensurable with the square on $AG$,

and twice the rectangle $AG$, $GB$ with the rectangle $AG$, $GB$,

therefore the squares on $AG$, $GB$ are incommensurable with twice the rectangle $AG$, $GB$ [X. 13]

But $CL$ is equal to the squares on $AG$, $GB$,

and $FL$ is equal to twice the rectangle $AG$, $GB$,

therefore $CL$ is also incommensurable with $FL$

But as $CL$ is to $FL$, so is $CM$ to $FM$, [VI. 1]

therefore $CM$ is incommensurable in length with $FM$ [X. 11]

And both are rational,

therefore $CM$, $MF$ are rational straight lines commensurable in square only,

*therefore $CF$ is an apotome* [X. 73]

I say next that it is also a third apotome

For since the square on $AG$ is commensurable with the square on $GB$,

therefore $CH$ is also commensurable with $KL$,

so that $CK$ is also commensurable with $KM$ [VI. 1, X. 11]

And since the rectangle $AG$, $GB$ is a mean proportional between the squares on $AG$, $GB$,

and $CH$ is equal to the square on $AG$,

$KL$ *equal to the square on* $GB$,

and $NL$ equal to the rectangle $AG$, $GB$,

therefore $NL$ is also a mean proportional between $CH$, $KL$,

therefore as $CH$ is to $NL$ so is $NL$ to $KL$

But, as $CH$ is to $NL$ so is $CK$ to $NM$

and, as $NL$ is to $KL$, so is $NM$ to $KM$, [VI 1]

therefore as $CK$ is to $MN$, so is $MN$ to $KM$, [V 11]

therefore the rectangle $CK$, $KM$ is equal to [the square on $MN$, that is to] the fourth part of the square on $FM$

Since then $CM$, $MF$ are two unequal straight lines and a parallelogram equal to the fourth part of the square on $FM$ and deficient by a square figure has been applied to $CM$ and divides it into commensurable parts,

therefore the square on $CM$ is greater than the square on $MF$ by the square on a straight line commensurable with $CM$ [X 17]

And neither of the straight lines $CM$, $MF$ is commensurable in length with the rational straight line $CD$ set out

therefore $CF$ is a third apotome [X Deff III 3]

Q E D

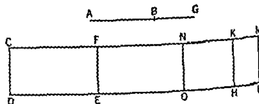
Therefore etc

## Proposition 100

*The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome*

Let $AB$ be a minor and $CD$ a rational straight line and to the rational straight line $CD$ let $CE$ be applied equal to the square on $AB$ and producing $CF$ as breadth

I say that $CF$ is a fourth apotome




For let $BG$ be the annex to $AB$

therefore $AG$, $GB$ are straight lines incommensurable in square which make the sum of the squares on $AG$, $GB$ rational but twice the rectangle $AG$, $GB$ medial [X 76]

To $CD$ let there be applied $CH$ equal to the square on $AG$ and producing $CK$ as breadth

and $KL$ equal to the square on $BG$ producing $KM$ as breadth,

therefore the whole $CL$ is equal to the squares on $AG$, $GB$

And the sum of the squares on $AG$, $GB$ is rational

therefore $CL$ is also rational

And it is applied to the rational straight line $CD$ producing $CM$ as breadth,

therefore $CM$ is also rational and commensurable in length with $CD$ [X 20]

And since the whole $CL$ is equal to the squares on $AG$, $GB$ and in these, $CE$ is equal to the square on $AB$

therefore the remainder $FL$ is equal to twice the rectangle $AG$, $GB$ [II 7]

Let then $FM$ be bisected at the point $N$

and let $NO$ be drawn through $N$ parallel to either of the straight lines $CD$, $ML$,

therefore each of the rectangles $FO$, $NL$ is equal to the rectangle $AG$, $GB$

And since twice the rectangle $AG$, $GB$ is medial and is equal to $FL$

therefore $FL$ is also medial

And it is applied to the rational straight line $FE$ producing $FM$ as breadth

therefore $FM$ is rational and incommensurable in length with $CD$ [x 22]

And since the sum of the squares on $AG$ $GB$ is rational while twice the rectangle $AG$ $GB$ is medial,

the squares on $AG$ $GB$ are incommensurable with twice the rectangle $AG$ $GB$

But $GL$ is equal to the squares on $AG$, $GB$,

and $FL$ equal to twice the rectangle $AG$ $GB$,

therefore $CL$ is incommensurable with $FL$

But as $CL$ is to $FL$ so is $CM$ to $MF$, [vi 1]

therefore $CM$ is incommensurable in length with $MF$ [x 11]

And both are rational,

therefore $CM$, $MF$ are rational straight lines commensurable in square only

therefore $CF$ is an apotome [x 73]

I say that it is also a fourth apotome

For, since $AG$, $GB$ are incommensurable in square

therefore the square on $AG$ is also incommensurable with the square on $GB$

And $CH$ is equal to the square on $AG$,

and $KL$ equal to the square on $GB$,

therefore $CH$ is incommensurable with $KL$

But as $CH$ is to $KL$, so is $CK$ to $KM$, [vi 1]

therefore $CK$ is incommensurable in length with $KM$ [x 11]

And since the rectangle $AG$, $GB$ is a mean proportional between the squares on $AG$, $GB$,

and the square on $AG$ is equal to $CH$,

the square on $GB$ to $KL$

and the rectangle $AG$ $GB$ to $NL$

therefore $NL$ is a mean proportional between $CH$, $KL$

therefore as $CH$ is to $NL$, so is $NL$ to $KL$

But as $CH$ is to $NL$ so is $CK$ to $NM$

and as $NL$ is to $KL$ so is $NM$ to $KM$, [vi 1]

therefore as $CK$ is to $MN$ so is $MN$ to $KM$ [v 11]

therefore the rectangle $CK$, $KM$ is equal to the square on $MN$ [vi 17] that is to the fourth part of the square on $FM$

Since then $CM$ $MF$ are two unequal straight lines and the rectangle $CK$, $KM$ equal to the fourth part of the square on $MF$ and deficient by a square figure has been applied to $CM$ and divides it into incommensurable parts, therefore the square on $CM$ is greater than the square on $MF$ by the square on a straight line incommensurable with $CM$ [x 18]

And the whole $CM$ is commensurable in length with the rational straight line $CD$ set out,

therefore $CF$ is a fourth apotome [x Deff iii 4]

Therefore etc Q E D

PROPOSITION 101

*The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line produces as breadth a fifth apotome*

Let $AB$ be the straight line which produces with a rational area a medial whole and $CD$ a rational straight line and to $CD$ let $CE$ be applied equal to the square on $AB$ and producing $CF$ as breadth

I say that $CF$ is a fifth apotome

For let $BG$ be the annex to $AB$

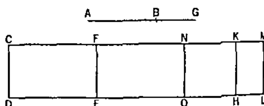
therefore $AG$, $GB$ are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational [x 78]

To $CD$ let there be applied $CH$ equal to the square on $AG$ and $KL$ equal to the square on $GB$,

therefore the whole $CL$ is equal to the squares on $AG$, $GB$

But the sum of the squares on $AG$ $GB$ together is medial,

therefore $CL$ is medial




And it is applied to the rational straight line $CD$, producing $CM$ as breadth

therefore $CM$ is rational and incommensurable with $CD$ [x 22]

And, since the whole $CL$ is equal to the squares on $AG$, $GB$,

and in these $CE$ is equal to the square on $AB$,

therefore the remainder $FL$ is equal to twice the rectangle $AG$, $GB$ [II 4]

Let then $FM$ be bisected at $N$,

and through $N$ let $NO$ be drawn parallel to either of the straight lines $CD$, $ML$,

therefore each of the rectangles $FO$ $NL$ is equal to the rectangle $AG$, $GB$

And, since twice the rectangle $AG$ $GB$ is rational and equal to $FL$

therefore $FL$ is rational

And it is applied to the rational straight line $EF$ producing $FM$ as breadth

therefore $FM$ is rational and commensurable in length with $CD$ [x 20]

Now since $CL$ is medial and $FL$ rational,

therefore $CL$ is incommensurable with $FL$

But as $CL$ is to $FL$ so is $CM$ to $MF$ [VI 1]

therefore $CM$ is incommensurable in length with $MF$ [x 11]

And both are rational,

therefore $CM$, $MF$ are rational straight lines commensurable in square only

therefore $CF$ is an apotome [x 73]

I say next that it is also a fifth apotome

For we can prove similarly that the rectangle $CK$ $KM$ is equal to the square on $NM$ that is to the fourth part of the square on $FM$

And since the square on $AG$ is incommensurable with the square on $GB$,

while the square on $AG$ is equal to $CH$,

and the square on $GB$ to $KL$

therefore $CH$ is incommensurable with $KL$

But as $CH$ is to $KL$ so is $CK$ to $KM$, [VI 1]

therefore $CK$ is incommensurable in length with $KM$ [x 11]

Since then $CM$ $MF$ are two unequal straight lines and a parallelogram equal to the fourth part of the square on $FM$ and deficient by a square figure has been applied to $CM$, and divides it into incommensurable parts

therefore the square on $CM$ is greater than the square on $MF$ by the square on a straight line incommensurable with $CM$ [x 18]

And the annex $FM$ is commensurable with the rational straight line $CD$ set out

therefore $CF$ is a fifth apotome [x Deff III 5]

Q E D

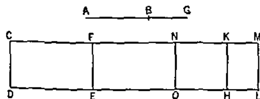
PROPOSITION 102

*The square on the straight line which produces with a medial area a medial whole if applied to a rational straight line produces as breadth a sixth apotome*

Let $AB$ be the straight line which produces with a medial area a medial whole and $CD$ a rational straight line,
and to $CD$ let $CE$ be applied equal to the square on $AB$ and producing $CF$ as breadth

I say that $CF$ is a sixth apotome

For let $BG$ be the annex to $AB$,




therefore $AG$ $GB$ are straight *lines incommensurable in* square which make the sum of the squares on them medial, twice the rectangle $AG$ $GB$ medial and the squares on $AG$ $GB$ incommensurable with twice the rectangle $AG$ $GB$ [x 78]

Now to $CD$ let there be applied $CH$ equal to the square on $AG$ and producing $CK$ as breadth

and $KL$ equal to the square on $BG$,

therefore the whole $CL$ is equal to the squares on $AG$ $GB$

therefore $CL$ is also medial

And it is applied to the rational straight line $CD$ producing $CM$ as breadth,
therefore $CM$ is rational and incommensurable in length with $CD$ [x 22]

Since now $CL$ is equal to the squares on $AG$ $GB$,

and in these $CE$ is equal to the square on $AB$

therefore the remainder $FL$ is equal to twice the rectangle $AG$ $GB$ [II 7]

And twice the rectangle $AG$ $GB$ is medial,

therefore $FL$ is also medial

And it is applied to the rational straight line $FE$ producing $FM$ as breadth
therefore $FM$ is rational and incommensurable in length with $CD$ [x 22]

And since the squares on $AG$ $GB$ are incommensurable with twice the rectangle $AG$, $GB$

*and $CL$ is equal to the squares on $AG$ $GB$*

and $FL$ equal to twice the rectangle $AG$ $GB$

therefore $CL$ is incommensurable with $FL$

But as $CL$ is to $FL$ so is $CM$ to $MF$ [VI 1]

therefore $CM$ is incommensurable in length with $MF$ [x 11]

And both are rational

Therefore $CM$ $MF$ are rational straight lines commensurable in square only

therefore $CF$ is an apotome [x 73]

I say next that it is also a sixth apotome

For since $FL$ is equal to twice the rectangle $AG$ $GB$

let $FM$ be bisected at $N$

and let $NO$ be drawn through $N$ parallel to $CD$

therefore each of the rectangles $FO$ $NL$ is equal to the rectangle $AG$, $GB$

And since $AG$ $GB$ are incommensurable in square

therefore the square on $AG$ is incommensurable with the square on $GB$

But $CH$ is equal to the square on $AG$,

and $KL$ is equal to the square on $GB$,

therefore $CH$ is incommensurable with $KL$

But as $CH$ is to $KL$ so is $CK$ to $KM$ [vi 1]

therefore $CK$ is incommensurable with $KM$ [x 11]

And since the rectangle $AG$, $GB$ is a mean proportional between the squares on $AG$, $GB$,

and $CH$ is equal to the square on $AG$,

$KL$ equal to the square on $GB$,

and $NL$ equal to the rectangle $AG$, $GB$,

therefore $NL$ is also a mean proportional between $CH$, $KL$,

therefore as $CH$ is to $NL$ so is $NL$ to $KL$

And for the same reason as before the square on $CM$ is greater than the square on $MF$ by the square on a straight line incommensurable with $CM$ [x 18]

And neither of them is commensurable with the rational straight line $CD$ set out

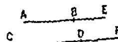
therefore $CF$ is a sixth apotome [x Deff III 6]

Q E D

## Proposition 103

*A straight line commensurable in length with an apotome is an apotome and the same in order*

Let $AB$ be an apotome




and let $CD$ be commensurable in length with $AB$

I say that $CD$ is also an apotome and the same in order with $AB$

For since $AB$ is an apotome let $BE$ be the annex to it

therefore $AE$, $EB$ are rational straight lines commensurable in square only [x 73]

Let it be contrived that the ratio of $BE$ to $DF$ is the same as the ratio of $AB$ to $CD$ [vi 12]

therefore also as one is to one so are all to all [v 12]

therefore also as the whole $AE$ is to the whole $CF$ so is $AB$ to $CD$

But $AB$ is commensurable in length with $CD$

Therefore $AE$ is also commensurable with $CF$ and $BE$ with $DF$ [x 11]

And $AE$, $EB$ are rational straight lines commensurable in square only

therefore $CF$, $FD$ are also rational straight lines commensurable in square only [x 13]

Now since as $AE$ is to $CF$ so is $BE$ to $DF$

alternately therefore as $AE$ is to $EB$ so is $CF$ to $FD$ [v 16]

And the square on $AE$ is greater than the square on $EB$ either by the square on a straight line commensurable with $AE$ or by the square on a straight line incommensurable with it

If then the square on $AE$ is greater than the square on $EB$ by the square on a straight line commensurable with $AE$ the square on $CF$ will also be greater than the square on $FD$ by the square on a straight line commensurable with $CF$ [x 14]

And, if $AE$ is commensurable in length with the rational straight line set out,

$CF$ is so also [x 12]

if $BE$ then $DF$ also [id ]

and if neither of the straight lines $AF$, $EB$, then neither of the straight lines $CF$, $FD$ [x 13]

But if the square on $AE$ is greater than the square on $FB$ by the square on a straight line incommensurable with $AE$,

the square on $CF$ will also be greater than the square on $FD$ by the square on a straight line incommensurable with $CF$ [x 14]

And if $AE$ is commensurable in length with the rational straight line set out,

$CF$ is so also

if $BF$ then $DF$ also [x 12]

and if neither of the straight lines $AE$, $EB$, then neither of the straight lines $CF$, $FD$ [x 13]

Therefore $CD$ is an apotome and the same in order with $AB$ Q E D

PROPOSITION 104

*A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order*

Let $AB$ be an apotome of a medial straight line,

and let $CD$ be commensurable in length with $AB$,

A B E
C D F

I say that $CD$ is also an apotome of a medial straight line and the same in order with $AB$

For, since $AB$ is an apotome of a medial straight line let $EB$ be the annex to it

Therefore $AE$ $EB$ are medial straight lines commensurable in square only [x 74 75]

Let it be contrived that as $AB$ is to $CD$, so is $BE$ to $DF$ [vi 12]

therefore $AE$ is also commensurable with $CF$ and $BE$ with $DF$ [v 12 x 11]

But $AE$ $EB$ are medial straight lines commensurable in square only

therefore $CF$ $FD$ are also medial straight lines [x 23] commensurable in square only [x 13]

therefore $CD$ is an apotome of a medial straight line [x 74 75]

I say next that it is also the same in order with $AB$

Since as $AE$ is to $EB$ so is $CF$ to $FD$

therefore also as the square on $AE$ is to the rectangle $AE$ $EB$ so is the square on $CF$ to the rectangle $CF$ $FD$

But the square on $AE$ is commensurable with the square on $CF$

therefore the rectangle $AE$, $EB$ is also commensurable with the rectangle $CF$ $FD$ [v 16 x 11]

Therefore, if the rectangle $AE$ $EB$ is rational the rectangle $CF$ $FD$ will also be rational [x Def 4]

and if the rectangle $AE$, $EB$ is medial the rectangle $CF$ $FD$ is also medial [x 23 Por ]

Therefore $CD$ is an apotome of a medial straight line and the same in order with $AB$ [x 74 75]

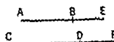
Q E D

PROPOSITION 105

*A straight line commensurable with a minor straight line is minor*

Let $AB$ be a minor straight line and $CD$ commensurable with $AB$,

I say that $CD$ is also minor




Let the same construction be made as before,

then since $AE$ $EB$ are incommensurable in square [x 76]

therefore $CF$ $FD$ are also incommensurable in square [x 13]

Now since as $AE$ is to $EB$ so is $CF$ to $FD$ [v 12 v 16]

therefore also as the square on $AE$ is to the square on $EB$ so is the square on $CF$ to the square on $FD$ [vi 22]

Therefore *componendo* as the squares on $AE$ $EB$ are to the square on $EB$ so are the squares on $CF$ $FD$ to the square on $FD$ [v 18]

But the square on $BE$ is commensurable with the square on $DF$

therefore the sum of the squares on $AE$, $EB$ is also commensurable with the sum of the squares on $CF$ $FD$ [v 16 x 11]

But the sum of the squares on $AE$ $EB$ is rational, [x 76]

therefore the sum of the squares on $CF$ $FD$ is also rational [x Def 4]

Again since as the square on $AE$ is to the rectangle $AE$, $EB$ so is the square on $CF$ to the rectangle $CF$ $FD$

while the square on $AE$ is commensurable with the square on $CF$

therefore the rectangle $AE$ $EB$ is also commensurable with the rectangle $CF$, $FD$

But the rectangle $AE$ $EB$ is medial [x 76]

therefore the rectangle $CF$ $FD$ is also medial, [x 23 Por]

therefore $CF$ $FD$ are straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial

Therefore $CD$ is minor [x 76]

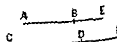
Q E D

PROPOSITION 106

*A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole*

Let $AB$ be a straight line which produces with a rational area a medial whole and $CD$ commensurable with $AB$

I say that $CD$ is also a straight line which produces with a rational area a medial whole




For let $BE$ be the annex to $AB$

therefore $AE$ $EB$ are straight lines incommensurable in square which make the sum of the squares on $AE$ $EB$ medial but the rectangle contained by them rational [x 77]

Let the same construction be made

Then we can prove in manner similar to the foregoing that $CF$ $FD$ are in the same ratio as $AE$ $EB$

the sum of the squares on $AE$ $EB$ is commensurable with the sum of the squares on $CF$ $FD$

and the rectangle $AE$ $EB$ with the rectangle $CF$ $FD$

so that $CF$, $FD$ are also straight lines incommensurable in square which make the sum of the squares on $CF$, $FD$ medial but the rectangle contained by them rational

Therefore $CD$ is a straight line which produces with a rational area a medial whole [x 77]

Q E D

## Proposition 107

*A straight line commensurable with that which produces with a medial area a medial whole is itself also a straight line which produces with a medial area a medial whole*

Let $AB$ be a straight line which produces with a medial area a medial whole

A B E

C D F

and let $CD$ be commensurable with $AB$

I say that $CD$ is also a straight line which produces with a medial area a medial whole

For let $BE$ be the annex to $AB$,

and let the same construction be made

therefore $AE$, $EB$ are straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and further the sum of the squares on them incommensurable with the rectangle contained by them [x 78]

Now, as was proved, $AE$ $EB$ are commensurable with $CF$, $FD$,

the sum of the squares on $AE$, $EB$ with the sum of the squares on $CF$, $FD$,

and the rectangle $AE$, $EB$ with the rectangle $CF$ $FD$,

therefore $CF$, $FD$ are also straight lines incommensurable in square which make the sum of the squares on them medial the rectangle contained by them medial and further, the sum of the squares on them incommensurable with the rectangle contained by them

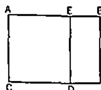
Therefore $CD$ is a straight line which produces with a medial area a medial whole [x 78]

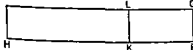
Q E D

## Proposition 108

*If from a rational area a medial area be subtracted the 'side' of the remaining area becomes one of two irrational straight lines either an apotome or a minor straight line*

For from the rational area $BC$ let the medial area $BD$ be subtracted




I say that the 'side' of the remainder $EC$ becomes one of two irrational straight lines either an apotome or a minor straight line

For let a rational straight line $FG$ be set out

to $FG$ let there be applied the rectangular parallelogram $GH$ equal to $BC$

and let $GK$ equal to $DB$ be subtracted

therefore the remainder $EC$ is equal to $LH$

Since then $BC$ is rational and $BD$ medial

while $BC$ is equal to $GH$, and $BD$ to $GK$,
therefore $GH$ is rational and $GK$ medial
And they are applied to the rational straight line $FG$,
therefore $FH$ is rational and commensurable in length with $FG$, [x 20]
while $FK$ is rational and incommensurable in length with $FG$, [x 22]
therefore $FH$ is incommensurable in length with $FK$ [x 13]
Therefore $FH$, $FK$ are rational straight lines commensurable in square only,
therefore $KH$ is an apotome [x 73] and $KF$ the annex to it

Now the square on $HF$ is greater than the square on $FK$ by the square on a straight line either commensurable with $HF$ or not commensurable

First let the square on it be greater by the square on a straight line commensurable with it

Now the whole $HF$ is commensurable in length with the rational straight line $FG$ set out,

therefore $KH$ is a first apotome [x Deff III 1]

But the "side" of the rectangle contained by a rational straight line and a first apotome is an apotome [x 91]

Therefore the "side" of $LH$ that is of $EC$ is an apotome

But if the square on $HF$ is greater than the square on $FK$ by the square on a straight line incommensurable with $HF$
while the whole $FH$ is commensurable in length with the rational straight line $FG$ set out

$KH$ is a fourth apotome [x Deff III 4]

But the "side" of the rectangle contained by a rational straight line and a fourth apotome is minor [x 94]

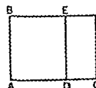
Q E D

## Proposition 109

*If from a medial area a rational area be subtracted there arise two other irrational straight lines either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole*

For from the medial area $BC$ let the rational area $BD$ be subtracted

I say that the "side" of the remainder $EC$ becomes one of two irrational straight lines either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole







For let a rational straight line $FG$ be set out and let the areas be similarly applied

It follows then that $FH$ is rational and incommensurable in length with $FG$
while $KF$ is rational and commensurable in length with $FG$
therefore $FH$, $FK$ are rational straight lines commensurable in square only [x 13]

therefore $KH$ is an apotome and $FK$ the annex to it [x 73]

Now the square on $HF$ is greater than the square on $FK$ either by the square on a straight line commensurable with $HF$ or by the square on a straight line incommensurable with it

If then the square on $HF$ is greater than the square on $FK$ by the square on a straight line commensurable with $HF$,

while the annex $FK$ is commensurable in length with the rational straight line $FG$ set out

$KH$ is a second apotome [x Deff III 2]

But $FG$ is rational,

so that the 'side' of $LH$, that is of $EC$, is a first apotome of a medial straight line [x 92]

But, if the square on $HF$ is greater than the square on $FK$ by the square on a straight line incommensurable with $HF$,

while the annex $FK$ is commensurable in length with the rational straight line $FG$ set out,

$KH$ is a fifth apotome, [x Deff III 5]

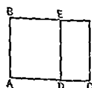
so that the 'side' of $EC$ is a straight line which produces with a rational area a medial whole [x 95]

Q E D

## Proposition 110

*If from a medial area there be subtracted a medial area incommensurable with the whole the two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole*

For as in the foregoing figures let there be subtracted from the medial area $BC$ the medial area $BD$ incommensurable with the whole







I say that the 'side' of $EC$ is one of two irrational straight lines either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole

For since each of the rectangles $BC$, $BD$ is medial

and $BC$ is incommensurable with $BD$

it follows that each of the straight lines $FH$, $FK$ will be rational and incommensurable in length with $FG$ [x 22]

And since $BC$ is incommensurable with $BD$

that is $GH$ with $GK$

$HF$ is also incommensurable with $FK$ [VI 1 x 11]

therefore $FH$ $FK$ are rational straight lines commensurable in square only

therefore $KH$ is an apotome [x 73]

If then the square on $FH$ is greater than the square on $FK$ by the square on a straight line commensurable with $FH$

while neither of the straight lines $FH$ $FK$ is commensurable in length with the rational straight line $FG$ set out

$KH$ is a third apotome [x Deff III 3]

But $KL$ is rational

and the rectangle contained by a rational straight line and a third apotome is irrational,

and the "side" of it is irrational and is called a second apotome of a medial straight line, [x 93]

so that the 'side' of $LH$, that is, of $EC$, is a second apotome of a medial straight line

But if *the square on $FH$ is greater than the square on $FK$ by the square on a straight line incommensurable with $FH$*

while neither of the straight lines $HF$ $FK$ is commensurable in length with $FG$

$KH$ is a sixth apotome [x Deff III 6]

But the "side' of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial whole [x 96]

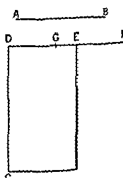
Therefore the "side of $LH$ that is of $EC$ is a straight line which produces with a medial area a medial whole Q E D

## PROPOSITION 111

*The apotome is not the same with the binomial straight line*

Let $AB$ be an apotome

I say that $AB$ is not the same with the binomial straight line




For if possible let it be so,

let a rational straight line $DC$ be set out and to $CD$ let there be applied the rectangle $CE$ equal to the square on $AB$ and producing $DE$ as breadth

Then since $AB$ is an apotome

$DE$ is a first apotome [x 97]

Let $EF$ be the annex to it

therefore $DF$ $FE$ are rational straight lines commensurable in square only

the square on $DF$ is greater than the square on $FE$ by the square on a straight line commensurable with $DF$

and $DF$ is commensurable in length with the rational straight line $DC$ set out [x Deff III 1]

Again since $AB$ is binomial

*therefore $DE$ is a first binomial straight line* [x 60]

Let it be divided into its terms at $G$

and let $DG$ be the greater term

therefore $DG$ $GE$ are rational straight lines commensurable in square only

*the square on $DG$ is greater than the square on $GE$ by the square on a straight* line commensurable with $DG$ and the greater term $DG$ is commensurable in length with the rational straight line $DC$ set out [x Deff II 1]

Therefore $DF$ is also commensurable in length with $DG$ [x 12]

therefore the remainder $GF$ is also commensurable in length with $DF$ [x 15]

But $DF$ is incommensurable in length with $EF$

therefore $FG$ is also incommensurable in length with $EF$ [x 13]

Therefore $GF$ $FE$ are rational straight lines commensurable in square only

therefore $EG$ is an apotome [x 73]

But it is also rational

which is impossible

Therefore the apotome is not the same with the binomial straight line

Q E D

*The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another*

For the square on a medial straight line if applied to a rational straight line produces as breadth a straight line rational and incommensurable in length with that to which it is applied [x 22]
while the square on an apotome, if applied to a rational straight line produces as breadth a first apotome, [x 97]
the square on a first apotome of a medial straight line if applied to a rational straight line produces as breadth a second apotome [x 98]
the square on a second apotome of a medial straight line if applied to a rational straight line produces as breadth a third apotome, [x 99]
the square on a minor straight line if applied to a rational straight line, produces as breadth a fourth apotome [x 100]
the square on the straight line which produces with a rational area a medial whole if applied to a rational straight line produces as breadth a fifth apotome [x 101]
and the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line produces as breadth a sixth apotome [x 102]

Since then the said breadths differ from the first and from one another from the first because it is rational, and from one another since they are not the same in order
it is clear that the irrational straight lines themselves also differ from one another

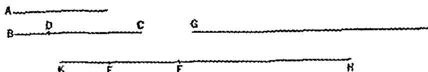
And, since the apotome has been proved not to be the same as the binomial straight line [x 111]
but if applied to a rational straight line the straight lines following the apotome produce as breadths each according to its own order, apotomes and those following the binomial straight line themselves also according to their order produce the binomials as breadths
therefore those following the apotome are different and those following the binomial straight line are different so that there are, in order, thirteen irrational straight lines in all

Medial
Binomial
First bimedial
Second bimedial
Major
Side of a rational plus a medial area
Side of the sum of two medial areas
*Apotome*
First apotome of a medial straight line
Second apotome of a medial straight line
Minor
Producing with a rational area a medial whole
Producing with a medial area a medial whole

PROPOSITION 112

*The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio and further the apotome so arising will have the same order as the binomial straight line*

Let $A$ be a rational straight line
let $BC$ be a binomial and let $DC$ be its greater term,
let the rectangle $BC$, $EF$ be equal to the square on $A$,




I say that $EF$ is an apotome the terms of which are commensurable with $CD$, $DB$ and in the same ratio and further $EF$ will have the same order as $BC$

For again let the rectangle $BD$ $G$ be equal to the square on $A$

Since then the rectangle $BC$ $EF$ is equal to the rectangle $BD$ $G$
therefore as $CB$ is to $BD$ so is $G$ to $EF$ [vi 16]

But $CB$ is greater than $BD$
therefore $G$ is also greater than $EF$ [v 16 v 14]

Let $EH$ be equal to $G$
therefore as $CB$ is to $BD$ so is $HE$ to $EF$
therefore *separando* as $CD$ is to $BD$ so is $HF$ to $FE$ [v 17]

Let it be contrived that as $HF$ is to $FE$ so is $FK$ to $KE$
therefore also the whole $HK$ is to the whole $KF$ as $FK$ is to $KE$
for as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [v 12]

But as $FK$ is to $KE$ so is $CD$ to $DB$ [v 11]
therefore also as $HK$ is to $KF$ so is $CD$ to $DB$ [id]

But the square on $CD$ is commensurable with the square on $DB$ [x 36]
therefore the square on $HK$ is also commensurable with the square on $KF$ [vi 22 x 11]

And as the square on $HK$ is to the square on $KF$ so is $HK$ to $KE$, since the three straight lines $HK$ $KF$ $KE$ are proportional [v Def 9]

Therefore $HK$ is commensurable in length with $KE$
so that $HE$ is also commensurable in length with $EK$ [x 15]

Now since the square on $A$ is equal to the rectangle $EH$ $BD$
while the square on $A$ is rational
therefore the rectangle $EH$ $BD$ is also rational

And it is applied to the rational straight line $BD$
therefore $EH$ is rational and commensurable in length with $BD$ [x 20]
so that $EK$ being commensurable with it is also rational and commensurable in length with $BD$

Since then as $CD$ is to $DB$ so is $FK$ to $KE$
while $CD$ $DB$ are straight lines commensurable in square only,
therefore $FK$ $KE$ are also commensurable in square only [x 11]

But $KE$ is rational,

therefore $FK$ is also rational

Therefore $FK$, $KE$ are rational straight lines commensurable in square only

therefore $EF$ is an apotome [x 73]

Now the square on $CD$ is greater than the square on $DB$ either by the square on a straight line commensurable with $CD$ or by the square on a straight line *incommensurable with it*

If then the square on $CD$ is greater than the square on $DB$ by the square on a straight line commensurable with $CD$ the square on $FK$ is also greater than the square on $KE$ by the square on a straight line commensurable with $FK$ [x 14]

And if $CD$ is commensurable in length with the rational straight line set out,

so also is $FK$ [x 11 12]

*if $BD$ is so commensurable*

so also is $KE$ [x 12]

but, if neither of the straight lines $CD$ $DB$ is so commensurable

neither of the straight lines $FK$, $KE$ is so

But if the square on $CD$ is greater than the square on $DB$ by the square on a straight line incommensurable with $CD$

the square on $FK$ is also greater than the square on $KE$ by the square on a straight line incommensurable with $FK$ [x 14]

And, if $CD$ is commensurable with the rational straight line set out,

so also is $FK$,

if $BD$ is so commensurable,

so also is $KE$,

but, if neither of the straight lines $CD$, $DB$ is so commensurable,

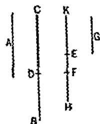
neither of the straight lines $FK$ $KE$ is so

so that $FE$ is an apotome the terms of which $FK$ $KE$ are commensurable with the terms $CD$ $DB$ of the binomial straight line and in the same ratio and it has the same order as $BC$ Q E D

## Proposition 113

*The square on a rational straight line if applied to an apotome produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio and further the binomial so arising has the same order as the apotome*

Let $A$ be a rational straight line and $BD$ an apotome and let the rectangle $BD$ $KH$ be equal to the square on $A$ so that the square on the rational straight line $A$ when applied to the apotome $BD$ produces $KH$ as breadth




I say that $KH$ is a binomial straight line the terms of which are commensurable with the terms of $BD$ and in the same ratio and further $KH$ has the same order as $BD$

For let $DC$ be the annex to $BD$

therefore $BC$ $CD$ are rational straight lines commensurable in square only [x 73]

Let the rectangle $BC$, $G$ be also equal to the square on $A$

But the square on $A$ is rational

therefore the rectangle $BC$, $G$ is also rational
And it has been applied to the rational straight line $BC$
therefore $G$ is rational and commensurable in length with $BC$ [x 20]
Since now the rectangle $BC$, $G$ is equal to the rectangle $BD$ $KH$,
therefore, proportionally, as $CB$ is to $BD$, so is $KH$ to $G$ [vi 16]
But $BC$ is greater than $BD$,
therefore $KH$ is also greater than $G$ [v 16 v 14]
Let $KE$ be made equal to $G$
therefore $KE$ is commensurable in length with $BC$
And since as $CB$ is to $BD$ so is $HK$ to $KE$
therefore *convertendo* as $BC$ is to $CD$ so is $KH$ to $HE$ [v 19 Por]
Let it be contrived that, as $KH$ is to $HE$ so is $HF$ to $FE$
therefore also the remainder $KF$ is to $FH$ as $KH$ is to $HE$ that is as $BC$ is to $CD$ [v 19]
But $BC$ $CD$ are commensurable in square only,
therefore $KF$ $FH$ are also commensurable in square only [x 11]
And since as $KH$ is to $HE$ so is $KF$ to $FH$
while as $KH$ is to $HE$ so is $HF$ to $FE$
therefore also as $KF$ is to $FH$ so is $HF$ to $FE$ [v 11]
so that also as the first is to the third so is the square on the first to the square on the second [v Def 9]
therefore also as $KF$ is to $FE$ so is the square on $KF$ to the square on $FH$
But the square on $KF$ is commensurable with the square on $FH$
for $KF$ $FH$ are commensurable in square,
therefore $KF$ is also commensurable in length with $FE$, [x 11]
so that $KF$ is also commensurable in length with $KE$ [x 15]
But $KE$ is rational and commensurable in length with $BC$,
therefore $KF$ is also rational and commensurable in length with $BC$ [x 12]
And since as $BC$ is to $CD$ so is $KF$ to $FH$
alternately as $BC$ is to $KF$ so is $DC$ to $FH$ [v 16]
But $BC$ is commensurable with $KF$
therefore $FH$ is also commensurable in length with $CD$ [x 11]
But $BC$ $CD$ are rational straight lines commensurable in square only
therefore $KF$ $FH$ are also rational straight lines (x Def 3) commensurable in square only
therefore $KH$ is binomial [x 36]
If now the square on $BC$ is greater than the square on $CD$ by the square on a straight line commensurable with $BC$
the square on $KF$ will also be greater than the square on $FH$ by the square on a straight line commensurable with $KF$ [x 14]
And, if $BC$ is commensurable in length with the rational straight line set out
so also is $KF$
if $CD$ is commensurable in length with the rational straight line set out,
so also is $FH$
but if neither of the straight lines $BC$ $CD$
then neither of the straight lines $KF$, $FH$
But, if the square on $BC$ is greater than the square on $CD$ by the square on a straight line incommensurable with $BC$
the square on $KF$ is also greater than the square on $FH$ by the square on a

straight line incommensurable with $KF$ [x 14]

And if $BC$ is commensurable with the rational straight line set out

so also is $KF$

if $CD$ is so commensurable

so also is $\Gamma H$

but if neither of the straight lines $BC$, $CD$,

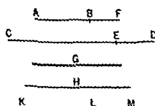
then neither of the straight lines $KF$, $\Gamma H$

Therefore $KH$ is a binomial straight line the terms of which $KF$, $FH$ are commensurable with the terms $BC$ $CD$ of the apotome and in the same ratio and further $KH$ has the same order as $BD$ Q E D

## Proposition 114

*If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio the ' side of the area is rational*

For let an area the rectangle $AB$ $CD$ be contained by the apotome $AB$ and the binomial straight line $CD$




and let $CE$ be the greater term of the latter let the terms $CE$ $ED$ of the binomial straight line be commensurable with the terms $AF$ $FB$ of the apotome and in the same ratio and let the side of the rectangle $AB$ $CD$ be $G$

I say that $G$ is rational

For let a rational straight line $H$ be set out and to $CD$ let there be applied a rectangle equal to the square on $H$ and producing $KL$ as breadth

Therefore $KL$ is an apotome

Let its terms be $KM$ $ML$ commensurable with the terms $CE$ $ED$ of the binomial straight line and in the same ratio [x 112]

But $CE$ $ED$ are also commensurable with $AF$, $FB$ and in the same ratio

therefore as $AF$ is to $FB$ so is $KM$ to $ML$

Therefore alternately as $AF$ is to $KM$ so is $BF$ to $LM$

therefore also the remainder $AB$ is to the remainder $KL$ as $AF$ is to $KM$ [v 19]

But $AF$ is commensurable with $KM$ [x 12]

therefore $AB$ is also commensurable with $KL$ [x 11]

And as $AB$ is to $KL$ so is the rectangle $CD$ $AB$ to the rectangle $CD$ $KL$ [vi 1]

therefore the rectangle $CD$ $AB$ is also commensurable with the rectangle $CD$ $KL$ [x 11]

But the rectangle $CD$ $KL$ is equal to the square on $H$

therefore the rectangle $CD$ $AB$ is commensurable with the square on $H$

But the square on $G$ is equal to the rectangle $CD$ $AB$

therefore the square on $G$ is commensurable with the square on $H$

But the square on $H$ is rational

therefore the square on $G$ is also rational

therefore $G$ is rational

And it is the side of the rectangle $CD$ $AB$

Therefore etc

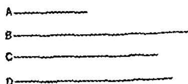
PORISM And it is made manifest to us by this also that it is possible for a rational area to be contained by irrational straight lines Q E D

## PROPOSITION 115

*From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding*

Let $A$ be a medial straight line

I say that from $A$ there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding




Let a rational straight line $B$ be set out and let the square on $C$ be equal to the rectangle $B$ $A$,

therefore $C$ is irrational [x Def 4]

for that which is contained by an irrational and a rational straight line is irrational [deduction from x 20]

And it is not the same with any of the preceding for the square on none of the preceding if applied to a rational straight line produces as breadth a medial straight line

Again let the square on $D$ be equal to the rectangle $B$, $C$,

therefore the square on $D$ is irrational [deduction from x 20]

Therefore $D$ is irrational [x Def 4]

and it is not the same with any of the preceding for the square on none of the preceding if applied to a rational straight line produces $C$ as breadth

Similarly if this arrangement proceeds *ad infinitum*, it is manifest that from the medial straight line there arise irrational straight lines infinite in number and none is the same with any of the preceding Q E D

# BOOK ELEVEN

## DEFINITIONS

1 A *solid* is that which has length breadth and depth

2 An extremity of a solid is a surface

3 A *straight line is at right angles to a plane* when it makes right angles with all the straight lines which meet it and are in the plane

4 A *plane is at right angles to a plane* when the straight lines drawn in one of the planes at right angles to the common section of the planes are at right angles to the remaining plane

5 The *inclination of a straight line to a plane* is assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane and a straight line joined from the point thus arising to the ex tremity of the straight line which is in the plane the angle contained by the straight line so drawn and the straight line standing up

6 The *inclination of a plane to a plane* is the acute angle contained by the straight lines drawn at right angles to the common section at the same point one in each of the planes

7 A plane is said to be *similarly inclined* to a plane as another is to another when the said angles of the inclinations are equal to one another

8 *Parallel planes* are those which do not meet

9 *Similar solid figures* are those contained by similar planes equal in multi tude

10 *Equal and similar solid figures* are those contained by similar planes equal in multitude and in magnitude

11 A *solid angle* is the inclination constituted by more than two lines which meet one another and are not in the same surface towards all the lines

Otherwise A *solid angle* is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point

12 A *pyramid* is a solid figure contained by planes which is constructed from one plane to one point

13 A *prism* is a solid figure contained by planes two of which namely those which are opposite are equal similar and parallel while the rest are parallelo-grams

14 When the diameter of a semicircle remaining fixed the semicircle is car ried round and restored again to the same position from which it began to be moved the figure so comprehended is a *sphere*

15 The *axis of the sphere* is the straight line which remains fixed and about which the semicircle is turned

16 The *centre of the sphere* is the same as that of the semicircle

17 A *diameter of the sphere* is any straight line drawn through the centre and

terminated in both directions by the surface of the sphere

18 When one side of those about the right angle in a right angled triangle remaining fixed the triangle is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *cone*

And if the straight line which remains fixed be equal to the remaining side about the right angle which is carried round, the cone will be *right-angled*, if less *obtuse angled*, and if greater *acute-angled*

19 The *axis of the cone* is the straight line which remains fixed and about which the triangle is turned

20 And the *base* is the circle described by the straight line which is carried round

21 When one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved the figure so comprehended is a *cylinder*

22 The *axis of the cylinder* is the straight line which remains fixed and about which the parallelogram is turned

23 And the *bases* are the circles described by the two sides opposite to one another which are carried round

24 *Similar cones and cylinders* are those in which the axes and the diameters of the bases are proportional

25 A *cube* is a solid figure contained by six equal squares

26 An *octahedron* is a solid figure contained by eight equal and equilateral triangles

27 An *icosahedron* is a solid figure contained by twenty equal and equilateral triangles

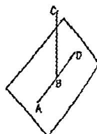
28 A *dodecahedron* is a solid figure contained by twelve equal equilateral and equiangular pentagons

## BOOK XI PROPOSITIONS

### Proposition I

*A part of a straight line cannot be in the plane of reference and a part in a plane more elevated*

For if possible let a part $AB$ of the straight line $ABC$ be in the plane of reference and a part $BC$ in a plane more elevated




There will then be in the plane of reference some straight line continuous with $AB$ in a straight line

Let it be $BD$

therefore $AB$ is a common segment of the two straight lines $ABC$ $ABD$

which is impossible inasmuch as if we describe a circle with centre $B$ and distance $AB$ the diameters will cut off unequal circumferences of the circle

Therefore a part of a straight line cannot be in the plane of reference and a part in a plane more elevated

Q E D

PROPOSITION 2

*If two straight lines cut one another, they are in one plane and every triangle is in one plane*

For let the two straight lines $AB$, $CD$ cut one another at the point $E$,

I say that $AB$ $CD$ are in one plane and every triangle is in one plane

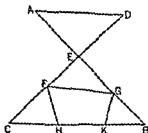
For let points $F$, $G$ be taken at random on $EC$, $EB$,

let $CB$, $FG$ be joined

and let $FH$, $GK$ be drawn across

I say first that the triangle $ECB$ is in one plane

For if part of the triangle $ECB$ either $FHC$ or $GBK$ is in the plane of reference and the rest in another




a part also of one of the straight lines $EC$, $EB$ will be in the plane of reference and a part in another

But if the part $FCBG$ of the triangle $ECB$ be in the plane of reference and the rest in another a part also of both the straight lines $EC$ $EB$ will be in the plane of reference and a part in another which was proved absurd [XI 1]

Therefore the triangle $ECB$ is in one plane

But in whatever plane the triangle $ECB$ is in that plane also is each of the straight lines $EC$, $EB$ and in whatever plane each of the straight lines $EC$ $EB$ is in that plane are $AB$ $CD$ also [XI 1]

Therefore the straight lines $AB$ $CD$ are in one plane and every triangle is in one plane Q E D

PROPOSITION 3

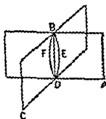
*If two planes cut one another their common section is a straight line*

For let the two planes $AB$ $BC$ cut one another

and let the line $DB$ be their common section

I say that the line $DB$ is a straight line

For if not from $D$ to $B$ let the straight line $DEB$ be joined in the plane $AB$,




and in the plane $BC$ the straight line $DFB$

Then the two straight lines $DEB$ $DFB$ will have the same extremities and will clearly enclose an area which is absurd

Therefore $DEB$ $DFB$ are not straight lines

Similarly we can prove that neither will there be any other straight line joined from $D$ to $B$ except $DB$ the common section of the planes $AB$ $BC$

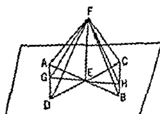
Therefore etc Q E D

PROPOSITION 4

*If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section it will also be at right angles to the plane through them*

For let a straight line $EF$ be set up at right angles to the two straight lines $AB$, $CD$, which cut one another at the point $E$, from $E$;

I say that $EF$ is also at right angles to the plane through $AB$, $CD$.




For let $AE$, $EB$, $CE$, $ED$ be cut off equal to one another,

and let any straight line $GEH$ be drawn across through $E$, at random;

let $AD$, $CB$ be joined,

and further, let $FA$, $FG$, $FD$, $FC$, $FH$, $FB$ be joined from the point $F$ taken at random <on $EF$>.

Now, since the two straight lines $AE$, $ED$ are equal to the two straight lines $CE$, $EB$, and contain equal angles, [I. 15]

therefore the base $AD$ is equal to the base $CB$,

and the triangle $AED$ will be equal to the triangle $CEB$; [I. 4]

so that the angle $DAE$ is also equal to the angle $EBC$.

But the angle $AEG$ is also equal to the angle $BEH$; [I. 15]

therefore $AGE$, $BEH$ are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely that adjacent to the equal angles, that is to say $AE$ to $EB$;

therefore they will also have the remaining sides equal to the remaining sides. [I. 26]

Therefore $GE$ is equal to $EH$, and $AG$ to $BH$.

And, since $AE$ is equal to $EB$,

while $FE$ is common and at right angles,

therefore the base $FA$ is equal to the base $FB$. [I. 4]

For the same reason

$FC$ is also equal to $FD$.

And, since $AD$ is equal to $CB$,

and $FA$ is also equal to $FB$,

the two sides $FA$, $AD$ are equal to the two sides $FB$, $BC$ respectively;

and the base $FD$ was proved equal to the base $FC$;

therefore the angle $FAD$ is also equal to the angle $FBC$. [I. 8]

And since, again, $AG$ was proved equal to $BH$,

and further $FA$ also equal to $FB$,

the two sides $FA$, $AG$ are equal to the two sides $FB$, $BH$.

And the angle $FAG$ was proved equal to the angle $FBH$;

therefore the base $FG$ is equal to the base $FH$. [I. 4]

Now since, again, $GE$ was proved equal to $EH$,

and $EF$ is common,

the two sides $GE$, $EF$ are equal to the two sides $HE$, $EF$;

and the base $FG$ is equal to the base $FH$;

therefore the angle $GEF$ is equal to the angle $HEF$. [I. 8]

Therefore each of the angles $GEF$, $HEF$ is right.

Therefore $FE$ is at right angles to $GH$ drawn at random through $E$.

Similarly we can prove that $FE$ will also make right angles with all the straight lines which meet it and are in the plane of reference.

But a straight line is at right angles to a plane when it makes right angles

with all the straight lines which meet it and are in that same plane, [XI Def 3]
therefore $FE$ is at right angles to the plane of reference

But the plane of reference is the plane through the straight lines $AB$ $CD$

Therefore $FE$ is at right angles to the plane through $AB$, $CD$

Therefore etc Q E D

PROPOSITION 5

*If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section the three straight lines are in one plane*

For let a straight line $AB$ be set up at right angles to the three straight lines $BC$, $BD$, $BE$, at their point of meeting at $B$,

I say that $BC$ $BD$, $BE$ are in one plane

For suppose they are not but, if possible, let $BD$, $BE$ be in the plane of reference and $BC$ in one more elevated

let the plane through $AB$, $BC$ be produced,
it will thus make as common section in the plane of reference, a straight line [XI 3]

Let it make $BF$

Therefore the three straight lines $AB$, $BC$, $BF$ are in one plane, namely that drawn through $AB$ $BC$

Now, since $AB$ is at right angles to each of the straight lines $BD$, $BE$ therefore $AB$ is also at right angles to the plane through $BD$, $BE$ [XI 4]

But the plane through $BD$, $BE$ is the plane of reference,

therefore $AB$ is at right angles to the plane of reference

Thus $AB$ will also make right angles with all the straight lines which meet it and are in the plane of reference [XI Def 3]

But $BF$ which is in the plane of reference meets it,

therefore the angle $ABF$ is right

But by hypothesis the angle $ABC$ is also right,

therefore the angle $ABF$ is equal to the angle $ABC$

And they are in one plane

which is impossible

Therefore the straight line $BC$ is not in a more elevated plane,

therefore the three straight lines $BC$, $BD$ $BE$ are in one plane

Therefore if a straight line be set up at right angles to three straight lines, at their point of meeting, the three straight lines are in one plane Q E D

PROPOSITION 6

*If two straight lines be at right angles to the same plane the straight lines will be parallel*

For let the two straight lines $AB$ $CD$ be at right angles to the plane of reference

I say that $AB$ is parallel to $CD$

For let them meet the plane of reference at the points $B$, $D$

let the straight line $BD$ be joined

let $DE$ be drawn in the plane of reference at right angles to $BD$,

let $DE$ be made equal to $AB$

and let $BE$, $AE$, $AD$ be joined

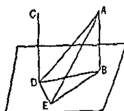
Now, since $AB$ is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [XI Def 3]

But each of the straight lines $BD$, $BE$ is in the plane of reference and meets $AB$,

therefore each of the angles $ABD$ $ABE$ is right

For the same reason

each of the angles $CDB$, $CDE$ is also right




And since $AB$ is equal to $DE$

and $BD$ is common,

the two sides $AB$, $BD$ are equal to the two sides $ED$, $DB$,

and they include right angles,

therefore the base $AD$ is equal to the base $BE$ [I 4]

And since $AB$ is equal to $DE$,

while $AD$ is also equal to $BE$,

the two sides $AB$, $BE$ are equal to the two sides $ED$, $DA$,

and $AE$ is their common base,

therefore the angle $ABE$ is equal to the angle $EDA$ [I 8]

But the angle $ABE$ is right

therefore the angle $EDA$ is also right,

therefore $ED$ is at right angles to $DA$

But it is also at right angles to each of the straight lines $BD$, $DC$,

therefore $ED$ is set up at right angles to the three straight lines $BD$ $DA$ $DC$ at their point of meeting

therefore the three straight lines $BD$ $DA$, $DC$ are in one plane [XI 5]

But, in whatever plane $DB$ $DA$ are in that plane is $AB$ also

for every triangle is in one plane, [XI 2]

therefore the straight lines $AB$ $BD$, $DC$ are in one plane

And each of the angles $ABD$ $BDC$ is right

therefore $AB$ is parallel to $CD$ [I 28]

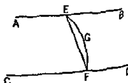
Therefore etc Q E D

## Proposition 7

*If two straight lines be parallel and points be taken at random on each of them the straight line joining the points is in the same plane with the parallel straight lines*

Let $AB$ $CD$ be two parallel straight lines

and let points $E$, $F$ be taken at random on them respectively




I say that the straight line joining the points $E$, $F$ is in the same plane with the parallel straight lines

For suppose it is not, but if possible let it be in a more elevated plane as $EGF$

and let a plane be drawn through $EGF$

it will then make as section in the plane of reference a straight line [XI 3]

Let it make it as $EF$,

therefore the two straight lines $EGF$ $EF$ will enclose an area which is impossible

Therefore the straight line joined from $E$ to $F$ is not in a plane more elevated, therefore the straight line joined from $E$ to $F$ is in the plane through the parallel straight lines $AB$, $CD$

Therefore etc. Q. E. D.

## Proposition 8

*If two straight lines be parallel and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane*

Let $AB$, $CD$ be two parallel straight lines,

and let one of them, $AB$, be at right angles to the plane of reference,

I say that the remaining one, $CD$, will also be at right angles to the same plane

For let $AB$, $CD$ meet the plane of reference at the points $B$, $D$,

and let $BD$ be joined,

therefore $AB$, $CD$, $BD$ are in one plane [XI 7]

Let $DE$ be drawn in the plane of reference, at right angles to $BD$,

let $DE$ be made equal to $AB$

and let $BE$, $AE$, $AD$ be joined

Now since $AB$ is at right angles to the plane of reference therefore $AB$ is also at right angles to all the straight lines which meet it and are in the plane of reference [XI Def 3]

therefore each of the angles $ABD$, $ABE$ is right

And since the straight line $BD$ has fallen on the parallels $AB$, $CD$,

therefore the angles $ABD$, $CDB$ are equal to two right angles [I 29]

But the angle $ABD$ is right

therefore the angle $CDB$ is also right

therefore $CD$ is at right angles to $BD$

And since $AB$ is equal to $DE$

and $BD$ is common

the two sides $AB$, $BD$ are equal to the two sides $ED$, $DB$,

and the angle $ABD$ is equal to the angle $EDB$

for each is right

therefore the base $AD$ is equal to the base $BE$

And, since $AB$ is equal to $DE$,

and $BE$ to $AD$,

the two sides $AB$, $BE$ are equal to the two sides $ED$, $DA$ respectively,

and $AE$ is their common base,

therefore the angle $ABE$ is equal to the angle $EDA$

But the angle $ABE$ is right

therefore the angle $EDA$ is also right

therefore $ED$ is at right angles to $AD$

But it is also at right angles to $DB$

therefore $ED$ is also at right angles to the plane through $BD$, $DA$ [XI 4]

Therefore $ED$ will also make right angles with all the straight lines which meet it and are in the plane through $BD$, $DA$

But $DC$ is in the plane through $BD$, $DA$, inasmuch as $AB$, $BD$ are in the

plane through $BD$, $DA$, [XI 4]

and $DC$ is also in the plane in which $AB$, $BD$ are

Therefore $ED$ is at right angles to $DC$,

so that $CD$ is also at right angles to $DE$

But $CD$ is also at right angles to $BD$

Therefore $CD$ is set up at right angles to the two straight lines $DE$, $DB$ which cut one another from the point of section at $D$,

so that $CD$ is also at right angles to the plane through $DE$, $DB$ [XI 4]

But the plane through $DE$, $DB$ is the plane of reference,

therefore $CD$ is at right angles to the plane of reference

Therefore etc Q E D

PROPOSITION 9

*Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another*

For let each of the straight lines $AB$, $CD$ be parallel to $EF$, not being in the same plane with it

I say that $AB$ is parallel to $CD$

For let a point $G$ be taken at random on $EF$

and from it let there be drawn $GH$ in the plane through $EF$, $AB$, at right angles to $EF$ and $GK$ in the plane through $FE$, $CD$ again at right angles to $EF$

Now since $EF$ is at right angles to each of the straight lines $GH$, $GK$

therefore $EF$ is also at right angles to the plane through $GH$, $GK$ [XI 4]

And $EF$ is parallel to $AB$

therefore $AB$ is also at right angles to the plane through $HG$, $GK$ [XI 8]

For the same reason

$CD$ is also at right angles to the plane through $HG$, $GK$,

therefore each of the straight lines $AB$, $CD$ is at right angles to the plane through $HG$, $GK$

But if two straight lines be at right angles to the same plane the straight lines are parallel [XI 6]

therefore $AB$ is parallel to $CD$ Q E D

PROPOSITION 10

*If two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane they will contain equal angles*

For let the two straight lines $AB$, $BC$ meeting one another be parallel to the two straight lines $DE$, $EF$ meeting one another not in the same plane,

I say that the angle $ABC$ is equal to the angle $DEF$

For let $BA$, $BC$, $ED$, $EF$ be cut off equal to one another and let $AD$, $CF$, $BE$, $AC$, $DF$ be joined

Now since $BA$ is equal and parallel to $ED$

therefore $AD$ is also equal and parallel to $BE$ [I 33]

For the same reason

$CF$ is also equal and parallel to $BE$

Therefore each of the straight lines $AD$, $CF$ is equal and parallel to $BE$

But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another, [XI 9]

therefore $AD$ is parallel and equal to $CF$

And $AC$, $DF$ join them,

therefore $AC$ is also equal and parallel to $DF$ [I 33]

Now, since the two sides $AB$, $BC$ are equal to the two sides $DE$, $EF$,

and the base $AC$ is equal to the base $DF$

therefore the angle $ABC$ is equal to the angle $DEF$ [I 8]

Therefore etc Q E D

## Proposition 11

*From a given elevated point to draw a straight line perpendicular to a given plane*

Let $A$ be the given elevated point and the plane of reference the given plane,

thus it is required to draw from the point $A$ a straight line perpendicular to the plane of reference

Let any straight line $BC$ be drawn at random in the plane of reference

and let $AD$ be drawn from the point $A$ perpendicular to $BC$ [I 12]

If then $AD$ is also perpendicular to the plane of reference that which was enjoined will have been done

*But if not let $DE$ be drawn from the point $D$ at right angles to $BC$ and in the plane of reference,* [I 11]

let $AF$ be drawn from $A$ perpendicular to $DE$ [I 12]

and let $GH$ be drawn through the point $F$ parallel to $BC$ [I 31]

Now since $BC$ is at right angles to each of the straight lines $DA$, $DE$,

therefore $BC$ is also at right angles to the plane through $ED$, $DA$ [XI 4]

And $GH$ is parallel to it,

but if two straight lines be parallel and one of them be at right angles to any plane the remaining one will also be at right angles to the same plane [XI 8]

therefore $GH$ is also at right angles to the plane through $ED$, $DA$

Therefore $GH$ is also at right angles to all the straight lines which meet it and are in the plane through $ED$ $DA$ [XI Def 3]

But $AF$ meets it and is in the plane through $ED$ $DA$

therefore $GH$ is at right angles to $FA$,

so that $FA$ is also at right angles to $GH$

But *$AF$ is also at right angles to $DE$*

therefore $AF$ is at right angles to each of the straight lines $GH$ $DE$

But if a straight line be set up at right angles to two straight lines which cut one another at the point of section it will also be at right angles to the plane through them [XI 4]

therefore $FA$ is at right angles to the plane through $ED$ $GH$

But the plane through $ED$ $GH$ is the plane of reference

therefore $AF$ is at right angles to the plane of reference

Therefore from the given elevated point $A$ the straight line $AF$ has been drawn perpendicular to the plane of reference

Q E F

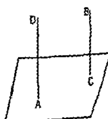
### Proposition 12

*To set up a straight line at right angles to a given plane from a given point in it*

Let the plane of reference be the given plane,

and $A$ the point in it,

thus it is required to set up from the point $A$ a straight line at right angles to the plane of reference




Let any elevated point $B$ be conceived

from $B$ let $BC$ be drawn perpendicular to the plane of reference [XI 11]

and through the point $A$ let $AD$ be drawn parallel to $BC$ [I 31]

Then since $AD$ $CB$ are two parallel straight lines while one of them $BC$ is at right angles to the plane of reference therefore the remaining one $AD$ is also at right angles to the plane of reference [XI 8]

Therefore $AD$ has been set up at right angles to the given plane from the point $A$ in it

Q E F

### Proposition 13

*From the same point two straight lines cannot be set up at right angles to the same plane on the same side*

For if possible from the same point $A$ let the two straight lines $AB$, $AC$ be set up at right angles to the plane of reference and on the same side

and let a plane be drawn through $BA$ $AC$

it will then make as section through $A$ in the plane of reference a straight line [XI 3]

Let it make $DAE$

therefore the straight lines $AB$ $AC$, $DAE$ are in one plane

And since $CA$ is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [XI Def 3]

But $DAE$ meets it and is in the plane of reference

therefore the angle $CAE$ is right

For the same reason

the angle $BAE$ is also right

therefore the angle $CAE$ is equal to the angle $BAE$

And they are in one plane

which is impossible

Therefore etc

Q E D

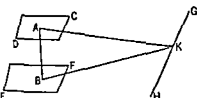
### Proposition 14

*Planes to which the same straight line is at right angles will be parallel*

For let any straight line $AB$ be at right angles to each of the planes $CD$, $EF$,

I say that the planes are parallel

For, if not they will meet when produced




Let them meet,

they will then make as common section, a straight line [XI 3]

Let them make *GH*,

let a point *K* be taken at random on *GH*, and let *AK*, *BK* be joined

Now, since *AB* is at right angles to the plane *EF*

therefore *AB* is also at right angles to *BK* which is a straight line in the plane *EF* produced, [XI Def 3]

therefore the angle *ABK* is right

For the same reason

the angle *BAK* is also right

Thus in the triangle *ABK* the two angles *ABK* *BAK* are equal to two right angles

which is impossible [I 17]

Therefore the planes *CD* *EF* will not meet when produced

therefore the planes *CD* *EF* are parallel [XI Def 8]

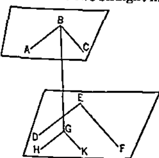
Therefore planes to which the same straight line is at right angles are parallel

Q E D

## Proposition 15

*If two straight lines meeting one another be parallel to two straight lines meeting one another not being in the same plane, the planes through them are parallel*

For let the two straight lines *AB*, *BC* meeting one another be parallel to the two straight lines *DE* *EF* meeting one another not being in the same plane, I say that the planes produced through *AB* *BC* and *DE*, *EF* will not meet one another




For let *BG* be drawn from the point *B* perpendicular to the plane through *DE* *EF* [XI 11], and let it meet the plane at the point *G*, through *G* let *GH* be drawn parallel to *ED* and *GK* parallel to *EF* [I 31]

Now since *BG* is at right angles to the plane through *DE* *EF*

therefore it will also make right angles with all the straight lines which meet it and are in the plane through *DE* *EF* [XI Def 3]

But each of the straight lines *GH* *GK* meets it and is in the plane through *DE* *EF*

therefore each of the angles *BGH*, *BGK* is right

And since *BA* is parallel to *GH*, [XI 9]

therefore the angles *GBA*, *BGH* are equal to two right angles [I 29]

But the angle *BGH* is right

therefore the angle *GBA* is also right

therefore *GB* is at right angles to *BA*

For the same reason

$GB$ is also at right angles to $BC$

Since then the straight line $GB$ is set up at right angles to the two straight lines $BA$, $BC$ which cut one another, therefore $GB$ is also at right angles to the plane through $BA$, $BC$ [XI 4]

But planes to which the same straight line is at right angles are parallel, [XI 14]

therefore the plane through $AB$, $BC$ is parallel to the plane through $DE$, $EF$

Therefore if two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, the planes through them are parallel

Q E D

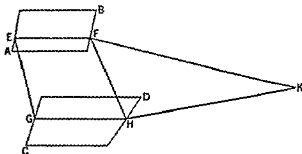
PROPOSITION 16

*If two parallel planes be cut by any plane their common sections are parallel*

For let the two parallel planes $AB$, $CD$ be cut by the plane $EFGH$,

and let $EF$, $GH$ be their common sections,

I say that $EF$ is parallel to $GH$




For if not $EF$ $GH$ will when produced meet either in the direction of $F$ $H$ or of $E$ $G$

Let them be produced as in the direction of $F$ $H$ and let them first, meet at $K$

Now since $EFK$ is in the plane $AB$

therefore all the points on $EFK$ are also in the plane $AB$ [XI 1]

But $K$ is one of the points on the straight line $EFK$,

therefore $K$ is in the plane $AB$

For the same reason

$K$ is also in the plane $CD$

therefore the planes $AB$ $CD$ will meet when produced

But they do not meet because they are by hypothesis parallel

therefore the straight lines $EF$ $GH$ will not meet when produced in the direction of $F$ $H$

Similarly we can prove that neither will the straight lines $EF$, $GH$ meet when produced in the direction of $E$ $G$

But straight lines which do not meet in either direction are parallel [I Def 23]

Therefore $EF$ is parallel to $GH$

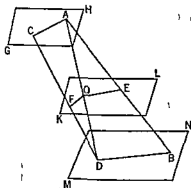
Therefore etc

Q E D

### PROPOSITION 17

*If two straight lines be cut by parallel planes they will be cut in the same ratios*

For let the two straight lines $AB$ $CD$ be cut by the parallel planes $GH$, $KL$, $MN$ at the points $A$, $E$, $B$ and $C$, $F$, $D$,




I say that, as the straight line $AE$ is to $EB$ so is $CF$ to $FD$

For let $AC$ $BD$, $AD$ be joined, let $AD$ meet the plane $KL$ at the point $O$, and let $EO$ $OF$ be joined

Now, since the two parallel planes $KL$ $MN$ are cut by the plane $EBDO$, their common sections $EO$ $BD$ are parallel [XI 16]

For the same reason since the two parallel planes $GH$ $KL$ are cut by the plane $AOFC$, their common sections $AC$, $OF$ are parallel [id]

And, since the straight line $EO$ has been drawn parallel to $BD$, one of the sides of the triangle $ABD$

therefore proportionally, as $AE$ is to $EB$ so is $AO$ to $OD$ [VI 2]

Again since the straight line $OF$ has been drawn parallel to $AC$ one of the sides of the triangle $ADC$

proportionally as $AO$ is to $OD$, so is $CF$ to $FD$ [id]

But it was also proved that as $AO$ is to $OD$ so is $AE$ to $EB$,

therefore also as $AE$ is to $EB$, so is $CF$ to $FD$ [V 11]

Therefore etc Q E D

### PROPOSITION 18

*If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane*

For let any straight line $AB$ be at right angles to the plane of reference

I say that all the planes through $AB$ are also at right angles to the plane of reference

For let the plane $DE$ be drawn through $AB$ let $CE$ be the common section of the plane $DE$ and the plane of reference

let a point $F$ be taken at random on $CE$ and from $F$ let $FG$ be drawn in the plane $DE$ at right angles to $CE$ [I 11]

Now since $AB$ is at right angles to the plane of reference $AB$ is also at right angles to all the straight lines which meet it and are in the plane of reference [XI Def 3]

so that it is also at right angles to $CE$

therefore the angle $ABF$ is right

But the angle $GFB$ is also right

therefore $AB$ is parallel to $FG$ [I 28]

But $AB$ is at right angles to the plane of reference,

therefore $FG$ is also at right angles to the plane of reference [XI 8]

Now a plane is at right angles to a plane when the straight lines drawn in one of the planes at right angles to the common section of the planes are at right angles to the remaining plane [XI Def 4]

And $FG$ drawn in one of the planes $DE$ at right angles to $CE$ the common section of the planes was proved to be at right angles to the plane of reference therefore the plane $DE$ is at right angles to the plane of reference

Similarly also it can be proved that all the planes through $AB$ are at right angles to the plane of reference

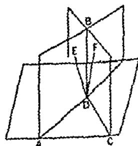
Therefore etc Q E D

PROPOSITION 19

*If two planes which cut one another be at right angles to any plane their common section will also be at right angles to the same plane*

For let the two planes $AB$ $BC$ be at right angles to the plane of reference and let $BD$ be their common section

I say that $BD$ is at right angles to the plane of reference




For suppose it is not and from the point $D$ let $DE$ be drawn in the plane $AB$ at right angles to the straight line $AD$ and $DF$ in the plane $BC$ at right angles to $CD$

Now since the plane $AB$ is at right angles to the plane of reference

and $DE$ has been drawn in the plane $AB$ at right angles to $AD$ their common section

therefore $DF$ is at right angles to the plane of reference [XI Def 4]

Similarly we can prove that

$DF$ is also at right angles to the plane of reference

Therefore from the same point $D$ two straight lines have been set up at right angles to the plane of reference on the same side

which is impossible [XI 13]

Therefore no straight line except the common section $DB$ of the planes $AB$, $BC$ can be set up from the point $D$ at right angles to the plane of reference

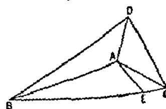
Therefore etc Q E D

PROPOSITION 20

*If a solid angle be contained by three plane angles any two taken together in any manner are greater than the remaining one*

For let the solid angle at $A$ be contained by the three plane angles $BAC$ $CAD$ $DAB$

I say that any two of the angles $BAC$ $CAD$ $DAB$ taken together in any manner are greater than the remaining one




If now the angles $BAC$ $CAD$ $DAB$ are equal to one another it is manifest that any two are greater than the remaining one

But if not let $BAC$ be greater

and on the straight line $AB$ and at the point $A$ on it let the angle $BAE$ be

constructed in the plane through $BA$ $AC$, equal to the angle $DAB$

let $AE$ be made equal to $AD$,

and let $BEC$, drawn across through the point $E$ cut the straight lines $AB$, $AC$ at the points $B$ $C$,

let $DB$, $DC$ be joined

Now since $DA$ is equal to $AE$,

and $AB$ is common

two sides are equal to two sides,

and the angle $DAB$ is equal to the angle $BAE$,

therefore the base $DB$ is equal to the base $BE$ [I 4]

And, since the two sides $BD$, $DC$ are greater than $BC$ [I 20]

and of these $DB$ was proved equal to $BE$

therefore the remainder $DC$ is greater than the remainder $EC$

Now, since $DA$ is equal to $AE$,

and $AC$ is common

and the base $DC$ is greater than the base $EC$,

therefore the angle $DAC$ is greater than the angle $EAC$ [I 25]

But the angle $DAB$ was made equal to the angle $BAE$,

therefore the angles $DAB$, $DAC$ are greater than the angle $BAC$

Similarly we can prove that the remaining angles also taken together two and two, are greater than the remaining one

Therefore etc Q E D

## Proposition 21

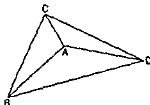
*Any solid angle is contained by plane angles less than four right angles*

Let the angle at $A$ be a solid angle contained by the plane angles $BAC$, $CAD$ $DAB$,

I say that the angles $BAC$ $CAD$ $DAB$ are less than four right angles

For let points $B$, $C$, $D$ be taken at random on the straight lines $AB$, $AC$ $AD$ respectively,

and let $BC$, $CD$, $DB$ be joined




Now since the solid angle at $B$ is contained by the three plane angles $CBA$ $ABD$ $CBD$ any two are greater than the remaining one [XI 20]

therefore the angles $CBA$ $ABD$ are greater than the angle $CBD$

For the same reason

the angles $BCA$ $ACD$ are also greater than the angle $BCD$, and the angles $CDA$ $ADB$ are greater than the angle $CDB$

therefore the six angles $CBA$ $ABD$ $BCA$ $ACD$ $CDA$ $ADB$ are greater than the three angles $CBD$ $BCD$ $CDB$

But the three angles $CBD$ $BDC$ $BCD$ are equal to two right angles [I 32]

therefore the six angles $CBA$ $ABD$ $BCA$, $ACD$ $CDA$ $ADB$ are greater than two right angles

And since the three angles of each of the triangles $ABC$ $ACD$ $ADB$ are equal to two right angles

therefore the nine angles of the three triangles the angles $CBA$ $ACB$, $BAC$, $ACD$ $CDA$ $CAD$ $ADB$, $DBA$ $BAD$ are equal to six right angles

and of them the six angles $ABC$, $BCA$, $ACD$, $CDA$, $ADB$, $DBA$ are greater than two right angles,
therefore the remaining three angles $BAC$, $CAD$, $DAB$ containing the solid angle are less than four right angles.

Therefore etc. Q. E. D.

## Proposition 22

*If there be three plane angles of which two taken together in any manner are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines.*

Let there be three plane angles $ABC$, $DEF$, $GHK$, of which two taken together in any manner are greater than the remaining one, namely

the angles $ABC$, $DEF$ greater than the angle $GHK$,

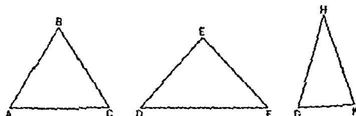
the angles $DEF$, $GHK$ greater than the angle $ABC$,

and further, the angles $GHK$, $ABC$ greater than the angle $DEF$;

let the straight lines $AB$, $BC$, $DE$, $EF$, $GH$, $HK$ be equal,

and let $AC$, $DF$, $GK$ be joined;

I say that it is possible to construct a triangle out of straight lines equal to $AC$, $DF$, $GK$, that is, that any two of the straight lines $AC$, $DF$, $GK$ are greater than the remaining one.



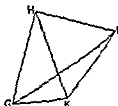

Now, if the angles $ABC$, $DEF$, $GHK$ are equal to one another, it is manifest that, $AC$, $DF$, $GK$ being equal also, it is possible to construct a triangle out of straight lines equal to $AC$, $DF$, $GK$.

But, if not, let them be unequal,

and on the straight line $HK$, and at the point $H$ on it, let the angle $KHL$ be constructed equal to the angle $ABC$;

let $HL$ be made equal to one of the straight lines $AB$, $BC$, $DE$, $EF$, $GH$, $HK$,

and let $KL$, $GL$ be joined.




Now, since the two sides $AB$, $BC$ are equal to the two sides $KH$, $HL$,

and the angle at $B$ is equal to the angle $KHL$,

therefore the base $AC$ is equal to the base $KL$. [I. 4]

And, since the angles $ABC$, $GHK$ are greater than the angle $DEF$,

while the angle $ABC$ is equal to the angle $KHL$,

therefore the angle $GHL$ is greater than the angle $DEF$.

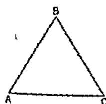
And, since the two sides $GH$, $HL$ are equal to the two sides $DE$, $EF$,

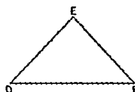
and the angle $GHL$ is greater than the angle $DEF$,
therefore the base $GL$ is greater than the base $DF$ [I 24]
But $GK$, $KL$ are greater than $GL$
Therefore $GK$, $KL$ are much greater than $DF$
But $KL$ is equal to $AC$,
therefore $AC$, $GK$ are greater than the remaining straight line $DF$
Similarly we can prove that
$AC$, $DF$ are greater than $GK$,
and further $DF$, $GK$ are greater than $AC$
Therefore it is possible to construct a triangle out of straight lines equal to $AC$, $DF$, $GK$ Q E D

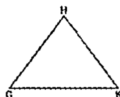
## Proposition 23

*To construct a solid angle out of three plane angles two of which taken together in any manner, are greater than the remaining one thus the three angles must be less than four right angles*

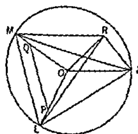
Let the angles $ABC$, $DEF$, $GHK$ be the three given plane angles and let two of these taken together in any manner be greater than the remaining one, while further the three are less than four right angles
thus it is required to construct a solid angle out of angles equal to the angles $ABC$ $DEF$, $GHK$




Let $AB$, $BC$, $DE$, $EF$ $GH$ $HK$ be cut off equal to one another
and let $AC$ $DF$ $GK$ be joined
it is therefore possible to construct a triangle out of straight lines equal to $AC$, $DF$, $GK$ [XI 22]




Let $LMN$ be so constructed that $AC$ is equal to $LM$, $DF$ to $MN$ and further, $GK$ to $NL$,
let the circle $LMN$ be described about the triangle $LMN$
let its centre be taken and let it be $O$,
let $LO$ $MO$ $NO$ be joined
I say that $AB$ is greater than $LO$
For if not $AB$ is either equal to $LO$, or less
First let it be equal
Then since $AB$ is equal to $LO$
while $AB$ is equal to $BC$ and $OL$ to $OM$
the two sides $AB$ $BC$ are equal to the two sides $LO$ $OM$ respectively,
and by hypothesis the base $AC$ is equal to the base $LM$,
therefore the angle $ABC$ is equal to the angle $LOM$ [I 8]
For the same reason

the angle $DEF$ is also equal to the angle $MON$,
and further the angle $GHK$ to the angle $NOL$,
therefore the three angles $ABC$, $DEF$, $GHK$ are equal to the three angles $LOM$, $MON$, $NOL$.

But the three angles $LOM$, $MON$, $NOL$ are equal to four right angles,
therefore the angles $ABC$, $DEF$, $GKH$ are equal to four right angles.

But they are also, by hypothesis, less than four right angles:
which is absurd.

Therefore $AB$ is not equal to $LO$.

I say next that neither is $AB$ less than $LO$.

For, if possible, let it be so,
and let $OP$ be made equal to $AB$, and $OQ$ equal to $BC$,
and let $PQ$ be joined.

Then, since $AB$ is equal to $BC$,
$OP$ is also equal to $OQ$,
so that the remainder $LP$ is equal to $QM$.

Therefore $LM$ is parallel to $PQ$, [VI. 2]
and $LMO$ is equiangular with $PQO$, [I. 29]
therefore, as $OL$ is to $LM$, so is $OP$ to $PQ$; [VI. 4]
and alternately, as $LO$ is to $OP$, so is $LM$ to $PQ$. [V. 16]

But $LO$ is greater than $OP$;
therefore $LM$ is also greater than $PQ$.

But $LM$ was made equal to $AC$;
therefore $AC$ is also greater than $PQ$.

Since then the two sides $AB$, $BC$ are equal to the two sides $PO$, $OQ$,
and the base $AC$ is greater than the base $PQ$,
therefore the angle $ABC$ is greater than the angle $POQ$. [I. 25]

Similarly we can prove that
the angle $DEF$ is also greater than the angle $MON$,
and the angle $GHK$ greater than the angle $NOL$.

Therefore the three angles $ABC$, $DEF$, $GHK$ are greater than the three angles $LOM$, $MON$, $NOL$.

But, by hypothesis, the angles $ABC$, $DEF$, $GHK$ are less than four right angles;
therefore the angles $LOM$, $MON$, $NOL$ are much less than four right angles.

But they are also equal to four right angles:
which is absurd.

Therefore $AB$ is not less than $LO$.

And it was proved that neither is it equal;
therefore $AB$ is greater than $LO$.

Let then $OR$ be set up from the point $O$ at right angles to the plane of the circle $LMN$, [XI. 12]
and let the square on $OR$ be equal to that area by which the square on $AB$ is greater than the square on $LO$; [Lemma]
let $RL$, $RM$, $RN$ be joined.

Then, since $RO$ is at right angles to the plane of the circle $LMN$,
therefore $RO$ is also at right angles to each of the straight lines $LO$, $MO$, $NO$.

And since $LO$ is equal to $OM$,
while $OR$ is common and at right angles,

therefore the base $RL$ is equal to the base $RM$ [I 4]

For the same reason

$RN$ is also equal to each of the straight lines $RL$, $RM$,

therefore the three straight lines $RL$, $RM$, $RN$ are equal to one another

Next, since by hypothesis the square on $OR$ is equal to that area by which the square on $AB$ is greater than the square on $LO$

therefore the square on $AB$ is equal to the squares on $LO$, $OR$

But the square on $LR$ is equal to the squares on $LO$, $OR$, for the angle $LOR$ is right, [I 47]

therefore the square on $AB$ is equal to the square on $RL$,

therefore $AB$ is equal to $RL$

But each of the straight lines $BC$, $DE$, $EF$, $GH$, $HK$ is equal to $AB$,

while each of the straight lines $RM$, $RN$ is equal to $RL$,

therefore each of the straight lines $AB$, $BC$, $DE$, $EF$, $GH$, $HK$

is equal to each of the straight lines $RL$, $RM$, $RN$

And since the two sides $LR$, $RM$ are equal to the two sides $AB$, $BC$,

and the base $LM$ is by hypothesis equal to the base $AC$,

therefore the angle $LRM$ is equal to the angle $ABC$ [I 8]

For the same reason

the angle $MRN$ is also equal to the angle $DEF$,

and the angle $LRN$ to the angle $GHK$

Therefore, out of the three plane angles $LRM$, $MRN$, $LRN$, which are equal to the three given angles $ABC$, $DEF$, $GHK$, the solid angle at $R$ has been constructed which is contained by the angles $LRM$, $MRN$, $LRN$ Q E F

LEMMA

But how it is possible to take the square on $OR$ equal to that area by which the square on $AB$ is greater than the square on $LO$ we can show as follows

Let the straight lines $AB$, $LO$ be set out,

and let $AB$ be the greater,

let the semicircle $ABC$ be described on $AB$

and into the semicircle $ABC$ let $AC$ be fitted equal to the straight line $LO$ not being greater than the diameter $AB$ [IV 1]

let $CB$ be joined

Since then the angle $ACB$ is an angle in the semicircle $ACB$ [III 31]

therefore the angle $ACB$ is right

Therefore the square on $AB$ is equal to the squares on $AC$, $CB$ [I 47]

Hence the square on $AB$ is greater than the square on $AC$ by the square on $CB$

But $AC$ is equal to $LO$

Therefore the square on $AB$ is greater than the square on $LO$ by the square on $CB$

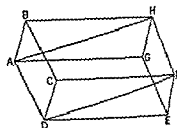
If then we cut off $OR$ equal to $BC$ the square on $AB$ will be greater than the square on $LO$ by the square on $OR$

PROPOSITION 24

*If a solid be contained by parallel planes the opposite planes are parallelogrammic*

For let the solid $CDHG$ be contained by the parallel planes $AC$, $GF$ $AH$ $DF$ $BF$, $AE$,

I say that the opposite planes in it are equal and parallelogrammic




For since the two parallel planes $BG$ $CE$ are cut by the plane $AC$ their common sections are parallel [XI 16]

Therefore $AB$ is parallel to $DC$

Again since the two parallel planes $BF$ $AE$ are cut by the plane $AC$ their common sections are parallel [XI 16]

Therefore $BC$ is parallel to $AD$

But $AB$ was also proved parallel to $DC$

therefore $AC$ is a parallelogram

Similarly we can prove that each of the planes $DF$, $FG$, $GB$, $BF$, $AE$ is a parallelogram

Let $AH$ $DF$ be joined

Then since $AB$ is parallel to $DC$ and $BH$ to $CF$ the two straight lines $AB$ $BH$ which meet one another are parallel to the two straight lines $DC$ $CF$ which meet one another not in the same plane, [XI 10]

therefore they will contain equal angles,

therefore the angle $ABH$ is equal to the angle $DCF$

And since the two sides $AB$ $BH$ are equal to the two sides $DC$ $CF$, [I 34]

and the angle $ABH$ is equal to the angle $DCF$

therefore the base $AH$ is equal to the base $DF$

and the triangle $ABH$ is equal to the triangle $DCF$ [I 4]

And the parallelogram $BG$ is double of the triangle $ABH$ and the parallelogram $CE$ double of the triangle $DCF$ [I 34]

therefore the parallelogram $BG$ is equal to the parallelogram $CE$

Similarly we can prove that

$AC$ is also equal to $GF$

and $AE$ to $BF$

Therefore etc Q E D

## Proposition 25

*If a parallelepipedal solid be cut by a plane which is parallel to the opposite planes then as the base is to the base so will the solid be to the solid*

For let the parallelepipedal solid $ABCD$ be cut by the plane $FG$ which is parallel to the opposite planes $RA$ $DH$

I say that as the base $AEFV$ is to the base $EHCF$ so is the solid $ABFU$ to the solid $EGCD$

For let $AH$ be produced in each direction

let any number of straight lines whatever $AK$ $KL$ be made equal to $AE$,

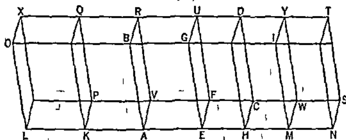
and any number whatever $HM$ $MN$ equal to $EH$,

and let the parallelograms $LP$ $KV$ $HW$ $MS$ and the solids $LQ$, $KR$, $DM$, $MT$ be completed

Then since the straight lines $LK$ $KA$ $AE$ are equal to one another

the parallelograms $LP$ $KV$ $AF$ are also equal to one another,

$KO$, $KB$, $AG$ are equal to one another,
and further, $LX$, $KQ$, $AR$ are equal to one another, for they are opposite. [XI 24]




For the same reason
the parallelograms $EC$, $HW$, $MS$ are also equal to one another,
$HG$, $HI$, $IN$ are equal to one another,
and further, $DH$, $MY$, $NT$ are equal to one another.
Therefore in the solids $LQ$, $KR$, $AU$ three planes are equal to three planes.
But the three planes are equal to the three opposite,
therefore the three solids $LQ$, $KR$, $AU$ are equal to one another.
For the same reason
the three solids $ED$, $DM$, $MT$ are also equal to one another.
Therefore, whatever multiple the base $LF$ is of the base $AF$, the same multiple also is the solid $LU$ of the solid $AU$.
For the same reason
whatever multiple the base $NF$ is of the base $FH$, the same multiple also is the solid $NU$ of the solid $HU$.
And if the base $LF$ is equal to the base $NF$, the solid $LU$ is also equal to the solid $NU$;
if the base $LF$ exceeds the base $NF$, the solid $LU$ also exceeds the solid $NU$,
and, if one falls short, the other falls short.
Therefore, there being four magnitudes, the two bases $AF$, $FH$, and the two solids $AU$, $UH$,
equimultiples have been taken of the base $AF$ and the solid $AU$, namely the base $LF$ and the solid $LU$,
and equimultiples of the base $HF$ and the solid $HU$, namely the base $NF$ and the solid $NU$,
and it has been proved that, if the base $LF$ exceeds the base $FN$, the solid $LU$ also exceeds the solid $NU$,
if the bases are equal, the solids are equal,
and if the base falls short, the solid falls short.
Therefore, as the base $AF$ is to the base $FH$, so is the solid $AU$ to the solid $UH$. [V Def 5]

Q E D

PROPOSITION 26

*On a given straight line, and at a given point on it, to construct a solid angle equal to a given solid angle.*

Let $AB$ be the given straight line, $A$ the given point on it and the angle at $D$ contained by the angles $EDC$, $EDF$, $FDC$, the given solid angle,

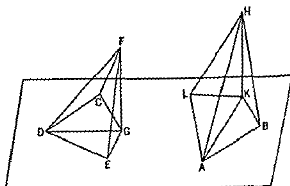
thus it is required to construct on the straight line $AB$, and at the point $A$ on it a solid angle equal to the solid angle at $D$

For let a point $F$ be taken at random on $DF$

let $FG$ be drawn from $F$ perpendicular to the plane through $ED$ $DC$, and let it meet the plane at $G$ [XI 11]

let $DG$ be joined,

let there be constructed on the straight line $AB$ and at the point $A$ on it the angle $BAL$ equal to the angle $EDC$, and the angle $BAK$ equal to the angle $EDG$ [I 23]




let $AK$ be made equal to $DG$

let $KH$ be set up from the point $K$ at right angles to the plane through $BA$, $AL$, [XI 12]

let $KH$ be made equal to $GF$

and let $HA$ be joined

I say that the solid angle at $A$, contained by the angles $BAL$ $BAH$, $HAL$ is equal to the solid angle at $D$ contained by the angles $EDC$, $EDF$, $FDC$

For let $AB$ $DE$ be cut off equal to one another

and let $HB$ $KB$ $FE$ $GE$ be joined

Then since $FG$ is at right angles to the plane of reference it will also make right angles with all the straight lines which meet it and are in the plane of reference [XI Def 3]

therefore each of the angles $FGD$ $FGE$ is right

For the same reason

each of the angles $HKA$ $HKB$ is also right

And since the two sides $KA$ $AB$ are equal to the two sides $GD$ $DE$ respectively

and they contain equal angles

therefore the base $KB$ is equal to the base $GE$ [I 4]

But $KH$ is also equal to $GF$

and they contain right angles

therefore $HB$ is also equal to $FE$ [I 4]

Again since the two sides $AK$ $KH$ are equal to the two sides $DG$ $GF$,

and they contain right angles

*therefore the base AH is equal to the base FD* [I 4]

But $AB$ is also equal to $DE$

therefore the two sides $HA$ $AB$ are equal to the two sides $DF$ $DE$

And the base $HB$ is equal to the base $FE$

therefore the angle $BAH$ is equal to the angle $EDF$ [I 8]

For the same reason

the angle $HAL$ is also equal to the angle $FDC$

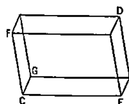
And the angle $BAL$ is also equal to the angle $EDC$

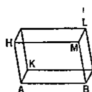
Therefore on the straight line $AB$, and at the point $A$ on it a solid angle has been constructed equal to the given solid angle at $D$ Q E F

PROPOSITION 27

*On a given straight line to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid*

Let $AB$ be the given straight line and $CD$ the given parallelepipedal solid thus it is required to describe on the given straight line $AB$ a parallelepipedal solid similar and similarly situated to the given parallelepipedal solid $CD$




For on the straight line $AB$ and at the point $A$ on it let the solid angle contained by the angles $BAH$ $HAK$ $KAB$, be constructed equal to the solid angle at $C$, so that the angle $BAH$ is equal to the angle $ECF$ the angle $BAK$ equal to the angle $ECG$, and the angle $KAH$ to the angle $GCF$

and let it be contrived that

as $EC$ is to $CG$, so is $BA$ to $AK$,

and as $GC$ is to $CF$, so is $KA$ to $AH$ [VI 12]

Therefore also *ex aequali*

as $EC$ is to $CF$, so is $BA$ to $AH$ [V 22]

Let the parallelogram $HB$ and the solid $AL$ be completed

Now since, as $EC$ is to $CG$ so is $BA$ to $AK$,

and the sides about the equal angles $ECG$ $BAK$ are thus proportional

therefore the parallelogram $GE$ is similar to the parallelogram $KB$

For the same reason

the parallelogram $KH$ is also similar to the parallelogram $GF$ and further $FE$ to $HB$,

therefore three parallelograms of the solid $CD$ are similar to three parallelograms of the solid $AL$

But the former three are both equal and similar to the three opposite parallelograms

and the latter three are both equal and similar to the three opposite parallelograms,

therefore the whole solid $CD$ is similar to the whole solid $AL$ [XI Def 9]

Therefore on the given straight line $AB$ there has been described $AL$ similar and similarly situated to the given parallelepipedal solid $CD$ Q E F

PROPOSITION 28

*If a parallelepipedal solid be cut by a plane through the diagonals of the opposite planes the solid will be bisected by the plane*

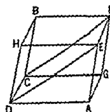
For let the parallelepipedal solid $AB$ be cut by the plane $CDEF$ through the diagonals $CF$, $DE$ of opposite planes

I say that the solid $AB$ will be bisected by the plane $CDEF$

For since the triangle $CGF$ is equal to the triangle $CFB$, [I 34]

and $ADE$ to $DEH$,

while the parallelogram $CA$ is also equal to the parallelogram $EB$, for they are opposite and $GE$ to $CH$, therefore the prism contained by the two triangles $CGF$, $ADE$ and the three parallelograms $GE$ $AC$, $CE$ is also equal to the prism contained by the two triangles $CFB$ $DEH$ and the three parallelograms $CH$, $BE$, $CE$, for they are contained by planes equal both in multitude and in magnitude [XI Def 10]



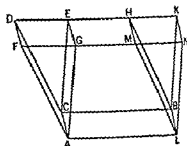

Hence the whole solid $AB$ is bisected by the plane $CDEF$ Q E D

## Proposition 29

*Parallelepipedal solids which are on the same base and of the same height and in which the extremities of the sides which stand up are on the same straight lines are equal to one another*

Let $CM$ $CN$ be parallelepipedal solids on the same base $AB$ and of the same height

and let the extremities of their sides which stand up namely $AG$ $AF$ $LM$ $LN$, $CD$ $CE$ $BH$ $BK$, be on the same straight lines $FN$ $DK$




I say that the solid $CM$ is equal to the solid $CN$

For since each of the figures $CH$ $CK$ is a parallelogram $CB$ is equal to each of the straight lines $DH$ $EK$ [I 34]

hence $DH$ is also equal to $EK$

Let $EH$ be subtracted from each

therefore the remainder $DE$ is equal to the remainder $HK$

Hence the triangle $DCE$ is also equal to the triangle $HBK$ [I 8 4]

and the parallelogram $DG$ to the parallelogram $HN$ [I 36]

For the same reason

the triangle $AFG$ is also equal to the triangle $MLN$

But the parallelogram $CF$ is equal to the parallelogram $BM$ and $CG$ to $BN$, for they are opposite

therefore the prism contained by the two triangles $AFG$ $DCE$ and the three parallelograms $AD$ $DG$ $CG$ is equal to the prism contained by the two triangles $MLN$ $HBK$ and the three parallelograms $BM$ $HN$ $BN$

Let there be added to each the solid of which the parallelogram $AB$ is the base and $GEHM$ its opposite

therefore the whole parallelepipedal solid $CM$ is equal to the whole parallelepipedal solid $CN$

Therefore etc Q E D

## Proposition 30

*Parallelepipedal solids which are on the same base and of the same height and in which the extremities of the sides which stand up are not on the same straight lines are equal to one another*

Let *CM*, *CN* be parallelepipedal solids on the same base *AB* and of the same height,

and let the extremities of their sides which stand up, namely *AF*, *AG*, *LM*, *LN*, *CD*, *CE*, *BH*, *BK*, not be on the same straight lines;

I say that the solid *CM* is equal to the solid *CN*.

For let *NK*, *DH* be produced and meet one another at *R*,

and further, let *FM*, *GE* be produced to *P*, *Q*;

let *AO*, *LP*, *CQ*, *BR* be joined.

Then the solid *CM*, of which the parallelogram *ACBL* is the base, and *FDHM* its opposite, is equal to the solid *CP*, of which the parallelogram *ACBL* is the base and *OQRP* its opposite;

for they are on the same base *ACBL* and of the same height, and the extremities of their sides which stand up, namely *AF*, *AO*, *LM*, *LP*, *CD*, *CQ*, *BH*, *BR*, are on the same straight lines *FP*, *DR*. [XI. 29]

*But the solid CP, of which the parallelogram ACBL is the base and OQRP* its opposite, is equal to the solid *CN*, of which the parallelogram *ACBL* is the base and *GEKN* its opposite,

for they are again on the same base *ACBL* and of the same height, and the extremities of their sides which stand up, namely *AG*, *AO*, *CE*, *CQ*, *LN*, *LP*, *BK*, *BR*, are on the same straight lines *GQ*, *NR*.

Hence the solid *CM* is also equal to the solid *CN*.

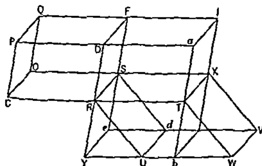
Therefore etc. Q. E. D.

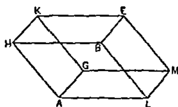
## Proposition 31

*Parallelepipedal solids which are on equal bases and of the same height are equal to one another.*

Let the parallelepipedal solids *AE*, *CF*, of the same height, be on equal bases *AB*, *CD*;

I say that the solid *AE* is equal to the solid *CF*.




First let the sides which stand up, *HK*, *BE*, *AG*, *LM*, *PQ*, *DF*, *CO*, *RS*, be at right angles to the bases *AB*, *CD*;

let the straight line $RT$ be produced in a straight line with $CR$;
on the straight line $RT$, and at the point $R$ on it, let the angle $TRU$ be constructed equal to the angle $ALB$, [I. 23]

let $RT$ be made equal to $AL$, and $RU$ equal to $LB$,
and let the base $RW$ and the solid $XU$ be completed.

Now, since the two sides $TR$, $RU$ are equal to the two sides $AL$, $LB$,
and they contain equal angles,
therefore the parallelogram $RW$ is equal and similar to the parallelogram $HL$.

Since again $AL$ is equal to $RT$, and $LM$ to $RS$,
and they contain right angles,
therefore the parallelogram $RX$ is equal and similar to the parallelogram $AM$.

For the same reason
$LE$ is also equal and similar to $SU$;
therefore three parallelograms of the solid $AE$ are equal and similar to three parallelograms of the solid $XU$.

But the former three are equal and similar to the three opposite, and the latter three to the three opposite; [XI. 24]
therefore the whole parallelepipedal solid $AE$ is equal to the whole parallelepipedal solid $XU$. [XI. Def. 10]

Let $DR$, $WU$ be drawn through and meet one another at $Y$,
let $aTb$ be drawn through $T$ parallel to $DY$,
let $PD$ be produced to $a$,
and let the solids $YX$, $RI$ be completed.

Then the solid $XY$, of which the parallelogram $RX$ is the base and $Yc$ its opposite, is equal to the solid $XU$ of which the parallelogram $RX$ is the base and $UV$ its opposite,
for they are on the same base $RX$ and of the same height, and the extremities of their sides which stand up, namely $RY$, $RU$, $Tb$, $TW$, $Se$, $Sd$, $Xc$, $XV$, are on the same straight lines $YW$, $eV$. [XI. 29]

But the solid $XU$ is equal to $AE$;
therefore the solid $XY$ is also equal to the solid $AE$.

And, since the parallelogram $RUWT$ is equal to the parallelogram $YT$,
for they are on the same base $RT$ and in the same parallels $RT$, $YW$, [I. 35]
while $RUWT$ is equal to $CD$, since it is also equal to $AB$,
therefore the parallelogram $YT$ is also equal to $CD$.

But $DT$ is another parallelogram;
therefore, as the base $CD$ is to $DT$, so is $YT$ to $DT$. [V. 7]

And, since the parallelepipedal solid $CI$ has been cut by the plane $RF$ which is parallel to opposite planes,
as the base $CD$ is to the base $DT$, so is the solid $CF$ to the solid $RI$. [XI. 25]

For the same reason,
since the parallelepipedal solid $YI$ has been cut by the plane $RX$ which is parallel to opposite planes,
as the base $YT$ is to the base $TD$, so is the solid $YX$ to the solid $RI$. [XI. 25]

But, as the base $CD$ is to $DT$, so is $YT$ to $DT$;
therefore also, as the solid $CF$ is to the solid $RI$, so is the solid $YX$ to $RI$. [V. 11]

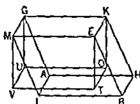
Therefore each of the solids $CF$, $YX$ has to $RI$ the same ratio;
therefore the solid $CF$ is equal to the solid $YX$. [V. 9]

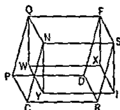
But Y X was proved equal to $AE$;

therefore $AE$ is also equal to $CF$.

Next, let the sides standing up $AG$, $HK$, $BE$, $LM$, $CN$, $PQ$, $DF$, $RS$ not be at right angles to the bases $AB$, $CD$;

I say again that the solid $AE$ is equal to the solid $CF$.




For from the points K, $E$, $G$, $M$, $Q$, $F$, $N$, $S$ let $KO$, $ET$, $GU$, $MV$, $QW$, $FX$, $NY$, $SI$ be drawn perpendicular to the plane of reference, and let them meet the plane at the points $O$, $T$, $U$, $V$, $W$, $X$, $Y$, $I$,

and let $OT$, $OU$, $UV$, $TV$, $WX$, $WY$, $YI$, $IX$ be joined.

Then the solid $KV$ is equal to the solid $QI$,

for they are on the equal bases $KM$, $QS$ and of the same height, and their sides which stand up are at right angles to their bases. [First part of this Prop.]

But the solid $KV$ is equal to the solid $AE$,

and $QI$ to $CF$;

for they are on the same base and of the same height, while the extremities of their sides which stand up are not on the same straight lines. [XI. 30]

Therefore the solid $AE$ is also equal to the solid $CF$.

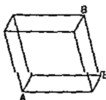
Therefore etc. Q. E. D.

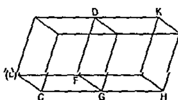
PROPOSITION 32.

*Parallelepipedal solids which are of the same height are to one another as their bases.*

Let $AB$, $CD$ be parallelepipedal solids of the same height;

I say that the parallelepipedal solids $AB$, $CD$ are to one another as their bases, that is, that, as the base $AE$ is to the base $CF$, so is the solid $AB$ to the solid $CD$.




For let $FH$ equal to $AE$ be applied to $FG$, [I. 45]

and on $FH$ as base, and with the same height as that of $CD$, let the parallelepipedal solid $GK$ be completed.

Then the solid $AB$ is equal to the solid $GK$;

for they are on equal bases $AE$, $FH$ and of the same height. [XI. 31]

And, since the parallelepipedal solid $CK$ is cut by the plane $DG$ which is parallel to opposite planes,

But the former three parallelograms are equal and similar to their opposites and the latter three to their opposites, [XI 24]

therefore the whole solid $KP$ is equal and similar to the whole solid $CD$ [XI Def 10]

Let the parallelogram $GK$ be completed,

and on the parallelograms $GK$, $KL$ as bases and with the same height as that of $AB$, let the solids $EO$ $LQ$ be completed

Then since owing to the similarity of the solids $AB$, $CD$

as $AE$ is to $CF$ so is $EG$ to $FN$, and $EH$ to $FR$

while $CF$ is equal to $EK$, $FN$ to $EL$, and $FR$ to $EM$

therefore as $AE$ is to $EK$, so is $GE$ to $EL$, and $HE$ to $EM$

But, as $AE$ is to $EK$ so is $AG$ to the parallelogram $GK$,

as $GE$ is to $EL$, so is $GK$ to $KL$

and as $HE$ is to $EM$, so is $QE$ to $KM$, [VI 1]

therefore also as the parallelogram $AG$ is to $GK$ so is $GK$ to $KL$ and $QE$ to $KM$

But, as $AG$ is to $GK$, so is the solid $AB$ to the solid $EO$,

as $GK$ is to $KL$ so is the solid $OE$ to the solid $QL$

and, as $QE$ is to $KM$, so is the solid $QL$ to the solid $KP$, [XI 32]

therefore also as the solid $AB$ is to $EO$ so is $EO$ to $QL$ and $QL$ to $KP$

But if four magnitudes be continuously proportional, the first has to the fourth the ratio triplicate of that which it has to the second [V Def 10]

therefore the solid $AB$ has to $KP$ the ratio triplicate of that which $AB$ has to $EO$

But as $AB$ is to $EO$, so is the parallelogram $AG$ to $GK$ and the straight line $AE$ to $EK$ [VI 1]

hence the solid $AB$ has also to $KP$ the ratio triplicate of that which $AE$ has to $EK$

But the solid $KP$ is equal to the solid $CD$,

and the straight line $EK$ to $CF$

therefore the solid $AB$ has also to the solid $CD$ the ratio triplicate of that which the corresponding side of it $AE$ has to the corresponding side $CF$

Therefore etc Q E D

PORISM From this it is manifest that if four straight lines be <continuously> proportional as the first is to the fourth so will a parallelepipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second inasmuch as the first has to the fourth the ratio triplicate of that which it has to the second

PROPOSITION 34

*In equal parallelepipedal solids the bases are reciprocally proportional to the heights and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal*

Let $AB$ $CD$ be equal parallelepipedal solids,

I say that in the parallelepipedal solids $AB$ $CD$ the bases are reciprocally proportional to the heights

that is as the base $EH$ is to the base $NQ$ so is the height of the solid $CD$ to the height of the solid $AB$

First, let the sides which stand up namely $AG$, $EF$, $LB$ $HK$, $CM$ $NO$ $PD$

$QR$ be at right angles to their bases,

I say that, as the base $EH$ is to the base $NQ$, so is $CM$ to $AG$

If now the base $EH$ is equal to the base $NQ$

while the solid $AB$ is also equal to the solid $CD$,

$CM$ will also be equal to $AG$

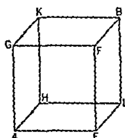
For parallelepipedal solids of the same height are to one another as the bases, [XI 32]

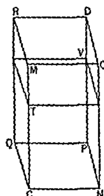
and as the base $EH$ is to $NQ$ so will $CM$ be to $AG$

and it is manifest that in the parallelepipedal solids $AB$, $CD$ the bases are reciprocally proportional to the heights

Next, let the base $EH$ not be equal to the base $NQ$,

but let $EH$ be greater




Now the solid $AB$ is equal to the solid $CD$

therefore $CM$ is also greater than $AG$

Let then $CT$ be made equal to $AG$,

and let the parallelepipedal solid $VC$ be completed on $NQ$ as base and with $CT$ as height

Now since the solid $AB$ is equal to the solid $CD$

and $CV$ is outside them

while equals have to the same the same ratio [V 7]

therefore as the solid $AB$ is to the solid $CV$ so is the solid $CD$ to the solid $CV$

But as the solid $AB$ is to the solid $CV$ so is the base $EH$ to the base $NQ$

for the solids $AB$ $CV$ are of equal height [XI 32]

and as the solid $CD$ is to the solid $CV$ so is the base $MQ$ to the base $TQ$ [XI 25] and $CM$ to $CT$ [VI 1]

therefore also as the base $EH$ is to the base $NQ$ so is $MC$ to $CT$

But $CT$ is equal to $AG$

therefore also as the base $EH$ is to the base $NQ$ so is $MC$ to $AG$

Therefore in the parallelepipedal solids $AB$, $CD$ the bases are reciprocally proportional to the heights

Again in the parallelepipedal solids $AB$, $CD$ let the bases be reciprocally proportional to the heights that is as the base $EH$ is to the base $NQ$ so let the height of the solid $CD$ be to the height of the solid $AB$,

I say that the solid $AB$ is equal to the solid $CD$

Let the sides which stand up be again at right angles to the bases

Now if the base $EH$ is equal to the base $NQ$

and as the base $EH$ is to the base $NQ$ so is the height of the solid $CD$ to the height of the solid $AB$,

therefore the height of the solid $CD$ is also equal to the height of the solid $AB$

But parallelepipedal solids on equal bases and of the same height are equal to one another, [XI 31]

therefore the solid $AB$ is equal to the solid $CD$

Next let the base $EH$ not be equal to the base $NQ$,

but let $EH$ be greater,

therefore the height of the solid $CD$ is also greater than the height of the solid $AB$,

that is $CM$ is greater than $AG$

Let $CT$ be again made equal to $AG$

and let the solid $CV$ be similarly completed

Since, as the base $EH$ is to the base $NQ$ so is $MC$ to $AG$

while $AG$ is equal to $CT$

therefore as the base $EH$ is to the base $NQ$ so is $CM$ to $CT$

But as the base $EH$ is to the base $NQ$, so is the solid $AB$ to the solid $CV$,

for the solids $AB$, $CV$ are of equal height, [XI 32]

and, as $CM$ is to $CT$, so is the base $MQ$ to the base $QT$ [VI 1]

and the solid $CD$ to the solid $CV$ [XI 25]

Therefore also as the solid $AB$ is to the solid $CV$, so is the solid $CD$ to the solid $CV$,

therefore each of the solids $AB$ $CD$ has to $CV$ the same ratio

Therefore the solid $AB$ is equal to the solid $CD$ [V 9]

Now let the sides which stand up $FE$, $BL$, $GA$, $HK$, $ON$, $DP$, $MC$, $RQ$ not be at right angles to their bases

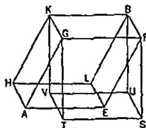
let perpendiculars be drawn from the points $F$ $G$, $B$, $K$, $O$ $M$, $D$, $R$ to the planes through $EH$, $NQ$

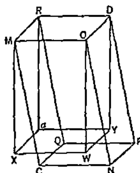
and let them meet the planes at $S$ $T$, $U$, $V$ $W$ $X$, $Y$, $a$

and let the solids $FV$ $Oa$ be completed,

I say that in this case too if the solids $AB$ $CD$ are equal the bases are reciprocally proportional to the heights that is as the base $EH$ is to the base $NQ$, so is the height of the solid $CD$ to the height of the solid $AB$

Since the solid $AB$ is equal to the solid $CD$




while $AB$ is equal to $BT$,

for they are on the same base $FK$ and of the same height [XI 29 30]

and the solid $CD$ is equal to $DX$;

for they are again on the same base $RO$ and of the same height, [id.]

therefore the solid $BT$ is also equal to the solid $DX$.

Therefore, as the base $FK$ is to the base $OR$, so is the height of the solid $DX$ to the height of the solid $BT$. [Part 1.]

But the base $FK$ is equal to the base $EH$,

and the base $OR$ to the base $NQ$;

therefore, as the base $EH$ is to the base $NQ$, so is the height of the solid $DX$ to the height of the solid $BT$.

But the solids $DX$, $BT$ and the solids $DC$, $BA$ have the same heights respectively;

therefore, as the base $EH$ is to the base $NQ$, so is the height of the solid $DC$ to the height of the solid $AB$.

Therefore in the parallelepipedal solids $AB$, $CD$ the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids $AB$, $CD$ let the bases be reciprocally proportional to the heights,

that is, as the base $EH$ is to the base $NQ$, so let the height of the solid $CD$ be to the height of the solid $AB$;

I say that the solid $AB$ is equal to the solid $CD$.

For, with the same construction,

since, as the base $EH$ is to the base $NQ$, so is the height of the solid $CD$ to the height of the solid $AB$,

while the base $EH$ is equal to the base $FK$,

and $NQ$ to $OR$,

therefore, as the base $FK$ is to the base $OR$, so is the height of the solid $CD$ to the height of the solid $AB$.

But the solids $AB$, $CD$ and $BT$, $DX$ have the same heights respectively;

therefore, as the base $FK$ is to the base $OR$, so is the height of the solid $DX$ to the height of the solid $BT$.

Therefore in the parallelepipedal solids $BT$, $DX$ the bases are reciprocally proportional to the heights;

therefore the solid $BT$ is equal to the solid $DX$. [Part 1.]

But $BT$ is equal to $BA$,

for they are on the same base $FK$ and of the same height, [XI. 29, 30]

and the solid $DX$ is equal to the solid $DC$. [id.]

Therefore the solid $AB$ is also equal to the solid $CD$. Q. E. D.

## Proposition 35

*If there be two equal plane angles, and on their vertices there be set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points be taken at random and perpendiculars be drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines be joined to the vertices of the original angles, they will contain with the elevated straight lines equal angles.*

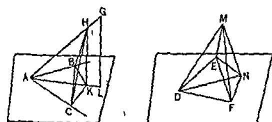
Let the angles $BAC$, $EDF$ be two equal rectilineal angles, and from the points $A$, $D$ let the elevated straight lines $AG$, $DM$ be set up containing with the original straight lines equal angles respectively, namely the angle $MDE$ to the angle $GAB$ and the angle $MDF$ to the angle $GAC$,

let points $G$, $M$ be taken at random on $AC$, $DM$,

let $GL$, $MN$ be drawn from the points $G$, $M$ perpendicular to the planes through $BA$, $AC$ and $ED$, $DF$, and let them meet the planes at $L$, $N$,

and let $LA$, $ND$ be joined

I say that the angle $GAL$ is equal to the angle $MDN$




Let $AH$ be made equal to $DM$,

and let $HK$ be drawn through the point $H$ parallel to $GL$

But $GL$ is perpendicular to the plane through $BA$, $AC$,

therefore $HK$ is also perpendicular to the plane through $BA$, $AC$ [XI 8]

From the points $K$, $N$ let $KC$, $NF$ $KB$, $NE$ be drawn perpendicular to the straight lines $AC$, $DF$, $AB$, $DE$

and let $HC$, $CB$ $MF$ $FE$ be joined

Since the square on $HA$ is equal to the squares on $HK$, $KA$,

and the squares on $KC$, $CA$ are equal to the square on $KA$, [I 47]

therefore the square on $HA$ is also equal to the squares on $HK$, $KC$, $CA$

But the square on $HC$ is equal to the squares on $HK$, $KC$, [I 47]

therefore the square on $HA$ is equal to the squares on $HC$, $CA$

Therefore the angle $HCA$ is right [I 48]

For the same reason

the angle $DFM$ is also right

Therefore the angle $ACH$ is equal to the angle $DFM$

But the angle $HAC$ is also equal to the angle $MDF$

Therefore $MDF$ $HAC$ are two triangles which have two angles equal to two angles respectively and one side equal to one side namely that subtending one of the equal angles that is, $HA$ equal to $MD$,

therefore they will also have the remaining sides equal to the remaining sides respectively [I 26]

Therefore $AC$ is equal to $DF$

Similarly we can prove that $AB$ is also equal to $DE$

Since then $AC$ is equal to $DF$ and $AB$ to $DE$

the two sides $CA$ $AB$ are equal to the two sides $FD$ $DE$

But the angle $CAB$ is also equal to the angle $FDE$

therefore the base $BC$ is equal to the base $EF$, the triangle to the triangle, and the remaining angles to the remaining angles [I 4]

therefore the angle $ACB$ is equal to the angle $DFE$

But the right angle $ACK$ is also equal to the right angle $DFN$

therefore the remaining angle $BCK$ is also equal to the remaining angle $EFN$

For the same reason

the angle $CBK$ is also equal to the angle $FEN$

Therefore $BCK$ $EFN$ are two triangles which have two angles equal to two angles respectively and one side equal to one side namely that adjacent to the equal angles that is $BC$ equal to $EF$

therefore they will also have the remaining sides equal to the remaining sides [I 26]

Therefore $CK$ is equal to $FN$

But $AC$ is also equal to $DF$,

therefore the two sides $AC$, $CK$ are equal to the two sides $DF$, $FN$, and they contain right angles

Therefore the base $AK$ is equal to the base $DN$ [I 4]

And since $AH$ is equal to $DM$

the square on $AH$ is also equal to the square on $DM$

But the squares on $AK$, $KH$ are equal to the square on $AH$, for the angle $AKH$ is right, [I 47]

and the squares on $DN$, $NM$ are equal to the square on $DM$ for the angle $DNM$ is right, [I 47]

therefore the squares on $AK$, $KH$ are equal to the squares on $DN$, $NM$,

and of these the square on $AK$ is equal to the square on $DN$

therefore the remaining square on $KH$ is equal to the square on $NM$

therefore $HK$ is equal to $MN$

And since the two sides $HA$, $AK$ are equal to the two sides $MD$, $DN$ respectively,

and the base $HK$ was proved equal to the base $MN$,

therefore the angle $HAK$ is equal to the angle $MDN$ [I 8]

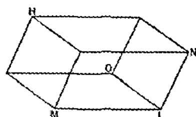
Therefore etc

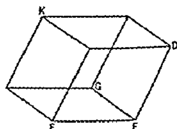
PORISM From this it is manifest that if there be two equal plane angles and if there be set up on them elevated straight lines which are equal and contain equal angles with the original straight lines respectively, the perpendiculars drawn from their extremities to the planes in which are the original angles are equal to one another Q E D

## PROPOSITION 36

*If three straight lines be proportional the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral but equiangular with the aforesaid solid*

Let $A$, $B$, $C$ be three straight lines in proportion so that as $A$ is to $B$ so is $B$ to $C$




I say that the solid formed out of $A$, $B$, $C$ is equal to the solid on $B$ which is equilateral but equiangular with the aforesaid solid

Let there be set out the solid angle at $E$ contained by the angles $DEG$, $GEF$, $FED$,

let each of the straight lines $DE$, $GE$, $EF$ be made equal to $B$, and let the parallelepipedal solid $EK$ be completed

let $LM$ be made equal to $A$,

and on the straight line $LM$, and at the point $L$ on it, let there be constructed a solid angle equal to the solid angle at $E$ namely that contained by $NLO$, $OLM$, $MLN$

let $LO$ be made equal to $B$, and $LN$ equal to $C$

Now since as $A$ is to $B$ so is $B$ to $C$

while $A$ is equal to $LM$ $B$ to each of the straight lines $LO$ $ED$ and $C$ to $LN$,

therefore as $LM$ is to $EF$ so is $DE$ to $LN$

Thus the sides about the equal angles $NLM$, $DEF$ are reciprocally proportional,

*therefore the parallelogram $MN$ is equal to the parallelogram $DF$* [VI 14]

And since the angles $DEF$, $NLM$ are two plane rectilineal angles and on them the elevated straight lines $LO$, $EG$ are set up which are equal to one another and contain equal angles with the original straight lines respectively

therefore the perpendiculars drawn from the points $G$ $O$ to the planes through $NL$, $LM$ and $DE$ $EF$ are equal to one another, [XI 35 Por]

hence the solids $LH$ $EK$ are of the same height

But parallelepipedal solids on equal bases and of the same height are equal to one another [XI 31]

therefore the solid $HL$ is equal to the solid $EK$

And $LH$ is the solid formed out of $A$ $B$, $C$ and $EK$ the solid on $B$

therefore the parallelepipedal solid formed out of $A$ $B$, $C$ is equal to the solid on $B$ which is equilateral but equiangular with the aforesaid solid

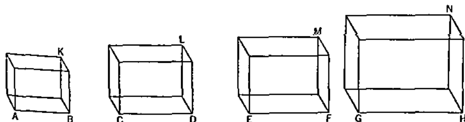
Q E D

## Proposition 37

*If four straight lines be proportional the parallelepipedal solids on them which are similar and similarly described will also be proportional and if the parallelepipedal solids on them which are similar and similarly described be proportional the straight lines will themselves also be proportional*

Let $AB$, $CD$ $EF$, $GH$ be four straight lines in proportion so that as $AB$ is to $CD$ so is $EF$ to $GH$,

and let there be described on $AB$, $CD$, $EF$ $GH$ the similar and similarly situ




ated parallelepipedal solids $KA$ $LC$, $ME$ $NG$,

I say that as $KA$ is to $LC$ so is $ME$ to $NG$

For since the parallelepipedal solid $KA$ is similar to $LC$, therefore $KA$ has to $LC$ the ratio triplicate of that which $AB$ has to $CD$ [XI 33]

For the same reason $ME$ also has to $NG$ the ratio triplicate of that which $EF$ has to $GH$ [id]

And as $AB$ is to $CD$ so is $EF$ to $GH$

Therefore also as $AK$ is to $LC$, so is $ME$ to $NG$

Next as the solid $AK$ is to the solid $LC$ so let the solid $ME$ be to the solid $NG$,

I say that as the straight line $AB$ is to $CD$ so is $EF$ to $GH$

For since again $KA$ has to $LC$ the ratio triplicate of that which $AB$ has to $CD$ [XI 33]

and $ME$ also has to $NG$ the ratio triplicate of that which $EF$ has to $GH$ [id]

and as $KA$ is to $LC$, so is $ME$ to $NG$

therefore also as $AB$ is to $CD$ so is $EF$ to $GH$

Therefore etc Q E D

### Proposition 38

*If the sides of the opposite planes of a cube be bisected and planes be carried through the points of section the common section of the planes and the diameter of the cube bisect one another*

For let the sides of the opposite planes $CF$ $AH$ of the cube $AF$ be bisected at the points $K$ $L$ $M$ $N$ $O$ $Q$ $P$ $R$ and through the points of section let the planes $KN$ $OR$ be carried

let $US$ be the common section of the planes and $DG$ the diameter of the cube $AF$

I say that $UT$ is equal to $TS$ and $DT$ to $TG$

For let $DU$ $UE$ $BS$ $SG$ be joined

Then since $DO$ is parallel to $PE$

the alternate angles $DOU$ $UPE$ are equal to one another [I 29]

And since $DO$ is equal to $PE$ and $OU$ to $UP$

and they contain equal angles

therefore the base $DU$ is equal to the base $UE$

the triangle $DOU$ is equal to the triangle $PUE$

and the remaining angles are equal to the remaining angles [I 4]

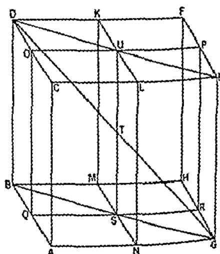
therefore the angle $OUD$ is equal to the angle $PUE$

For this reason $DUE$ is a straight line [I 14]

For the same reason $BSG$ is also a straight line

and $BS$ is equal to $SG$

Now since $CA$ is equal and parallel to $DB$,




while $CA$ is also equal and parallel to $EG$

therefore $DB$ is also equal and parallel to $EG$ [XI 9]

And the straight lines $DE$, $BG$ join their extremities,

therefore $DE$ is parallel to $BG$ [I 33]

Therefore the angle $EDT$ is equal to the angle $BGT$,

for they are alternate, [I 29]

and the angle $DTU$ is equal to the angle $GTS$ [I 15]

Therefore $DTU$, $GTS$ are two triangles which have two angles equal to two angles and one side equal to one side namely that subtending one of the equal angles that is $DU$ equal to $GS$

for they are the halves of $DE$, $BG$

therefore they will also have the remaining sides equal to the remaining sides [I 26]

Therefore $DT$ is equal to $TG$ and $UT$ to $TS$

Therefore etc Q E D

PROPOSITION 39

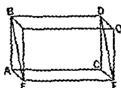
*If there be two prisms of equal height and one have a parallelogram as base and the other a triangle and if the parallelogram be double of the triangle the prisms will be equal*

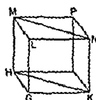
Let $ABCDEF$, $GHKLMN$ be two prisms of equal height

let one have the parallelogram $AF$ as base and the other the triangle $GHK$,

and let the parallelogram $AF$ be double of the triangle $GHK$

I say that the prism $ABCDEF$ is equal to the prism $GHKLMN$




For let the solids $AO$, $GP$ be completed

Since the parallelogram $AF$ is double of the triangle $GHK$

while the parallelogram $HK$ is also double of the triangle $GHK$ [I 34]

therefore the parallelogram $AF$ is equal to the parallelogram $HK$

But parallelepipedal solids which are on equal bases and of the same height are equal to one another [XI 31]

therefore the solid $AO$ is equal to the solid $GP$

And the prism $ABCDEF$ is half of the solid $AO$

and the prism $GHKLMN$ is half of the solid $GP$ [XI 28]

therefore the prism $ABCDEF$ is equal to the prism $GHKLMN$

Therefore etc Q E D

# BOOK TWELVE

## PROPOSITIONS

### Proposition 1

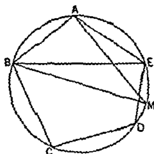
*Similar polygons inscribed in circles are to one another as the squares on the diameters*

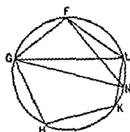
Let *ABC*, *FGH* be circles,
let *ABCDE*, *FGHKL* be similar polygons inscribed in them, and let *BM*, *GN* be diameters of the circles;

*I say that*, as the square on *BM* is to the square on *GN*, so is the polygon *ABCDE* to the polygon *FGHKL*.

For let *BE*, *AM*, *GL*, *FN* be joined.

Now, since the polygon *ABCDE* is similar to the polygon *FGHKL*,




the angle *BAE* is equal to the angle *GFL*,
and as *BA* is to *AE*, so is *GF* to *FL*. [VI. Def. 1]

Thus *BAE*, *GFL* are two triangles which have one angle equal to one angle, namely the angle *BAE* to the angle *GFL*, and the sides about the equal angles proportional;

therefore the triangle *ABE* is equiangular with the triangle *FGL*. [VI. 6]

Therefore the angle *AEB* is equal to the angle *FLG*.

But the angle *AEB* is equal to the angle *AMB*,
for they stand on the same circumference; [III. 27]
and the angle *FLG* to the angle *FNG*;
therefore the angle *AMB* is also equal to the angle *FNG*.

But the right angle *BAM* is also equal to the right angle *GFN*; [III. 31]
therefore the remaining angle is equal to the remaining angle. [I. 32]

Therefore the triangle *ABM* is equiangular with the triangle *FGN*.

Therefore, proportionally, as *BM* is to *GN*, so is *BA* to *GF*. [VI. 4]

But the ratio of the square on $BM$ to the square on $GN$ is duplicate of the ratio of $BM$ to $GN$;

and the ratio of the polygon $ABCDE$ to the polygon $FGHKL$ is duplicate of the ratio of $BA$ to $GF$, [VI. 20]

therefore also, as the square on $BM$ is to the square on $GN$, so is the polygon $ABCDE$ to the polygon $FGHKL$.

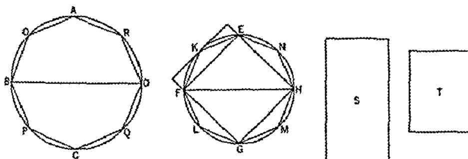
Therefore etc. Q. E. D.

## Proposition 2

*Circles are to one another as the squares on the diameters.*

Let $ABCD$, $EFGH$ be circles, and $BD$, $FH$ their diameters;

I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the square on $BD$ to the square on $FH$.




For, if the square on $BD$ is not to the square on $FH$ as the circle $ABCD$ is to the circle $EFGH$,

then, as the square on $BD$ is to the square on $FH$, so will the circle $ABCD$ be either to some less area than the circle $EFGH$, or to a greater.

First, let it be in that ratio to a less area $S$.

Let the square $EFGH$ be inscribed in the circle $EFGH$; then the inscribed square is greater than the half of the circle $EFGH$, inasmuch as, if through the points $E$, $F$, $G$, $H$ we draw tangents to the circle, the square $EFGH$ is half the square circumscribed about the circle, and the circle is less than the circumscribed square;

hence the inscribed square $EFGH$ is greater than the half of the circle $EFGH$.

Let the circumferences $EF$, $FG$, $GH$, $HE$ be bisected at the points $K$, $L$, $M$, $N$,

and let $EK$, $KF$, $FL$, $LG$, $GM$, $MH$, $HN$, $NE$ be joined;

therefore each of the triangles $EKF$, $FLG$, $GMH$, $HNE$ is also greater than the half of the segment of the circle about it, inasmuch as, if through the points $K$, $L$, $M$, $N$ we draw tangents to the circle and complete the parallelograms on the straight lines $EF$, $FG$, $GH$, $HE$, each of the triangles $EKF$, $FLG$, $GMH$, $HNE$ will be half of the parallelogram about it,

while the segment about it is less than the parallelogram;

hence each of the triangles $EKF$, $FLG$, $GMH$, $HNE$ is greater than the half of the segment of the circle about it.

Thus, by bisecting the remaining circumferences and joining straight lines

and by doing this continually we shall leave some segments of the circle which will be less than the excess by which the circle $EFGH$ exceeds the area $S$

For it was proved in the first theorem of the tenth book that if two unequal magnitudes be set out and if from the greater there be subtracted a magnitude greater than the half and from that which is left a greater than the half and if this be done continually there will be left some magnitude which will be less than the lesser magnitude set out

Let segments be left such as described and let the segments of the circle $EFGH$ on $EK$, $KF$, $FL$, $LG$, $GM$, $MH$, $HN$, $NE$ be less than the excess by which the circle $EFGH$ exceeds the area $S$

Therefore the remainder the polygon $EKFLGMHN$ is greater than the area $S$

Let there be inscribed also in the circle $ABCD$ the polygon $AOBPCQDR$ similar to the polygon $EKFLGMHN$

therefore as the square on $BD$ is to the square on $FH$ so is the polygon $AOBPCQDR$ to the polygon $EKFLGMHN$ [XII 1]

But as the square on $BD$ is to the square on $FH$ so also is the circle $ABCD$ to the area $S$

therefore also as the circle $ABCD$ is to the area $S$ so is the polygon $AOBPCQDR$ to the polygon $EKFLGMHN$ [V 11]

therefore alternately as the circle $ABCD$ is to the polygon inscribed in it so is the area $S$ to the polygon $EKFLGMHN$ [V 16]

But the circle $ABCD$ is greater than the polygon inscribed in it

therefore the area $S$ is also greater than the polygon $EKFLGMHN$

But it is also less

which is impossible

Therefore as the square on $BD$ is to the square on $FH$ so is not the circle $ABCD$ to any area less than the circle $EFGH$

Similarly we can prove that neither is the circle $EFGH$ to any area less than the circle $ABCD$ as the square on $FH$ is to the square on $BD$

I say next that neither is the circle $ABCD$ to any area greater than the circle $EFGH$ as the square on $BD$ is to the square on $FH$

For if possible let it be in that ratio to a greater area $S$

Therefore inversely as the square on $FH$ is to the square on $DB$ so is the area $S$ to the circle $ABCD$

But as the area $S$ is to the circle $ABCD$ so is the circle $EFGH$ to some area less than the circle $ABCD$

therefore also as the square on $FH$ is to the square on $BD$ so is the circle $EFGH$ to some area less than the circle $ABCD$ [V 11]

which was proved impossible

Therefore as the square on $BD$ is to the square on $FH$ so is not the circle $ABCD$ to any area greater than the circle $EFGH$

And it was proved that neither is it in that ratio to any area less than the circle $EFGH$

therefore as the square on $BD$ is to the square on $FH$ so is the circle $ABCD$ to the circle $EFGH$

Therefore etc Q E D

LEMMA

I say that the area $S$ being greater than the circle $EFGH$, as the area $S$ is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$

For let it be contrived that as the area $S$ is to the circle $ABCD$ so is the circle $EFGH$ to the area $T$

I say that the area $T$ is less than the circle $ABCD$

For since as the area $S$ is to the circle $ABCD$, so is the circle $EFGH$, to the area $T$,

therefore alternately, as the area $S$ is to the circle $EFGH$, so is the circle $ABCD$ to the area $T$ [V 16]

But the area $S$ is greater than the circle $EFGH$,

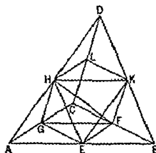
therefore the circle $ABCD$ is also greater than the area $T$

Hence as the area $S$ is to the circle $ABCD$ so is the circle $EFGH$ to some area less than the circle $ABCD$ Q E D

PROPOSITION 3

*Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another similar to the whole and having triangular bases and into two equal prisms and the two prisms are greater than the half of the whole pyramid*

Let there be a pyramid of which the triangle $ABC$ is the base and the point $D$ the vertex




I say that the pyramid $ABCD$ is divided into two pyramids equal to one another having triangular bases and similar to the whole pyramid and into two equal prisms and the two prisms are greater than the half of the whole pyramid

For let $AB$ $BC$ $CA$ $AD$ $DB$ $DC$ be bisected at the points $E$ $F$ $G$ $H$, $K$ $L$ and let $HE$, $EG$ $GH$ $HK$ $KL$ $LH$ $KF$ $FG$ be joined

Since $AE$ is equal to $EB$ and $AH$ to $DH$

therefore $EH$ is parallel to $DB$ [VI 2]

For the same reason

$HK$ is also parallel to $AB$

Therefore $HEBK$ is a parallelogram

therefore $HK$ is equal to $EB$ [I 34]

But $EB$ is equal to $EA$

therefore $AE$ is also equal to $HK$

But $AH$ is also equal to $HD$

therefore the two sides $EA$ $AH$ are equal to the two sides $KH$ $HD$ respectively

and the angle $EAH$ is equal to the angle $KHD$,

therefore the base $EH$ is equal to the base $KD$ [I 4]

Therefore the triangle $AEH$ is equal and similar to the triangle $HKD$

For the same reason

the triangle $AHG$ is also equal and similar to the triangle $HLD$

Now since two straight lines $EH$ $HG$ meeting one another are parallel to two straight lines $KD$ $DL$ meeting one another and are not in the same plane they will contain equal angles [XI 10]

Therefore the angle $EHG$ is equal to the angle $KDL$

And since the two straight lines $EH$, $HG$ are equal to the two $KD$ $DL$ respectively

and the angle $EHG$ is equal to the angle $KDL$

therefore the base $EG$ is equal to the base $KL$, [I 4]

therefore the triangle $EHG$ is equal and similar to the triangle $KDL$

*For the same reason*

the triangle $AEG$ is also equal and similar to the triangle $HKL$

Therefore the pyramid of which the triangle $AEG$ is the base and the point $H$ the vertex is equal and similar to the pyramid of which the triangle $HKL$ is the base and the point $D$ the vertex [XI Def 10]

And since $HK$ has been drawn parallel to $AB$, one of the sides of the triangle $ADB$

the triangle $ADB$ is equiangular to the triangle $DHK$ [I 29]

and they have their sides proportional

therefore the triangle $ADB$ is similar to the triangle $DHK$ [VI Def 1]

For the same reason

the triangle $DBC$ is also similar to the triangle $DKL$ and the triangle $ADC$ to the triangle $DLH$

Now since the two straight lines $BA$ $AC$ meeting one another are parallel to the two straight lines $KH$ $HL$ meeting one another not in the same plane, they will contain equal angles [XI 10]

Therefore the angle $BAC$ is equal to the angle $KHL$

And as $BA$ is to $AC$ so is $KH$ to $HL$

therefore the triangle $ABC$ is similar to the triangle $HKL$

Therefore also the pyramid of which the triangle $ABC$ is the base and the point $D$ the vertex is similar to the pyramid of which the triangle $HKL$ is the base and the point $D$ the vertex

But the pyramid of which the triangle $HKL$ is the base and the point $D$ the vertex was proved similar to the pyramid of which the triangle $AEG$ is the base and the point $H$ the vertex

Therefore each of the pyramids $AEGH$ $HKLD$ is similar to the whole pyramid $ABCD$

Next since $BF$ is equal to $FC$

the parallelogram $EBFG$ is double of the triangle $GFC$

And since if there be two prisms of equal height and one have a parallelogram as base and the other a triangle and if the parallelogram be double of the triangle the prisms are equal [XI 39]

therefore the prism contained by the two triangles $BKF$ $EHG$ and the three parallelograms $EBFG$ $EBKH$ $HKFG$ is equal to the prism contained by the two triangles $GFC$ $HKL$ and the three parallelograms $KFCL$ $LCGH$ $HKFG$

And it is manifest that each of the prisms namely that in which the parallelogram $EBFG$ is the base and the straight line $HK$ is its opposite and that in which the triangle $GFC$ is the base and the triangle $HKL$ its opposite is greater than each of the pyramids of which the triangles $AEG$ $HKL$ are the bases and the points $H$ $D$ the vertices

inasmuch as if we join the straight lines $EF$ $EK$ the prism in which the parallelogram $EBFG$ is the base and the straight line $HK$ its opposite is greater than the pyramid of which the triangle $EBF$ is the base and the point $K$ the vertex

But the pyramid of which the triangle $EBF$ is the base and the point $K$ the vertex is equal to the pyramid of which the triangle $AEG$ is the base and the point $H$ the vertex

for they are contained by equal and similar planes

Hence also the prism in which the parallelogram $EBFG$ is the base and the straight line $HK$ its opposite is greater than the pyramid of which the triangle $AEG$ is the base and the point $H$ the vertex

But the prism in which the parallelogram $EBFG$ is the base and the straight line $HK$ its opposite is equal to the prism in which the triangle $GFC$ is the base and the triangle $HKL$ its opposite

and the pyramid of which the triangle $AEG$ is the base and the point $H$ the vertex is equal to the pyramid of which the triangle $HKL$ is the base and the point $D$ the vertex

Therefore the said two prisms are greater than the said two pyramids of which the triangles $AEG$ $HKL$ are the bases and the points $H$ $D$ the vertices

Therefore the whole pyramid of which the triangle $ABC$ is the base and the point $D$ the vertex has been divided into two pyramids equal to one another and into two equal prisms and the two prisms are greater than the half of the whole pyramid Q E D

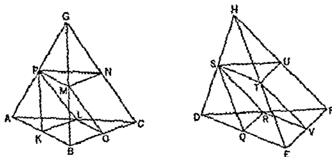
## Proposition 4

*If there be two pyramids of the same height which have triangular bases and each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms then as the base of the one pyramid is to the base of the other pyramid so will all the prisms in the one pyramid be to all the prisms being equal in multitude in the other pyramid*

Let there be two pyramids of the same height which have the triangular bases $ABC$ $DEF$ and vertices the points $G$ $H$

and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms [XII 3]

I say that as the base $ABC$ is to the base $DEF$ so are all the prisms in the pyramid $ABCG$ to all the prisms being equal in multitude in the pyramid $DEFH$




For since $BO$ is equal to $OC$ and $AL$ to $LC$

therefore $LO$ is parallel to $AB$

and the triangle $ABC$ is similar to the triangle $LOC$

For the same reason

the triangle $DEF$ is also similar to the triangle $RVF$

And since $BC$ is double of $CO$ and $EF$ of $FV$,

therefore as $BC$ is to $CO$ so is $EF$ to $FV$

And on $BC$ $CO$ are described the similar and similarly situated rectilineal figures $ABC$, $LOC$,

and on $EF$ $FV$ the similar and similarly situated figures $DEF$, $RVF$,

therefore as the triangle $ABC$ is to the triangle $LOC$ so is the triangle $DEF$ to the triangle $RVF$, [VI 22]

therefore alternately as the triangle $ABC$ is to the triangle $DEF$, so is the triangle $LOC$ to the triangle $RVF$ [V 16]

But as the triangle $LOC$ is to the triangle $RVF$ so is the prism in which the triangle $LOC$ is the base and $PMN$ its opposite to the prism in which the triangle $RVF$ is the base and $STU$ its opposite [Lemma following]

therefore also as the triangle $ABC$ is to the triangle $DEF$, so is the prism in which the triangle $LOC$ is the base and $PMN$ its opposite to the prism in which the triangle $RVF$ is the base and $STU$ its opposite

But as the said prisms are to one another so is the prism in which the parallelogram $KBOL$ is the base and the straight line $PM$ its opposite to the prism in which the parallelogram $QEVR$ is the base and the straight line $ST$ its opposite [XI 39 cf XII 3]

Therefore also the two prisms that in which the parallelogram $KBOL$ is the base and $PM$ its opposite and that in which the triangle $LOC$ is the base and $PMN$ its opposite are to the prisms in which $QEVR$ is the base and the straight line $ST$ its opposite and in which the triangle $RVF$ is the base and $STU$ its opposite in the same ratio [V 12]

Therefore also as the base $ABC$ is to the base $DEF$ so are the said two prisms to the said two prisms

And similarly if the pyramids $PMNG$ $STUH$ be divided into two prisms and two pyramids

as the base $PMN$ is to the base $STU$ so will the two prisms in the pyramid $PMNG$ be to the two prisms in the pyramid $STUH$

But as the base $PMN$ is to the base $STU$ so is the base $ABC$ to the base $DEF$

for the triangles $PMN$ $STU$ are equal to the triangles $LOC$ $RVF$ respectively

Therefore also as the base $ABC$ is to the base $DEF$ so are the four prisms to the four prisms

And similarly also if we divide the remaining pyramids into two pyramids and into two prisms then as the base $ABC$ is to base the $DEF$ so will all the prisms in the pyramid $ABCG$ be to all the prisms being equal in multitude in the pyramid $DEFH$ Q E D

LEMMA

But that as the triangle $LOC$ is to the triangle $RVF$ so is the prism in which the triangle $LOC$ is the base and $PMN$ its opposite to the prism in which the triangle $RVF$ is the base and $STU$ its opposite we must prove as follows

For in the same figure let perpendiculars be conceived drawn from $G$, $H$ to

the planes *ABC*, *DEF*, these are of course equal because by hypothesis, the pyramids are of equal height

Now, since the two straight lines *GC* and the perpendicular from *G* are cut by the parallel planes *ABC*, *PMN*

they will be cut in the same ratios [XI 17]

And *GC* is bisected by the plane *PMN* at *N*,

therefore the perpendicular from *G* to the plane *ABC* will also be bisected by the plane *PMN*

For the same reason

the perpendicular from *H* to the plane *DEF* will also be bisected by the plane *STU*

And the perpendiculars from *G* *H* to the planes *ABC*, *DEF* are equal,

therefore the perpendiculars from the triangles *PMN*, *STU* to the planes *ABC*, *DEF* are also equal

Therefore the prisms in which the triangles *LOC*, *RVF* are bases, and *PMN*, *STU* their opposites, are of equal height

Hence also the parallelepipedal solids described from the said prisms are of equal height and are to one another as their bases, [XI 32]

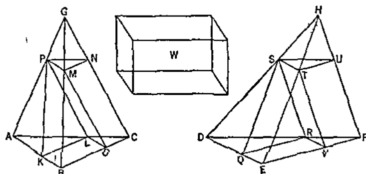
therefore their halves namely the said prisms, are to one another as the base *LOC* is to the base *RVF* Q E D

### Proposition 5

*Pyramids which are of the same height and have triangular bases are to one another as the bases*

Let there by pyramids of the same height of which the triangles *ABC*, *DEF* are the bases and the points *G* *H* the vertices,

I say that as the base *ABC* is to the base *DEF*, so is the pyramid *ABCG* to the pyramid *DEFH*




For if the pyramid *ABCG* is not to the pyramid *DEFH* as the base *ABC* is to the base *DEF*,

then as the base *ABC* is to the base *DEF* so will the pyramid *ABCG* be either to some solid less than the pyramid *DEFH* or to a greater

Let it first be in that ratio to a less solid *W* and let the pyramid *DEFH* be divided into two pyramids equal to one another and similar to the whole and into two equal prisms

then the two prisms are greater than the half of the whole pyramid [XII 3]

Again let the pyramids arising from the division be similarly divided and let this be done continually until there are left over from the pyramid *DEFH* some pyramids which are less than the excess by which the pyramid *DEFH* exceeds the solid *W* [X I]

Let such be left and let them be for the sake of argument, *DQRS STUH* therefore the remainders, the prisms in the pyramid *DEFH*, are greater than the solid *W*

Let the pyramid *ABCG* also be divided similarly, and a similar number of times with the pyramid *DEFH*

therefore as the base *ABC* is to the base *DEF* so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH* [XII 4]

But as the base *ABC* is to the base *DEF* so also is the pyramid *ABCG* to the solid *W*

therefore also as the pyramid *ABCG* is to the solid *W*, so are the prisms in the pyramid *ABCG* to the prisms in the pyramid *DEFH*, [V 11]

therefore alternately as the pyramid *ABCG* is to the prisms in it so is the solid *W* to the prisms in the pyramid *DEFH* [V 16]

But the pyramid *ABCG* is greater than the prisms in it,

therefore the solid *W* is also greater than the prisms in the pyramid *DEFH*

But it is also less

which is impossible

Therefore the prism *ABCG* is not to any solid less than the pyramid *DEFH* as the base *ABC* is to the base *DEF*

Similarly it can be proved that neither is the pyramid *DEFH* to any solid less than the pyramid *ABCG* as the base *DEF* is to the base *ABC*

I say next that neither is the pyramid *ABCG* to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*

For if possible let it be in that ratio to a greater solid *W*

therefore inversely as the base *DEF* is to the base *ABC*, so is the solid *W* to the pyramid *ABCG*

But as the solid *W* is to the solid *ABCG* so is the pyramid *DEFH* to some solid less than the pyramid *ABCG* as was before proved [XII 2 Lemma]

therefore also as the base *DEF* is to the base *ABC*, so is the pyramid *DEFH* to some solid less than the pyramid *ABCG* [V 11]

which was proved absurd

Therefore the pyramid *ABCG* is not to any solid greater than the pyramid *DEFH* as the base *ABC* is to the base *DEF*

But it was proved that neither is it in that ratio to a less solid

Therefore as the base *ABC* is to the base *DEF* so is the pyramid *ABCG* to the pyramid *DEFH* Q E D

PROPOSITION 6

*Pyramids which are of the same height and have polygonal bases are to one another as the bases*

Let there be pyramids of the same height of which the polygons *ABCDE FGHKL* are the bases and the points *M N* the vertices

I say that as the base *ABCDE* is to the base *FGHKL*, so is the pyramid *ABCDEM* to the pyramid *FGHKLN*

For let $AC$, $AD$, $FH$, $FK$ be joined.

Since then $ABCM$, $ACDM$ are two pyramids which have triangular bases and equal height,

they are to one another as the bases; [XII. 5]

therefore, as the base $ABC$ is to the base $ACD$, so is the pyramid $ABCM$ to the pyramid $ACDM$.

And, *componendo*, as the base $ABCD$ is to the base $ACD$, so is the pyramid $ABCDM$ to the pyramid $ACDM$. [V. 18]

But also, as the base $ACD$ is to the base $ADE$, so is the pyramid $ACDM$ to the pyramid $ADEM$. [XII. 5]

Therefore, *ex aequali*, as the base $ABCD$ is to the base $ADE$, so is the pyramid $ABCDM$ to the pyramid $ADEM$. [V. 22]

And again, *componendo*, as the base $ABCDE$ is to the base $ADE$, so is the pyramid $ABCDEM$ to the pyramid $ADEM$. [V. 18]

Similarly also it can be proved that, as the base $FGHKL$ is to the base $FGH$, so is the pyramid $FGHKLN$ to the pyramid $FGHN$.

And, since $ADEM$, $FGHN$ are two pyramids which have triangular bases and equal height,

therefore, as the base $ADE$ is to the base $FGH$, so is the pyramid $ADEM$ to the pyramid $FGHN$. [XII. 5]

But, as the base $ADE$ is to the base $ABCDE$, so was the pyramid $ADEM$ to the pyramid $ABCDEM$.

Therefore also, *ex aequali*, as the base $ABCDE$ is to the base $FGH$, so is the pyramid $ABCDEM$ to the pyramid $FGHN$. [V. 22]

But further, as the base $FGH$ is to the base $FGHKL$, so also was the pyramid $FGHN$ to the pyramid $FGHKLN$.

Therefore also, *ex aequali*, as the base $ABCDE$ is to the base $FGHKL$, so is the pyramid $ABCDEM$ to the pyramid $FGHKLN$. [V. 22]

Q. E. D.

PROPOSITION 7

*Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.*

Let there be a prism in which the triangle $ABC$ is the base and $DEF$ its opposite;

I say that the prism $ABCDEF$ is divided into three pyramids equal to one another which have triangular bases.

For let $BD$, $EC$, $CD$ be joined.

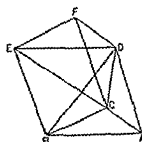
Since $ABED$ is a parallelogram, and $BD$ is its diameter,

therefore the triangle $ABD$ is equal to the triangle $EBD$; [I. 34]

therefore also the pyramid of which the triangle $ABD$ is the base and the point $C$ the vertex is equal to the pyramid of which the triangle $DEB$ is the base and

the point $C$ the vertex [XII 5]

But the pyramid of which the triangle $DEB$ is the base and the point $C$ the vertex is the same with the pyramid of which the triangle $EBC$ is the base and the point $D$ the vertex




for they are contained by the same planes

Therefore the pyramid of which the triangle $ABD$ is the base and the point $C$ the vertex is also equal to the pyramid of which the triangle $EBC$ is the base and the point $D$ the vertex

Again since $FCBE$ is a parallelogram

and $CE$ is its diameter

the triangle $CEF$ is equal to the triangle $CBE$ [I 34]

Therefore also the pyramid of which the triangle $BCE$ is the base and the point $D$ the vertex is equal to the pyramid of which the triangle $ECF$ is the base and the point $D$ the vertex [XII 5]

*But the pyramid of which the triangle BCE is the base and the point D* the vertex was proved equal to the pyramid of which the triangle $ABD$ is the base and the point $C$ the vertex

therefore also the pyramid of which the triangle $CEF$ is the base and the point $D$ the vertex is equal to the pyramid of which the triangle $ABD$ is the base and the point $C$ the vertex

therefore the prism $ABCDEF$ has been divided into three pyramids equal to one another which have triangular bases

And since the pyramid of which the triangle $ABD$ is the base and the point $C$ the vertex is the same with the pyramid of which the triangle $CAB$ is the base and the point $D$ the vertex

*for they are contained by the same planes*

while the pyramid of which the triangle $ABD$ is the base and the point $C$ the vertex was proved to be a third of the prism in which the triangle $ABC$ is the base and $DEF$ its opposite

therefore also the pyramid of which the triangle $ABC$ is the base and the point $D$ the vertex is a third of the prism which has the same base the triangle $ABC$, and $DEF$ as its opposite

PORISM From this it is manifest that any pyramid is a third part of the prism which has the same base with it and equal height Q E D

PROPOSITION 8

*Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides*

Let there be similar and similarly situated pyramids of which the triangles $ABC$ $DEF$ are the bases and the points $G$ $H$ the vertices

I say that the pyramid $ABCG$ has to the pyramid $DEFH$ the ratio triplicate of that which $BC$ has to $EF$

For let the parallelepipedal solids $BGML$ $EHQP$ be completed

Now since the pyramid $ABCG$ is similar to the pyramid $DEFH$

therefore the angle $ABC$ is equal to the angle $DEF$

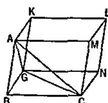
the angle $GBC$ to the angle $HEF$

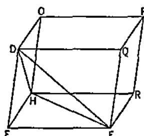
and the angle $ABG$ to the angle $DEH$

and as $AB$ is to $DE$, so is $BC$ to $EF$, and $BG$ to $EH$

And since as $AB$ is to $DE$, so is $BC$ to $EF$

and the sides are proportional about equal angles,




therefore the parallelogram $BM$ is similar to the parallelogram $EQ$

For the same reason

$BN$ is also similar to $ER$ and $BK$ to $EO$

therefore the three parallelograms $MB$ $BK$, $BN$ are similar to the three $EQ$, $EO$, $ER$

But the three parallelograms $MB$, $BK$ $BN$ are equal and similar to their three opposites

and the three $EQ$, $EO$ $ER$ are equal and similar to their three opposites [XI 24]

Therefore the solids $BGML$, $EHQP$ are contained by similar planes equal in multitude

Therefore the solid $BGML$ is similar to the solid $EHQP$

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides [XI 33]

Therefore the solid $BGML$ has to the solid $EHQP$ the ratio triplicate of that which the corresponding side $BC$ has to the corresponding side $EF$

But as the solid $BGML$ is to the solid $EHQP$, so is the pyramid $ABCG$ to the pyramid $DEFH$,

inasmuch as the pyramid is a sixth part of the solid because the prism which is half of the parallelepipedal solid [XI 28] is also triple of the pyramid [XII 7]

Therefore the pyramid $ABCG$ also has to the pyramid $DEFH$ the ratio triplicate of that which $BC$ has to $EF$ Q E D

PORISM From this it is manifest that similar pyramids which have polygonal bases are also to one another in the triplicate ratio of their corresponding sides

For, if they are divided into the pyramids contained in them which have triangular bases by virtue of the fact that the similar polygons forming their bases are also divided into similar triangles equal in multitude and corresponding to the wholes [VI 20]

then as the one pyramid which has a triangular base in the one complete pyramid is to the one pyramid which has a triangular base in the other complete pyramid so also will all the pyramids which have triangular bases contained in the one pyramid be to all the pyramids which have triangular bases contained in the other pyramid [V 12] that is the pyramid itself which has a polygonal base to the pyramid which has a polygonal base

But the pyramid which has a triangular base is to the pyramid which has a triangular base in the triplicate ratio of the corresponding sides

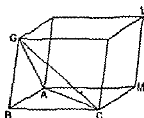
therefore also the pyramid which has a polygonal base has to the pyramid which has a similar base the ratio triplicate of that which the side has to the side

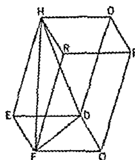
## Proposition 9

*In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights and those pyramids in which the bases are reciprocally proportional to the heights are equal*

For let there be equal pyramids which have the triangular bases *ABC* *DEF* and vertices the points *G* *H*

I say that in the pyramids *ABCG* *DEFH* the bases are reciprocally proportional to the heights that is as the base *ABC* is to the base *DEF* so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*




For let the parallelepipedal solids *BGML* *EHQP* be completed

Now since the pyramid *ABCG* is equal to the pyramid *DEFH*,

and the solid *BGML* is six times the pyramid *ABCG*

and the solid *EHQP* six times the pyramid *DEFH*

therefore the solid *BGML* is equal to the solid *EHQP*

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights [xi 34]

therefore as the base *BM* is to the base *EQ* so is the height of the solid *EHQP* to the height of the solid *BGML*

But as the base *BM* is to *EQ* so is the triangle *ABC* to the triangle *DEF* [i 34]

Therefore also as the triangle *ABC* is to the triangle *DEF* so is the height of the solid *EHQP* to the height of the solid *BGML* [v 11]

But the height of the solid *EHQP* is the same with the height of the pyramid *DEFH*

and the height of the solid *BGML* is the same with the height of the pyramid *ABCG*

therefore as the base *ABC* is to the base *DEF* so is the height of the pyramid *DEFH* to the height of the pyramid *ABCG*

Therefore in the pyramids *ABCG* *DEFH* the bases are reciprocally proportional to the heights

Next in the pyramids $ABCG$, $DEFH$ let the bases be reciprocally proportional to the heights,

that is as the base $ABC$ is to the base $DEF$ so let the height of the pyramid $DEFH$ be to the height of the pyramid $ABCG$,

I say that the pyramid $ABCG$ is equal to the pyramid $DEFH$

For with the same construction

since as the base $ABC$ is to the base $DEF$, so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$

while as the base $ABC$ is to the base $DEF$, so is the parallelogram $BM$ to the parallelogram $EQ$

therefore also, as the parallelogram $BM$ is to the parallelogram $EQ$, so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$ [v 11]

But the height of the pyramid $DEFH$ is the same with the height of the parallelepiped $EHQP$

and the height of the pyramid $ABCG$ is the same with the height of the parallelepiped $BGML$,

therefore as the base $BM$ is to the base $EQ$ so is the height of the parallelepiped $EHQP$ to the height of the parallelepiped $BGML$

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal, [xi 34]

therefore the parallelepipedal solid $BGML$ is equal to the parallelepipedal solid $EHQP$

And the pyramid $ABCG$ is a sixth part of $BGML$ and the pyramid $DEFH$ a sixth part of the parallelepiped $EHQP$,

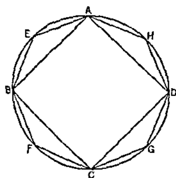
therefore the pyramid $ABCG$ is equal to the pyramid $DEFH$

Therefore etc Q E D

## Proposition 10

*Any cone is a third part of the cylinder which has the same base with it and equal height*

For let a cone have the same base, namely the circle $ABCD$, with a cylinder and equal height,




I say that the cone is a third part of the cylinder that is that the cylinder is triple of the cone

For if the cylinder is not triple of the cone the cylinder will be either greater than triple or less than triple of the cone

First let it be greater than triple

and let the square $ABCD$ be inscribed in the circle $ABCD$, [iv 6]

then the square $ABCD$ is greater than the half of the circle $ABCD$

From the square $ABCD$ let there be set up a prism of equal height with the cylinder

Then the prism so set up is greater than the half of the cylinder

inasmuch as if we also circumscribe a square about the circle $ABCD$ [iv 7] the square inscribed in the circle $ABCD$ is half of that circumscribed about it, and the solids set up from them are parallelepipedal prisms of equal height

while parallelepipedal solids which are of the same height are to one another as their bases [XI 32]

therefore also the prism set up on the square *ABCD* is half of the prism set up from the square circumscribed about the circle *ABCD*

[cf XI 28 or XII 6 and 7 Por]

and the cylinder is less than the prism set up from the square circumscribed about the circle *ABCD*

therefore the prism set up from the square *ABCD* and of equal height with the cylinder is *greater than the half of the cylinder*

Let the circumferences *AB BC CD DA* be bisected at the points *E, F G H*

and let *AE FB BF FC CG GD DH HA* be joined

then each of the triangles *AEB BFC CGD DHA* is greater than the half of that segment of the circle *ABCD* which is about it as we proved before [XII 2]

On each of the triangles *AEB BFC CGD DHA* let prisms be set up of equal height with the cylinder

*then each of the prisms so set up is greater than the half part of that segment of the cylinder which is about it*

inasmuch as if we draw through the points *E F G H* parallels to *AB BC, CD DA* complete the parallelograms on *AB BC CD DA* and set up from them parallelepipedal solids of equal height with the cylinder the prisms on the triangles *AEB BFC CGD DHA* are halves of the several solids set up and the segments of the cylinder are less than the parallelepipedal solids set up hence also the prisms on the triangles *AEB BFC CGD DHA* are greater than the half of the segments of the cylinder about them

Thus bisecting the circumferences that are left joining straight lines setting up on each of the triangles prisms of equal height with the cylinder,

and doing this continually

we shall leave some segments of the cylinder which will be less than the excess by which the cylinder exceeds the triple of the cone [X 1]

*Let such segments be left and let them be AE EB BF, FC CG, GD DH HA*

therefore the remainder the prism of which the polygon *AEBFCGDH* is the base and the height is the same as that of the cylinder is greater than triple of the cone

But the prism of which the polygon *AFBFCGDH* is the base and the height the same as that of the cylinder is triple of the pyramid of which the polygon *AEBFCGDH* is the base and the vertex is the same as that of the cone [XII 7 Por]

therefore also the pyramid of which the polygon *AEBFCGDH* is the base and the vertex is the same as that of the cone is greater than the cone which has the circle *ABCD* as base

But it is also less for it is enclosed by it

which is impossible

*Therefore the cylinder is not greater than triple of the cone*

I say next that neither is the cylinder less than triple of the cone

For if possible let the cylinder be less than triple of the cone

therefore inversely the cone is greater than a third part of the cylinder

Let the square $ABCD$ be inscribed in the circle $ABCD$,
therefore the square $ABCD$ is greater than the half of the circle $ABCD$

Now let there be set up from the square $ABCD$ a pyramid having the same vertex with the cone,
therefore the pyramid so set up is greater than the half part of the cone, seeing that as we proved before, if we circumscribe a square about the circle, the square $ABCD$ will be half of the square circumscribed about the circle, and if we set up from the squares parallelepipedal solids of equal height with the cone which are also called prisms the solid set up from the square $ABCD$ will be half of that set up from the square circumscribed about the circle,

for they are to one another as their bases [XI 32]

Hence also the thirds of them are in that ratio
therefore also the pyramid of which the square $ABCD$ is the base is half of the pyramid set up from the square circumscribed about the circle

And the pyramid set up from the square about the circle is greater than the cone,

for it encloses it

Therefore the pyramid of which the square $ABCD$ is the base and the vertex is the same with that of the cone is greater than the half of the cone

Let the circumferences $AB$ $BC$, $CD$, $DA$ be bisected at the points $E$ $F$ $G$, $H$,

and let $AE$ $EB$ $BF$ $FC$, $CG$, $GD$, $DH$, $HA$ be joined
therefore also each of the triangles $AEB$ $BFC$ $CGD$ $DHA$ is greater than the half part of that segment of the circle $ABCD$ which is about it

Now on each of the triangles $AEB$ $BFC$, $CGD$, $DHA$ let pyramids be set up which have the same vertex as the cone,
therefore also each of the pyramids so set up is in the same manner, greater than the half part of that segment of the cone which is about it

Thus by bisecting the circumferences that are left, joining straight lines setting up on each of the triangles a pyramid which has the same vertex as the cone

and doing this continually,
we shall leave some segments of the cone which will be less than the excess by which the cone exceeds the third part of the cylinder [X 1]

Let such be left and let them be the segments on $AE$ $EB$ $BF$, $FC$ $CG$, $GD$, $DH$ $HA$,
therefore the remainder the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex the same with that of the cone is greater than a third part of the cylinder

But the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex the same with that of the cone is a third part of the prism of which the polygon $AEBFCGDH$ is the base and the height is the same with that of the cylinder,
therefore the prism of which the polygon $AEBFCGDH$ is the base and the height is the same with that of the cylinder is greater than the cylinder of which the circle $ABCD$ is the base

But it is also less for it is enclosed by it

which is impossible

Therefore the cylinder is not less than triple of the cone

But it was proved that neither is it greater than triple,
therefore the cylinder is triple of the cone,
hence the cone is a third part of the cylinder

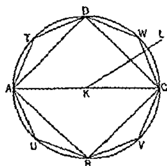
Therefore etc. Q. E. D.

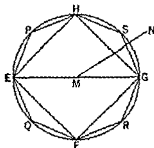
PROPOSITION 11

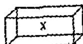
*Cones and cylinders which are of the same height are to one another as their bases.*

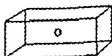
Let there be cones and cylinders of the same height,
let the circles $ABCD$, $EFGH$ be their bases, $KL$, $MN$ their axes and $AC$, $EG$ the diameters of their bases.

I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the cone $AL$ to the cone $EN$.










For, if not, then, as the circle $ABCD$ is to the circle $EFGH$, so will the cone $AL$ be either to some solid less than the cone $EN$ or to a greater.

First, let it be in that ratio to a less solid $O$, and let the solid $X$ be equal to that by which the solid $O$ is less than the cone $EN$;

therefore the cone $EN$ is equal to the solids $O$, $X$.

Let the square $EFGH$ be inscribed in the circle $EFGH$;

therefore the square is greater than the half of the circle.

Let there be set up from the square $EFGH$ a pyramid of equal height with the cone;

therefore the pyramid so set up is greater than the half of the cone, inasmuch as, if we circumscribe a square about the circle, and set up from it a pyramid of equal height with the cone, the inscribed pyramid is half of the circumscribed pyramid,

for they are to one another as their bases, [XII. 6]

while the cone is less than the circumscribed pyramid.

Let the circumferences $EF$, $FG$, $GH$, $HE$ be bisected at the points $P$, $Q$, $R$, $S$,
and let $HP$, $PE$, $EQ$, $QF$, $FR$, $RG$, $GS$, $SH$ be joined.

Therefore each of the triangles $HPE$, $EQF$, $FRG$, $GSH$ is greater than the half of that segment of the circle which is about it.

On each of the triangles $HPE$, $EQF$, $FRG$, $GSH$ let there be set up a pyramid of equal height with the cone;

therefore also each of the pyramids so set up is greater than the half of that segment of the cone which is about it.

Thus, bisecting the circumferences which are left, joining straight lines, set

ting up on each of the triangles pyramids of equal height with the cone

and doing this continually

we shall leave some segments of the cone which will be less than the solid X

[x 1]

Let such be left, and let them be the segments on *HP*, *PE*, *EQ*, *QF*, *FR*, *RG*, *GS*, *SH*,

therefore the remainder the pyramid of which the polygon *HPEQFRGS* is the base and the height the same with that of the cone, is greater than the solid *O*

Let there also be inscribed in the circle *ABCD* the polygon *DTAUBVCW* similar and similarly situated to the polygon *HPEQFRGS*,

and on it let a pyramid be set up of equal height with the cone *AL*

Since then, as the square on *AC* is to the square on *EG*, so is the polygon *DTAUBVCW* to the polygon *HPEQFRGS* [XII 1]

while as the square on *AC* is to the square on *EG*, so is the circle *ABCD* to the circle *EFGH*, [XII 2]

therefore also as the circle *ABCD* is to the circle *EFGH* so is the polygon *DTAUBVCW* to the polygon *HPEQFRGS*

But as the circle *ABCD* is to the circle *EFGH*, so is the cone *AL* to the solid *O*,

and as the polygon *DTAUBVCW* is to the polygon *HPEQFRGS*, so is the pyramid of which the polygon *DTAUBVCW* is the base and the point *L* the vertex to the pyramid of which the polygon *HPEQFRGS* is the base and the point *N* the vertex [XII 6]

Therefore also as the cone *AL* is to the solid *O*, so is the pyramid of which the polygon *DTAUBVCW* is the base and the point *L* the vertex to the pyramid of which the polygon *HPEQFRGS* is the base and the point *N* the vertex

[v 11]

therefore alternately as the cone *AL* is to the pyramid in it so is the solid *O* to the pyramid in the cone *EN* [v 16]

But the cone *AL* is greater than the pyramid in it

therefore the solid *O* is also greater than the pyramid in the cone *EN*

But it is also less

which is absurd

Therefore the cone *AL* is not to any solid less than the cone *EN* as the circle *ABCD* is to the circle *EFGH*

Similarly we can prove that neither is the cone *EN* to any solid less than the cone *AL* as the circle *EFGH* is to the circle *ABCD*

I say next that neither is the cone *AL* to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*

For if possible let it be in that ratio to a greater solid *O*

therefore inversely, as the circle *EFGH* is to the circle *ABCD* so is the solid *O* to the cone *AL*

But as the solid *O* is to the cone *AL* so is the cone *EN* to some solid less than the cone *AL*

therefore also as the circle *EFGH* is to the circle *ABCD* so is the cone *EN* to some solid less than the cone *AL*

which was proved impossible

Therefore the cone *AL* is not to any solid greater than the cone *EN* as the circle *ABCD* is to the circle *EFGH*

But it was proved that neither is it in this ratio to a less solid, therefore as the circle $ABCD$ is to the circle $EFGH$, so is the cone $AL$ to the cone $EN$

But as the cone is to the cone so is the cylinder to the cylinder,
for each is triple of each, [XII 10]

Therefore also as the circle $ABCD$ is to the circle $EFGH$ so are the cylinders on them which are of equal height

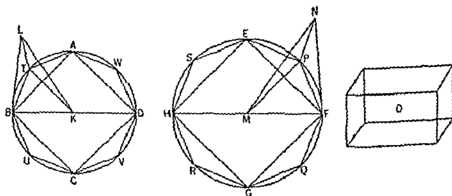
Therefore etc Q E D

PROPOSITION 12

*Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases*

Let there be similar cones and cylinders
let the circles $ABCD$, $EFGH$ be their bases $BD$ $FH$ the diameters of the bases, and $KL$ $MN$ the axes of the cones and cylinders,

I say that the cone of which the circle $ABCD$ is the base and the point $L$ the vertex has to the cone of which the circle $EFGH$ is the base and the point $N$ the vertex the ratio triplicate of that which $BD$ has to $FH$




For if the cone $ABCDL$ has not to the cone $EFGHN$ the ratio triplicate of that which $BD$ has to $FH$
the cone $ABCDL$ will have that triplicate ratio either to some solid less than the cone $EFGHN$ or to a greater

First let it have that triplicate ratio to a less solid $O$

Let the square $EFGH$ be inscribed in the circle $EFGH$ [IV 6]
therefore the square $EFGH$ is greater than the half of the circle $EFGH$

Now let there be set up on the square $EFGH$ a pyramid having the same vertex with the cone
therefore the pyramid so set up is greater than the half part of the cone

Let the circumferences $EF$ $FG$ $GH$ $HE$ be bisected at the points $P$ $Q$ $R$ $S$
and let $EP$ $PF$ $FQ$ $QG$ $GR$ $RH$ $HS$ $SE$ be joined

Therefore each of the triangles $EPF$ $FQG$ $GRH$ $HSE$ is also greater than the half part of that segment of the circle $EFGH$ which is about it

Now on each of the triangles $EPF$ $FQG$ $GRH$, $HSE$ let a pyramid be set up having the same vertex with the cone,

therefore each of the pyramids so set up is also greater than the half part of that segment of the cone which is about it

Thus bisecting the circumferences so left joining straight lines, setting up on each of the triangles pyramids having the same vertex with the cone,

and doing this continually,

we shall leave some segments of the cone which will be less than the excess by which the cone *EFGHN* exceeds the solid *O* [x 1]

Let such be left and let them be the segments on *EP*, *PF*, *FQ*, *QG*, *GR* *RH*, *HS* *SE*

therefore the remainder, the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex is greater than the solid *O*

Let there be also inscribed in the circle *ABCD* the polygon *ATBUCVDW* similar and similarly situated to the polygon *EPFQGRHS*

and let there be set up on the polygon *ATBUCVDW* a pyramid having the same vertex with the cone,

of the triangles containing the pyramid of which the polygon *ATBUCVDW* is the base and the point *L* the vertex let *LBT* be one,

and of the triangles containing the pyramid of which the polygon *EPFQGRHS* is the base and the point *N* the vertex let *NFP* be one,

and let *KT*, *MP* be joined

Now since the cone *ABCDL* is similar to the cone *EFGHN*,

therefore as *BD* is to *FH*, so is the axis *KL* to the axis *MN* [xi Def 24]

But, as *BD* is to *FH*, so is *BK* to *FM*,

therefore also as *BK* is to *FM* so is *KL* to *MN*

And alternately as *BK* is to *KL*, so is *FM* to *MN* [v 16]

And the sides are proportional about equal angles, namely the angles *BKL*, *FMN*,

therefore the triangle *BKL* is similar to the triangle *FMN* [vi 6]

Again, since as *BK* is to *KT*, so is *FM* to *MP*

and they are about equal angles namely the angles *BKT* *FMP*,

inasmuch as whatever part the angle *BKT* is of the four right angles at the centre *K* the same part also is the angle *FMP* of the four right angles at the centre *M*,

since then the sides are proportional about equal angles

therefore the triangle *BKT* is similar to the triangle *FMP* [vi 6]

Again since it was proved that, as *BK* is to *KL* so is *FM* to *MN*

while *BK* is equal to *KT* and *FM* to *PM*

therefore as *TK* is to *KL* so is *PM* to *MN*

and the sides are proportional about equal angles namely the angles *TKL* *PMN* for they are right,

therefore the triangle *LKT* is similar to the triangle *NMP* [vi 6]

And since owing to the similarity of the triangles *LKB* *NMF*

as *LB* is to *BK* so is *NF* to *FM*

and owing to the similarity of the triangles *BKT* *FMP*

as *KB* is to *BT* so is *MF* to *FP*

therefore *ex aequali* as *LB* is to *BT*, so is *NF* to *FP* [v 22]

Again since owing to the similarity of the triangles *LTK* *NPM*

as *LT* is to *TK* so is *NP* to *PM*

and owing to the similarity of the triangles *TKB*, *PMF*,

as $KT$ is to $TB$ so is $MP$ to $PF$,

therefore *ex aequali* as $LT$ is to $TB$ so is $NP$ to $PF$ [v 22]

But it was also proved that as $TB$ is to $BL$, so is $PF$ to $FN$

Therefore *ex aequali* as $TL$ is to $LB$, so is $PN$ to $NF$ [v 22]

Therefore in the triangles $LTB$, $NPF$ the sides are proportional,

therefore the triangles $LTB$ $NPF$ are equiangular [vi 5]

hence they are also similar [vi Def 1]

Therefore the pyramid of which the triangle $BKT$ is the base and the point $L$ the vertex is also similar to the pyramid of which the triangle $FMP$ is the base and the point $N$ the vertex

for they are contained by similar planes equal in multitude [xi Def 9]

But similar pyramids which have triangular bases are to one another in the triplicate ratio of their corresponding sides [xii 8]

Therefore the pyramid $BKTL$ has to the pyramid $FMPN$ the ratio triplicate of that which $BK$ has to $FM$

Similarly by joining straight lines from $A$, $W$ $D$, $V$ $C$, $U$ to $K$, and from $E$ $S$ $H$ $R$ $G$ $Q$ to $M$ and setting up on each of the triangles pyramids which have the same vertex with the cones

we can prove that each of the similarly arranged pyramids will also have to each similarly arranged pyramid the ratio triplicate of that which the corresponding side $BK$ has to the corresponding side $FM$ that is which $BD$ has to $FH$

And as one of the antecedents is to one of the consequents so are all the antecedents to all the consequents [v 12]

therefore also as the pyramid $BKTL$ is to the pyramid $FMPN$ so is the whole pyramid of which the polygon $ATBUCVDW$ is the base and the point $L$ the vertex to the whole pyramid of which the polygon $EPFQGRHS$ is the base and the point $N$ the vertex

hence also the pyramid of which $ATBUCVDW$ is the base and the point $L$ the vertex has to the pyramid of which the polygon $EPFQGRHS$ is the base and the point $N$ the vertex the ratio triplicate of that which $BD$ has to $FH$

But by hypothesis the cone of which the circle $ABCD$ is the base and the point $L$ the vertex has also to the solid $O$ the ratio triplicate of that which $BD$ has to $FH$

therefore as the cone of which the circle $ABCD$ is the base and the point $L$ the vertex is to the solid $O$ so is the pyramid of which the polygon $ATBUCVDW$ is the base and $L$ the vertex to the pyramid of which the polygon $EPFQGRHS$ is the base and the point $N$ the vertex

therefore alternately as the cone of which the circle $ABCD$ is the base and $L$ the vertex is to the pyramid contained in it of which the polygon $ATBUCVDW$ is the base and $L$ the vertex so is the solid $O$ to the pyramid of which the polygon $EPFQGRHS$ is the base and $N$ the vertex [v 16]

But the said cone is greater than the pyramid in it

for it encloses it

Therefore the solid $O$ is also greater than the pyramid of which the polygon $EPFQGRHS$ is the base and $N$ the vertex

But it is also less

which is impossible

Therefore the cone of which the circle $ABCD$ is the base and $L$ the vertex

has not to any solid less than the cone of which the circle $EFGH$ is the base and the point $N$ the vertex the ratio triplicate of that which $BD$ has to $FH$

Similarly we can prove that neither has the cone $EFGHN$ to any solid less than the cone $ABCDL$ the ratio triplicate of that which $FH$ has to $BD$

I say next that neither has the cone $ABCDL$ to any solid greater than the cone $EFGHN$ the ratio triplicate of that which $BD$ has to $FH$

For if possible let it have that ratio to a greater solid $O$

Therefore, inversely, the solid $O$ has to the cone $ABCDL$ the ratio triplicate of that which $FH$ has to $BD$

But as the solid $O$ is to the cone $ABCDL$ so is the cone $EFGHN$ to some solid less than the cone $ABCDL$

Therefore the cone $EFGHN$ also has to some solid less than the cone $ABCDL$ the ratio triplicate of that which $FH$ has to $BD$

which was proved impossible

Therefore the cone $ABCDL$ has not to any solid greater than the cone $EFGHN$ the ratio triplicate of that which $BD$ has to $FH$

But it was proved that neither has it this ratio to a less solid than the cone $EFGHN$

Therefore the cone $ABCDL$ has to the cone $EFGHN$ the ratio triplicate of that which $BD$ has to $FH$

But, as the cone is to the cone so is the cylinder to the cylinder, for the cylinder which is on the same base as the cone and of equal height with it is triple of the cone, [XII 10]

therefore the cylinder also has to the cylinder the ratio triplicate of that which $BD$ has to $FH$

Therefore etc Q E D

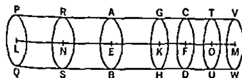
## Proposition 13

*If a cylinder be cut by a plane which is parallel to its opposite planes then, as the cylinder is to the cylinder so will the axis be to the axis*

For let the cylinder $AD$ be cut by the plane $GH$ which is parallel to the opposite planes $AB$ $CD$

and let the plane $GH$ meet the axis at the point $K$,

I say that as the cylinder $BG$ is to the cylinder $GD$, so is the axis $EK$ to the axis $KF$




For let the axis $EF$ be produced in both directions to the points $L$ $M$

and let there be set out any number whatever of axes $EN$ $NL$ equal to the axis $EK$

and any number whatever $FO$ $OM$ equal to $FK$

and let the cylinder $PW$ on the axis $LM$ be conceived of which the circles $PQ$ $VW$ are the bases

Let planes be carried through the points $N$, $O$ parallel to $AB$ $CD$ and to the bases of the cylinder $PW$

and let them produce the circles $RS$ $TU$ about the centres $N$, $O$

Then since the axes $LN$ $NE$ $EK$ are equal to one another,

therefore the cylinders $QR$ $RB$, $BG$ are to one another as their bases [XII 11]

But the bases are equal,

therefore the cylinders $QR$, $RB$ $BG$ are also equal to one another

Since then the axes $LN$, $NE$ $EK$ are equal to one another

and the cylinders $QR$ $RB$, $BG$ are also equal to one another,

and the multitude of the former is equal to the multitude of the latter, therefore whatever multiple the axis $KL$ is of the axis $EK$, the same multiple also will the cylinder $QG$ be of the cylinder $GB$

For the same reason whatever multiple the axis $MK$ is of the axis $KF$, the same multiple also is the cylinder $WG$ of the cylinder $GD$

And if the axis $KL$ is equal to the axis $KM$, the cylinder $QG$ will also be equal to the cylinder $GW$,

if the axis is greater than the axis the cylinder will also be greater than the cylinder

and if less less

Thus there being four magnitudes the axes $EK$ $KF$ and the cylinders $BG$ $GD$

there have been taken equimultiples of the axis $EK$ and of the cylinder $BG$, namely the axis $LK$ and the cylinder $QG$,

and equimultiples of the axis $KF$ and of the cylinder $GD$, namely the axis $KM$ and the cylinder $GW$,

and it has been proved that,

if the axis $KL$ is in excess of the axis $KM$ the cylinder $QG$ is also in excess of the cylinder $GW$,

if equal equal

and if less less

Therefore as the axis $EK$ is to the axis $KF$ so is the cylinder $BG$ to the cylinder $GD$ [V Def 5]

Q E D

PROPOSITION 14

*Cones and cylinders which are on equal bases are to one another as their heights*

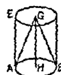
For let $EB$ $FD$ be cylinders on equal bases, the circles $AB$ $CD$

I say that as the cylinder $EB$ is to the cylinder $FD$ so is the axis $GH$ to the axis $KL$

For let the axis $KL$ be produced to the point $N$

let $LN$ be made equal to the axis $GH$,

and let the cylinder $CM$ be conceived about $LN$ as axis







Since then the cylinders $EB$ $CM$ are of the same height they are to one another as their bases [XII 11]

But the bases are equal to one another

therefore the cylinders $EB$ $CM$ are also equal

And since the cylinder $FM$ has been cut by the plane $CD$ which is parallel to its opposite planes

therefore as the cylinder $CM$ is to the cylinder $FD$ so is the axis $LN$ to the axis $KL$ [XII 13]

But the cylinder $CM$ is equal to the cylinder $EB$,

and the axis $LN$ to the axis $GH$,

therefore as the cylinder $EB$ is to the cylinder $FD$, so is the axis $GH$ to the axis $KL$.

But as the cylinder $EB$ is to the cylinder $FD$, so is the cone $ABG$ to the cone $CDK$. [XII 10]

Therefore also as the axis $GH$ is to the axis $KL$, so is the cone $ABG$ to the cone $CDK$ and the cylinder $EB$ to the cylinder $FD$. Q E D

PROPOSITION 15

*In equal cones and cylinders the bases are reciprocally proportional to the heights and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.*

Let there be equal cones and cylinders of which the circles $ABCD$, $EFGH$ are the bases,

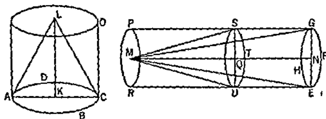
let $AC$, $EG$ be the diameters of the bases

and $KL$, $MN$ the axes which are also the heights of the cones or cylinders,

let the cylinders $AO$, $EP$ be completed.

I say that in the cylinders $AO$, $EP$ the bases are reciprocally proportional to the heights,

that is as the base $ABCD$ is to the base $EFGH$ so is the height $MN$ to the height $KL$.




For the height $LK$ is either equal to the height $MN$ or not equal.

First let it be equal.

Now the cylinder $AO$ is also equal to the cylinder $EP$.

But cones and cylinders which are of the same height are to one another as their bases, [XII 11]

therefore the base $ABCD$ is also equal to the base $EFGH$.

Hence also reciprocally as the base $ABCD$ is to the base $EFGH$, so is the height $MN$ to the height $KL$.

Next, let the height $LK$ not be equal to $MN$,

but let $MN$ be greater,

from the height $MN$ let $QN$ be cut off equal to $KL$,

through the point $Q$ let the cylinder $EP$ be cut by the plane $TUS$ parallel to the planes of the circles $EFGH$, $RP$,

and let the cylinder $ES$ be conceived erected from the circle $EFGH$ as base and with height $NQ$.

Now since the cylinder $AO$ is equal to the cylinder $EP$,

therefore as the cylinder $AO$ is to the cylinder $ES$ so is the cylinder $EP$ to the cylinder $ES$. [V 7]

But as the cylinder $AO$ is to the cylinder $ES$, so is the base $ABCD$ to the base $FFGH$,

for the cylinders $AO$, $ES$ are of the same height [XII 11]

and, as the cylinder $EP$ is to the cylinder $ES$ so is the height $MN$ to the height $QN$,

for the cylinder $EP$ has been cut by a plane which is parallel to its opposite planes [XII 13]

Therefore also as the base $ABCD$ is to the base $EFGH$, so is the height $MN$ to the height $QN$ [V 11]

But the height $QN$ is equal to the height $KL$,

therefore as the base $ABCD$ is to the base $EFGH$, so is the height $MN$ to the height $KL$

Therefore in the cylinders $AO$ $EP$ the bases are reciprocally proportional to the heights

Next in the cylinders $AO$, $EP$ let the bases be reciprocally proportional to the heights

that is as the base $ABCD$ is to the base $EFGH$, so let the height $MN$ be to the height $KL$

I say that the cylinder $AO$ is equal to the cylinder $EP$

For with the same construction

since as the base $ABCD$ is to the base $EFGH$ so is the height $MN$ to the height $KL$,

while the height $KL$ is equal to the height $QN$

therefore as the base $ABCD$ is to the base $EFGH$, so is the height $MN$ to the height $QN$

But as the base $ABCD$ is to the base $EFGH$ so is the cylinder $AO$ to the cylinder $ES$

for they are of the same height [XII 11]

and as the height $MN$ is to $QN$ so is the cylinder $EP$ to the cylinder $ES$ [XII 13]

therefore as the cylinder $AO$ is to the cylinder $ES$ so is the cylinder $EP$ to the cylinder $ES$ [V 11]

Therefore the cylinder $AO$ is equal to the cylinder $EP$ [V 9]

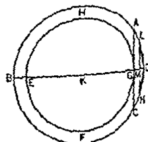
And the same is true for the cones also Q E D

PROPOSITION 16

*Given two circles about the same centre to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle*

Let $ABCD$ $EFGH$ be the two given circles about the same centre $K$

thus it is required to inscribe in the greater circle $ABCD$ an equilateral polygon with an even number of sides which does not touch the circle $EFGH$




For let the straight line $BKD$ be drawn through the centre $K$

and from the point $G$ let $GA$ be drawn at right angles to the straight line $BD$ and carried through to $C$

therefore $AC$ touches the circle $EFGH$ [III 16 Por]

Then bisecting the circumference $BAD$ bisecting the half of it and doing this continually, we shall leave a circumference less than $AD$ [X 1]

Let such be left and let it be $LD$,

from $L$ let $LM$ be drawn perpendicular to $BD$ and carried through to $N$

and let $LD$ $DN$ be joined

therefore $LD$ is equal to $DN$ [III 3, I 4]

Now, since $LN$ is parallel to $AC$,

and $AC$ touches the circle $EFGH$,

therefore $LN$ does not touch the circle $EFGH$

therefore $LD$ $DN$ are far from touching the circle $EFGH$

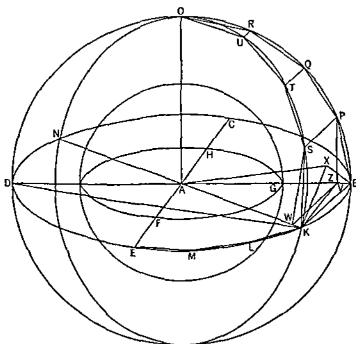
If then we fit into the circle $ABCD$ straight lines equal to the straight line $LD$ and placed continuously, there will be inscribed in the circle $ABCD$ an equilateral polygon with an even number of sides which does not touch the lesser circle $EFGH$ Q E F

## PROPOSITION 17

*Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface*

Let two spheres be conceived about the same centre $A$

thus it is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface




Let the spheres be cut by any plane through the centre

then the sections will be circles

inasmuch as the sphere was produced by the diameter remaining fixed and the semicircle being carried round it, [xi Def 14]
hence in whatever position we conceive the semicircle to be the plane carried through it will produce a circle on the circumference of the sphere

And it is manifest that this circle is the greatest possible
inasmuch as the diameter of the sphere which is of course the diameter both of the semicircle and of the circle is greater than all the straight lines drawn across in the circle or the sphere

Let then *BCDE* be the circle in the greater sphere
and *FGH* the circle in the lesser sphere
let two diameters in them *BD* *CE* be drawn at right angles to one another, then given the two circles *BCDE* *FGH* about the same centre, let there be inscribed in the greater circle *BCDE* an equilateral polygon with an even number of sides which does not touch the lesser circle *FGH*
let *BK* *KL* *LM* *ME* be its sides in the quadrant *BE*,
let *KA* be joined and carried through to *N*
let *AO* be set up from the point *A* at right angles to the plane of the circle *BCDE* and let it meet the surface of the sphere at *O*
and through *AO* and each of the straight lines *BD* *KN* let planes be carried they will then make greatest circles on the surface of the sphere for the reason stated

Let them make such
and in them let *BOD* *KON* be the semicircles on *BD* *KN*

Now since *OA* is at right angles to the plane of the circle *BCDE*
therefore all the planes through *OA* are also at right angles to the plane of the circle *BCDE* [xi 18]
hence the semicircles *BOD* *KON* are also at right angles to the plane of the circle *BCDE*

And since the semicircles *BED* *BOD* *KON* are equal
for they are on the equal diameters *BD* *KN*
therefore the quadrants *BE* *BO* *KO* are also equal to one another

Therefore there are as many straight lines in the quadrants *BO* *KO* equal to the straight lines *BK* *KL* *LM* *ME* as there are sides of the polygon in the quadrant *BE*

Let them be inscribed and let them be *BP* *PQ* *QR* *RO* and *KS* *ST*, *TU* *UO*
let *SP* *TQ* *UR* be joined
and from *P* *S* let perpendiculars be drawn to the plane of the circle *BCDE* [xi 11]
these will fall on *BD* *KN* the common sections of the planes
inasmuch as the planes of *BOD* *KON* are also at right angles to the plane of the circle *BCDE* [cf xi Def 4]

Let them so fall and let them be *PV* *SW*
and let *WV* be joined

Now since in the equal semicircles *BOD* *KON* equal straight lines *BP*, *KS* have been cut off
and the perpendiculars *PV* *SW* have been drawn
therefore *PV* is equal to *SW* and *BV* to *KW* [iii 27, i 26]

But the whole *BA* is also equal to the whole *KA*

therefore the remainder $VA$ is also equal to the remainder $WA$,

therefore as $BV$ is to $VA$ so is $KW$ to $WA$

therefore $WV$ is parallel to $KB$ [VI 2]

And since each of the straight lines $PV$, $SW$ is at right angles to the plane of the circle $BCDE$,

therefore $PV$ is parallel to $SW$ [XI 6]

But it was also proved equal to it

therefore $WV$, $SP$ are also equal and parallel [I 33]

And, since $WV$ is parallel to $SP$,

while $WV$ is parallel to $KB$,

therefore $SP$ is also parallel to $KB$ [XI 9]

And $BP$, $KS$ join their extremities,

therefore the quadrilateral $KBPS$ is in one plane

inasmuch as if two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels [XI 7]

For the same reason

each of the quadrilaterals $SPQT$, $TQRU$ is also in one plane

But the triangle $URO$ is also in one plane [XI 2]

If then we conceive straight lines joined from the points $P$, $S$ $Q$ $T$, $R$, $U$ to $A$, there will be constructed a certain polyhedral solid figure between the circumferences $BO$, $KO$, consisting of pyramids of which the quadrilaterals $KBPS$, $SPQT$, $TQRU$ and the triangle $URO$ are the bases and the point $A$ the vertex

And if we make the same construction in the case of each of the sides $KL$, $LM$ $ME$ as in the case of $BK$ and further in the case of the remaining three quadrants

there will be constructed a certain polyhedral figure inscribed in the sphere and contained by pyramids, of which the said quadrilaterals and the triangle $URO$, and the others corresponding to them are the bases and the point $A$ the vertex

I say that the said polyhedron will not touch the lesser sphere at the surface on which the circle $FGH$ is

Let $AX$ be drawn from the point $A$ perpendicular to the plane of the quadrilateral $KBPS$ and let it meet the plane at the point $X$ [XI 11]

let $XB$, $XK$ be joined

Then since $AX$ is at right angles to the plane of the quadrilateral $KBPS$, therefore it is also at right angles to all the straight lines which meet it and are in the plane of the quadrilateral [XI Def 3]

Therefore $AX$ is at right angles to each of the straight lines $BX$, $XK$

And since $AB$ is equal to $AK$

the square on $AB$ is also equal to the square on $AK$

And the squares on $AX$ $XB$ are equal to the square on $AB$

for the angle at $X$ is right [I 47]

and the squares on $AX$ $XK$ are equal to the square on $AK$ [id ]

Therefore the squares on $AX$ $XB$ are equal to the squares on $AX$ $XK$

Let the square on $AX$ be subtracted from each

therefore the remainder the square on $BX$ is equal to the remainder the square on $XK$,

therefore $BX$ is equal to $XK$

Similarly we can prove that the straight lines joined from X to $P$, $S$ are equal to each of the straight lines $BX$, $XK$.

Therefore the circle described with centre X and distance one of the straight lines $XB$, $XK$ will pass through $P$, $S$ also,

and $KBPS$ will be a quadrilateral in a circle.

Now, since $KB$ is greater than $WV$,

while $WV$ is equal to $SP$,

therefore $KB$ is greater than $SP$.

But $KB$ is equal to each of the straight lines $KS$, $BP$;

therefore each of the straight lines $KS$, $BP$ is greater than $SP$.

And, since $KBPS$ is a quadrilateral in a circle,

and $KB$, $BP$, $KS$ are equal, and $PS$ less,

and $BX$ is the radius of the circle,

therefore the square on $KB$ is greater than double of the square on $BX$.

Let $KZ$ be drawn from $K$ perpendicular to $BV$.

Then, since $BD$ is less than double of $DZ$,

and, as $BD$ is to $DZ$, so is the rectangle $DB$, $BZ$ to the rectangle $DZ$, $ZB$,

if a square be described upon $BZ$ and the parallelogram on $ZD$ be completed,

then the rectangle $DB$, $BZ$ is also less than double of the rectangle $DZ$, $ZB$.

And, if $KD$ be joined,

the rectangle $DB$, $BZ$ is equal to the square on $BK$,

and the rectangle $DZ$, $ZB$ equal to the square on $KZ$; [III. 31, VI. 8 and Por.]

therefore the square on $KB$ is less than double of the square on $KZ$.

But the square on $KB$ is greater than double of the square on $BX$;

therefore the square on $KZ$ is greater than the square on $BX$.

And, since $BA$ is equal to $KA$,

the square on $BA$ is equal to the square on $AK$.

And the squares on $BX$, $XA$ are equal to the square on $BA$,

and the squares on $KZ$, $ZA$ equal to the square on $KA$; [I. 47]

therefore the squares on $BX$, $XA$ are equal to the squares on $KZ$, $ZA$,

and of these the square on $KZ$ is greater than the square on $BX$;

therefore the remainder, the square on $ZA$, is less than the square on $XA$.

Therefore $AX$ is greater than $AZ$;

therefore $AX$ is much greater than $AG$.

And $AX$ is the perpendicular on one base of the polyhedron,

and $AG$ on the surface of the lesser sphere;

hence the polyhedron will not touch the lesser sphere on its surface.

Therefore, given two spheres about the same centre, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface. Q. E. F.

PORISM. But if in another sphere also a polyhedral solid be inscribed similar to the solid in the sphere $BCDE$,

the polyhedral solid in the sphere $BCDE$ has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere $BCDE$ has to the diameter of the other sphere.

For, the solids being divided into their pyramids similar in multitude and arrangement, the pyramids will be similar.

But similar pyramids are to one another in the triplicate ratio of their corresponding sides; [XII. 8, Por.]

therefore the pyramid of which the quadrilateral $KBPS$ is the base and the point $A$ the vertex has to the similarly arranged pyramid in the other sphere the ratio triplicate of that which the corresponding side has to the corresponding side that is of that which the radius $AB$ of the sphere about $A$ as centre has to the radius of the other sphere

Similarly also each pyramid of those in the sphere about $A$ as centre has to each similarly arranged pyramid of those in the other sphere the ratio triplicate of that which $AB$ has to the radius of the other sphere

And as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents, [V 12]

hence the whole polyhedral solid in the sphere about $A$ as centre has to the whole polyhedral solid in the other sphere the ratio triplicate of that which $AB$ has to the radius of the other sphere that is of that which the diameter $BD$ has to the diameter of the other sphere Q E D

PROPOSITION 18

*Spheres are to one another in the triplicate ratio of their respective diameters*

Let the spheres $ABC$, $DFF$ be conceived

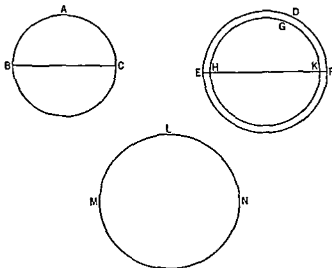
and let $BC$, $EF$ be their diameters,

I say that the sphere $ABC$ has to the sphere $DEF$ the ratio triplicate of that which $BC$ has to $EF$

For if the sphere $ABC$ has not to the sphere $DEF$ the ratio triplicate of that which $BC$ has to $EF$

then the sphere $ABC$ will have either to some less sphere than the sphere $DEF$, or to a greater the ratio triplicate of that which $BC$ has to $EF$

First let it have that ratio to a less sphere $GHK$,




let $DEF$ be conceived about the same centre with $GHK$,

let there be inscribed in the greater sphere $DEF$ a polyhedral solid which does not touch the lesser sphere $GHK$ at its surface [XII 17]

and let there also be inscribed in the sphere $ABC$ a polyhedral solid similar to the polyhedral solid in the sphere $DEF$,
therefore the polyhedral solid in $ABC$ has to the polyhedral solid in $DEF$ the ratio triplicate of that which $BC$ has to $EF$ [XII 17, Por ]

But the sphere $ABC$ also has to the sphere $GHK$ the ratio triplicate of that which $BC$ has to $EF$,
therefore as the sphere $ABC$ is to the sphere $GHK$, so is the polyhedral solid in the sphere $ABC$ to the polyhedral solid in the sphere $DEF$,
and alternately as the sphere $ABC$ is to the polyhedron in it so is the sphere $GHK$ to the polyhedral solid in the sphere $DEF$ [V 16]

But the sphere $ABC$ is greater than the polyhedron in it
therefore the sphere $GHK$ is also greater than the polyhedron in the sphere $DEF$

But it is also less

for it is enclosed by it

Therefore the sphere $ABC$ has not to a less sphere than the sphere $DEF$ the ratio triplicate of that which the diameter $BC$ has to $EF$

Similarly we can prove that neither has the sphere $DEF$ to a less sphere than the sphere $ABC$ the ratio triplicate of that which $EF$ has to $BC$

I say next that neither has the sphere $ABC$ to any greater sphere than the sphere $DEF$ the ratio triplicate of that which $BC$ has to $EF$

For if possible let it have that ratio to a greater $LMN$,
therefore inversely the sphere $LMN$ has to the sphere $ABC$ the ratio triplicate of that which the diameter $EF$ has to the diameter $BC$

But inasmuch as $LMN$ is greater than $DEF$
therefore as the sphere $LMN$ is to the sphere $ABC$ so is the sphere $DEF$ to some less sphere than the sphere $ABC$ as was before proved [XII 2 Lemma]

Therefore the sphere $DEF$ also has to some less sphere than the sphere $ABC$ the ratio triplicate of that which $EF$ has to $BC$

which was proved impossible

Therefore the sphere $ABC$ has not to any sphere greater than the sphere $DEF$ the ratio triplicate of that which $BC$ has to $EF$

But it was proved that neither has it that ratio to a less sphere

Therefore the sphere $ABC$ has to the sphere $DEF$ the ratio triplicate of that which $BC$ has to $EF$ Q E D

# BOOK THIRTEEN

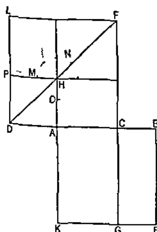
## PROPOSITIONS

### PROPOSITION 1

*If a straight line be cut in extreme and mean ratio the square on the greater segment added to the half of the whole is five times the square on the half*

For let the straight line $AB$ be cut in extreme and mean ratio at the point $C$, and let $AC$ be the greater segment,

let the straight line $AD$ be produced in a straight line with $CA$ and let $AD$ be made half of $AB$,




I say that the square on $CD$ is five times the square on $AD$

For let the squares $AE$, $DF$ be described on $AB$, $DC$

and let the figure in $DF$ be drawn,

let $FC$ be carried through to $G$

Now, since $AB$ has been cut in extreme and mean ratio at $C$,

therefore the rectangle $AB$ $BC$ is equal to the square on $AC$ [VI Def 3 VI 17]

And $CE$ is the rectangle $AB$, $BC$ and $FH$ the square on $AC$,

therefore $CE$ is equal to $FH$

And since $BA$ is double of $AD$,

while $BA$ is equal to $KA$ and $AD$ to $AH$,

therefore $KA$ is also double of $AH$

But as $KA$ is to $AH$ so is $CK$ to $CH$, [VI 1]

therefore $CK$ is double of $CH$

But $LH$, $HC$ are also double of $CH$

Therefore $KC$ is equal to $LH$ $HC$

But $CE$ was also proved equal to $HF$

therefore the whole square $AE$ is equal to the gnomon $MNO$

And since $BA$ is double of $AD$

the square on $BA$ is quadruple of the square on $AD$,

that is, $AE$ is quadruple of $DH$

But $AE$ is equal to the gnomon $MNO$

therefore the gnomon $MNO$ is also quadruple of $AP$

therefore the whole $DF$ is five times $AP$

And $DF$ is the square on $DC$, and $AP$ the square on $DA$,

therefore the square on $CD$ is five times the square on $DA$

Therefore etc Q E D

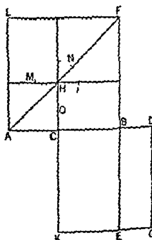
PROPOSITION 2

*If the square on a straight line be five times the square on a segment of it then, when the double of the said segment is cut in extreme and mean ratio the greater segment is the remaining part of the original straight line*

For let the square on the straight line $AB$ be five times the square on the segment $AC$ of it

and let $CD$ be double of $AC$,

I say that when $CD$ is cut in extreme and mean ratio the greater segment is $CB$




Let the squares $AF$ $CG$ be described on $AB$ $CD$ respectively

let the figure in $AF$ be drawn

and let $BE$ be drawn through

Now since the square on $BA$ is five times the square on $AC$

$AF$ is five times $AH$

Therefore the gnomon $MNO$ is quadruple of $AH$

And since $DC$ is double of $CA$

therefore the square on $DC$ is quadruple of the square on $CA$ that is $CG$ is quadruple of $AH$

But the gnomon $MNO$ was also proved quadruple of $AH$

therefore the gnomon $MNO$ is equal to $CG$

And since $DC$ is double of $CA$

while $DC$ is equal to $CK$ and $AC$ to $CH$

therefore $KB$ is also double of $BH$ [VI 1]

But $LH$ $HB$ are also double of $HB$

therefore $KB$ is equal to $LH$ $HB$

But the whole gnomon $MNO$ was also proved equal to the whole $CG$,

therefore the remainder $HF$ is equal to $BG$

And $BG$ is the rectangle $CD$, $DB$

for $CD$ is equal to $DG$,

and $HF$ is the square on $CB$

therefore the rectangle $CD$ $DB$ is equal to the square on $CB$

Therefore as $DC$ is to $CB$ so is $CB$ to $BD$

But $DC$ is greater than $CB$

therefore $CB$ is also greater than $BD$

Therefore when the straight line $CD$ is cut in extreme and mean ratio $CB$ is the greater segment

Therefore etc Q E D

LEMMA

That the double of $AC$ is greater than $BC$ is to be proved thus

If not let $BC$ be if possible double of $CA$

Therefore the square on $BC$ is quadruple of the square on $CA$

therefore the squares on $BC$ $CA$ are five times the square on $CA$

But by hypothesis the square on $BA$ is also five times the square on $CA$,

therefore the square on $BA$ is equal to the squares on $BC$, $CA$

which is impossible [II 4]

Therefore $CB$ is not double of $AC$

Similarly we can prove that neither is a straight line less than $CB$ double of $CA$,

for the absurdity is much greater

Therefore the double of $AC$ is greater than $CB$ Q E D

### Proposition 3

*If a straight line be cut in extreme and mean ratio the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment*

For let any straight line $AB$ be cut in extreme and mean ratio at the point $C$,

let $AC$ be the greater segment

and let $AC$ be bisected at $D$,

I say that the square on $BD$ is five times the square on $DC$

For let the square $AE$ be described on $AB$

and let the figure be drawn double

Since $AC$ is double of $DC$

therefore the square on $AC$ is quadruple of the square on $DC$,

that is $RS$ is quadruple of $FG$

And since the rectangle $AB$, $BC$ is equal to the square on $AC$

and $CE$ is the rectangle $AB$ $BC$

therefore $CE$ is equal to $RS$

But $RS$ is quadruple of $FG$

therefore $CE$ is also quadruple of $FG$

Again since $AD$ is equal to $DC$

$HK$ is also equal to $KF$

Hence the square $GF$ is also equal to the square $HL$

Therefore $GK$ is equal to $KL$ that is $MN$ to $NE$,

hence $MF$ is also equal to $FE$

But $MF$ is equal to $CG$,

therefore $CG$ is also equal to $FE$

Let $CN$ be added to each

therefore the gnomon $OPQ$ is equal to $CE$

But $CE$ was proved quadruple of $GF$,

therefore the gnomon $OPQ$ is also quadruple of the square $FG$

Therefore the gnomon $OPQ$ and the square $FG$ are five times $FG$

But the gnomon $OPQ$ and the square $FG$ are the square $DN$

And $DN$ is the square on $DB$ and $GF$ the square on $DC$

Therefore the square on $DB$ is five times the square on $DC$ Q E D

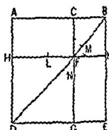
### Proposition 4

*If a straight line be cut in extreme and mean ratio the square on the whole and the square on the lesser segment together are triple of the square on the greater segment*

Let $AB$ be a straight line,

let it be cut in extreme and mean ratio at $C$, and let $AC$ be the greater segment,

I say that the squares on $AB$, $BC$ are triple of the square on $CA$




For let the square $ADEB$ be described on $AB$ and let the figure be drawn

Since then, $AB$ has been cut in extreme and mean ratio at $C$,

and $AC$ is the greater segment

therefore the rectangle $AB$, $BC$ is equal to the square on $AC$ [VI Def 3 VI 17]

And $AK$ is the rectangle $AB$ $BC$ and $HG$ the square on $AC$,

therefore $AK$ is equal to $HG$

And since $AF$ is equal to $FE$

let $CK$ be added to each,

therefore the whole $AK$ is equal to the whole $CE$,

therefore $AK$ $CE$ are double of $AK$

But $AK$ $CE$ are the gnomon $LMN$ and the square $CK$

therefore the gnomon $LMN$ and the square $CK$ are double of $AK$

But further $AK$ was also proved equal to $HG$

therefore the gnomon $LMN$ and the squares $CK$ $HG$ are triple of the square $HG$

And the gnomon $LMN$ and the squares $CK$ $HG$ are the whole square $AE$ and $CK$, which are the squares on $AB$ $BC$

while $HG$ is the square on $AC$

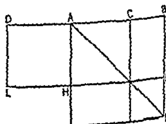
Therefore the squares on $AB$ $BC$ are triple of the square on $AC$

Q E D

## Proposition 5

*If a straight line be cut in extreme and mean ratio and there be added to it a straight line equal to the greater segment the whole straight line has been cut in extreme and mean ratio and the original straight line is the greater segment*

For let the straight line $AB$ be cut in extreme and mean ratio at the point $C$, let $AC$ be the greater segment and let $AD$ be equal to $AC$




I say that the straight line $DB$ has been cut in extreme and mean ratio at $A$ and the original straight line $AB$ is the greater segment

For let the square $AE$ be described on $AB$ and let the figure be drawn

Since $AB$ has been cut in extreme and mean ratio at C

therefore the rectangle $AB$ $BC$ is equal to the square on $AC$ [VI Def 3 VI 17]

And $CE$ is the rectangle $AB$ $BC$ and $CH$ the square on $AC$,

therefore $CE$ is equal to $HC$

But $HE$ is equal to $CE$

and $DH$ is equal to $HC$,
therefore $DH$ is also equal to $HE$
Therefore the whole $DK$ is equal to the whole $AE$
And $DK$ is the rectangle $BD$, $DA$,
for $AD$ is equal to $DL$,
and $AE$ is the square on $AB$,
therefore the rectangle $BD$, $DA$ is equal to the square on $AB$
Therefore as $DB$ is to $BA$, so is $BA$ to $AD$ [VI 17]
And $DB$ is greater than $BA$,
therefore $BA$ is also greater than $AD$ [V 14]
Therefore $DB$ has been cut in extreme and mean ratio at $A$ and $AB$ is the greater segment Q E D

PROPOSITION 6

*If a rational straight line be cut in extreme and mean ratio, each of the segments is the irrational straight line called apotome*

Let $AB$ be a rational straight line
let it be cut in extreme and mean ratio at $C$,
and let $AC$ be the greater segment
I say that each of the straight lines $AC$, $CB$ is the irrational straight line called apotome

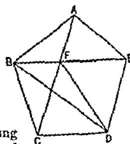
D A C B

For let $BA$ be produced and let $AD$ be made half of $BA$
Since then the straight line $AB$ has been cut in extreme and mean ratio and to the greater segment $AC$ is added $AD$ which is half of $AB$
therefore the square on $CD$ is five times the square on $DA$ [XIII 1]
Therefore the square on $CD$ has to the square on $DA$ the ratio which a number has to a number
therefore the square on $CD$ is commensurable with the square on $DA$ [X 6]
But the square on $DA$ is rational
for $DA$ is rational, being half of $AB$ which is rational
therefore the square on $CD$ is also rational, [X Def 4]
therefore $CD$ is also rational
And, since the square on $CD$ has not to the square on $DA$ the ratio which a square number has to a square number,
therefore $CD$ is incommensurable in length with $DA$, [X 9]
therefore $CD$ $DA$ are rational straight lines commensurable in square only
therefore $AC$ is an apotome [X 73]
Again since $AB$ has been cut in extreme and mean ratio
and $AC$ is the greater segment,
therefore the rectangle $AB$ $BC$ is equal to the square on $AC$ [VI Def 3 VI 17]
Therefore the square on the apotome $AC$ if applied to the rational straight line $AB$ produces $BC$ as breadth
But the square on an apotome if applied to a rational straight line produces as breadth a first apotome [X 97]
therefore $CB$ is a first apotome
And $CA$ was also proved to be an apotome
Therefore etc Q E D

PROPOSITION 7

*If three angles of an equilateral pentagon, taken either in order or not in order be equal, the pentagon will be equiangular*

For in the equilateral pentagon $ABCDE$ let, first three angles taken in order those at $A$, $B$ $C$ be equal to one another,




I say that the pentagon $ABCDE$ is equiangular

For let $AC$ $BE$ $FD$ be joined

Now since the two sides $CB$ $BA$ are equal to the two sides $BA$ $AE$ respectively,

and the angle $CBA$ is equal to the angle $BAE$

therefore the base $AC$ is equal to the base $BE$,

the triangle $ABC$ is equal to the triangle $ABE$

and the remaining angles will be equal to the remaining angles namely those which the equal sides subtend [I 4]

that is the angle $BCA$ to the angle $BEA$ and the angle $ABE$ to the angle $CAB$,

hence the side $AF$ is also equal to the side $BF$ [I 6]

But the whole $AC$ was also proved equal to the whole $BE$

therefore the remainder $FC$ is also equal to the remainder $FE$

But $CD$ is also equal to $DE$

Therefore the two sides $FC$ $CD$ are equal to the two sides $FE$ $ED$,

and the base $FD$ is common to them

therefore the angle $FCD$ is equal to the angle $FED$ [I 8]

But the angle $BCA$ was also proved equal to the angle $AEB$,

therefore the whole angle $BCD$ is also equal to the whole angle $AED$

But by hypothesis the angle $BCD$ is equal to the angles at $A$, $B$

therefore the angle $AED$ is also equal to the angles at $A$ $B$

Similarly we can prove that the angle $CDE$ is also equal to the angles at $A$, $B$ $C$

therefore the pentagon $ABCDE$ is equiangular

Next let the given equal angles not be angles taken in order, but let the angles at the points $A$ $C$ $D$ be equal

I say that in this case too the pentagon $ABCDE$ is equiangular

For let $BD$ be joined

Then since the two sides $BA$ $AE$ are equal to the two sides $BC$ $CD$

and they contain equal angles

therefore the base $BE$ is equal to the base $BD$

the triangle $ABE$ is equal to the triangle $BCD$

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend [I 4]

therefore the angle $AEB$ is equal to the angle $CDB$

But the angle $BED$ is also equal to the angle $BDE$

since the side $BE$ is also equal to the side $BD$ [I 5]

Therefore the whole angle $AED$ is equal to the whole angle $CDE$

But the angle $CDE$ is by hypothesis equal to the angles at $A$, $C$,

therefore the angle $AED$ is also equal to the angles at $A$ $C$

For the same reason

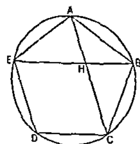
the angle $ABC$ is also equal to the angles at $A$ $C$ $D$

Therefore the pentagon $ABCDE$ is equiangular Q E D

PROPOSITION 8

*If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio and their greater segments are equal to the side of the pentagon*

For in the equilateral and equiangular pentagon $ABCDE$ let the straight lines $AC$, $BE$, cutting one another at the point $H$ subtend two angles taken in order, the angles at $A$ $B$




I say that each of them has been cut in extreme and mean ratio at the point $H$ and their greater segments are equal to the side of the pentagon

For let the circle $ABCDE$ be circumscribed about the pentagon $ABCDE$ [IV 14]

Then since the two straight lines $EA$, $AB$ are equal to the two $AB$ $BC$,
and they contain equal angles
therefore the base $BE$ is equal to the base $AC$
the triangle $ABE$ is equal to the triangle $ABC$
and the remaining angles will be equal to the remaining angles respectively namely those which the equal sides subtend [I 4]

Therefore the angle $BAC$ is equal to the angle $ABE$,
therefore the angle $AHE$ is double of the angle $BAH$ [I 32]

But the angle $EAC$ is also double of the angle $BAC$
inasmuch as the circumference $EDC$ is also double of the circumference $CB$ [III 28 VI 33]

therefore the angle $HAE$ is equal to the angle $AHE$
hence the straight line $HE$ is also equal to $EA$ that is to $AB$ [I 6]

And since the straight line $BA$ is equal to $AE$
the angle $ABE$ is also equal to the angle $AEB$ [I 5]

But the angle $ABE$ was proved equal to the angle $BAH$
therefore the angle $BEA$ is also equal to the angle $BAH$

And the angle $ABE$ is common to the two triangles $ABE$ and $ABH$,
therefore the remaining angle $BAE$ is equal to the remaining angle $AHB$ [I 32]
therefore the triangle $ABE$ is equiangular with the triangle $ABH$
therefore proportionally as $EB$ is to $BA$ so is $AB$ to $BH$ [VI 4]

But $BA$ is equal to $EH$
therefore as $BE$ is to $EH$ so is $EH$ to $HB$

And $BE$ is greater than $EH$
therefore $EH$ is also greater than $HB$ [V 14]

Therefore $BE$ has been cut in extreme and mean ratio at $H$ and the greater segment $HE$ is equal to the side of the pentagon

Similarly we can prove that $AC$ has also been cut in extreme and mean ratio at $H$ and its greater segment $CH$ is equal to the side of the pentagon Q E D

PROPOSITION 9

*If the side of the hexagon and that of the decagon inscribed in the same circle be added together the whole straight line has been cut in extreme and mean ratio and its greater segment is the side of the hexagon*

Let $ABC$ be a circle

of the figures inscribed in the circle $ABC$ let $BC$ be the side of a decagon $CD$ that of a hexagon

and let them be in a straight line

I say that the whole straight line $BD$ has been cut in extreme and mean ratio and $CD$ is its greater segment

For let the centre of the circle, the point $E$, be taken

let $EB$, $EC$ $ED$ be joined

and let $BE$ be carried through to $A$

Since $BC$ is the side of an equilateral decagon therefore the circumference $ACB$ is five times the circumference $BC$,

therefore the circumference $AC$ is quadruple of $CB$

But as the circumference $AC$ is to $CB$ so is the angle $AEC$ to the angle $CEB$ [VI 33]

therefore the angle $AEC$ is quadruple of the angle $CEB$

And since the angle $EBC$ is equal to the angle $ECB$ [I 5]

therefore the angle $AEC$ is double of the angle $ECB$ [I 32]

And since the straight line $EC$ is equal to $CD$

for each of them is equal to the side of the hexagon inscribed in the circle $ABC$ [IV 15 Por]

the angle $CED$ is also equal to the angle $CDE$ [I 5]

therefore the angle $ECB$ is double of the angle $EDC$ [I 32]

But the angle $AEC$ was proved double of the angle $ECB$,

therefore the angle $AEC$ is quadruple of the angle $EDC$

But the angle $AEC$ was also proved quadruple of the angle $BEC$

therefore the angle $EDC$ is equal to the angle $BEC$

But the angle $EBD$ is common to the two triangles $BEC$ and $BED$

therefore the remaining angle $BED$ is also equal to the remaining angle $ECB$, [I 32]

therefore the triangle $EBD$ is equiangular with the triangle $EBC$ [VI 4]

Therefore proportionally as $DB$ is to $BE$ so is $EB$ to $BC$

But $EB$ is equal to $CD$

Therefore as $BD$ is to $DC$ so is $DC$ to $CB$

And $BD$ is greater than $DC$

therefore $DC$ is also greater than $CB$

Therefore the straight line $BD$ has been cut in extreme and mean ratio and $DC$ is its greater segment Q E D

## Proposition 10

*If an equilateral pentagon be inscribed in a circle the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle*

Let $ABCDE$ be a circle

and let the equilateral pentagon $ABCDE$ be inscribed in the circle $ABCDE$

I say that the square on the side of the pentagon $ABCDE$ is equal to the

squares on the side of the hexagon and on that of the decagon inscribed in the circle $ABCDE$

For let the centre of the circle, the point $F$, be taken

let $AF$ be joined and carried through to the point $G$,

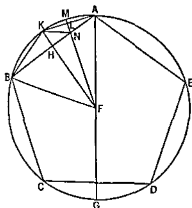
let $FB$ be joined

let $FH$ be drawn from $F$ perpendicular to $AB$ and be carried through to $K$,

let $AK$, $KB$ be joined

let $FL$ be again drawn from $F$ perpendicular to $AK$ and be carried through to $M$,

and let $KN$ be joined




Since the circumference $ABCG$ is equal to the circumference $AEDG$

and in them $ABC$ is equal to $AED$

therefore the remainder the circumference $CG$ is equal to the remainder $GD$

But $CD$ belongs to a pentagon,

therefore $CG$ belongs to a decagon

And since $FA$ is equal to $FB$

and $FH$ is perpendicular,

therefore the angle $AFK$ is also equal to the angle $KFB$ [I 5, I 26]

Hence the circumference $AK$ is also equal to $KB$ [III 26]

therefore the circumference $AB$ is double of the circumference $BK$,

therefore the straight line $AK$ is a side of a decagon

For the same reason

$AK$ is also double of $KM$

Now since the circumference $AB$ is double of the circumference $BK$,

while the circumference $CD$ is equal to the circumference $AB$

therefore the circumference $CD$ is also double of the circumference $BK$

But the circumference $CD$ is also double of $CG$

therefore the circumference $CG$ is equal to the circumference $BK$

But $BK$ is double of $KM$ since $KA$ is so also

therefore $CG$ is also double of $KM$

But, further the circumference $CB$ is also double of the circumference $BK$,

for the circumference $CB$ is equal to $BA$

Therefore the whole circumference $GB$ is also double of $BM$

hence the angle $GFB$ is also double of the angle $BFM$ [VI 33]

But the angle $GFB$ is also double of the angle $FAB$

for the angle $FAB$ is equal to the angle $ABF$

Therefore the angle $BFN$ is also equal to the angle $FAB$

But the angle $ABF$ is common to the two triangles $ABF$ and $BFN$,

therefore the remaining angle $AFB$ is equal to the remaining angle $BNF$ [I 32]

therefore the triangle $ABF$ is equiangular with the triangle $BFN$

Therefore proportionally as the straight line $AB$ is to $BF$ so is $FB$ to $BN$ [VI 4]

therefore the rectangle $AB$ $BN$ is equal to the square on $BF$ [VI 17]

Again since $AL$ is equal to $LK$,

while $LK$ is common and at right angles

therefore the base $KN$ is equal to the base $AN$ [I 4]

therefore the angle $LKN$ is also equal to the angle $LAN$

But the angle $LAN$ is equal to the angle $KBN$,

therefore the angle $LKN$ is also equal to the angle $KBN$

And the angle at $A$ is common to the two triangles $AKB$ and $AKN$

Therefore the remaining angle $AKB$ is equal to the remaining angle $KNA$ [I 32]

therefore the triangle $KBA$ is equiangular with the triangle $KNA$

Therefore proportionally as the straight line $BA$ is to $AK$, so is $KA$ to $AN$ [VI 4]

therefore the rectangle $BA$, $AN$ is equal to the square on $AK$ [VI 17]

But the rectangle $AB$, $BN$ was also proved equal to the square on $BF$

therefore the rectangle $AB$, $BN$ together with the rectangle $BA$, $AN$, that is the square on $BA$ [II 2] is equal to the square on $BF$ together with the square on $AK$

And $BA$ is a side of the pentagon, $BF$ of the hexagon [IV 15 Por] and $AK$ of the decagon

Therefore etc Q E D

PROPOSITION 11

*If in a circle which has its diameter rational an equilateral pentagon be inscribed the side of the pentagon is the irrational straight line called minor*

For in the circle $ABCDE$ which has its diameter rational let the equilateral pentagon $ABCDE$ be inscribed

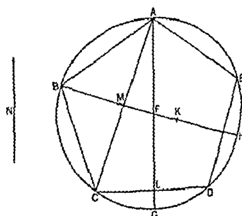
I say that the side of the pentagon is the irrational straight line called minor

For let the centre of the circle the point $F$ be taken

let $AF$, $FB$ be joined and carried through to the points $G$, $H$

let $AC$ be joined

and let $FK$ be made a fourth part of $AF$




Now $AF$ is rational

therefore $FK$ is also rational

But $BF$ is also rational

therefore the whole $BK$ is rational

And since the circumference $ACG$ is equal to the circumference $ADG$

and in them $ABC$ is equal to $AED$

therefore the remainder $CG$ is equal to the remainder $GD$

And if we join $AD$ we conclude that the angles at $L$ are right and $CD$ is double of $CL$

For the same reason

the angles at $M$ are also right and $AC$ is double of $CM$

Since then the angle $ALC$ is equal to the angle $AMF$,

and the angle $LAC$ is common to the two triangles $ACL$ and $AMF$
therefore the remaining angle $ACL$ is equal to the remaining angle $MFA$, [I 32]

therefore the triangle $ACL$ is equiangular with the triangle $AMF$,

therefore, proportionally as $LC$ is to $CA$, so is $MF$ to $FA$

And the doubles of the antecedents may be taken,
therefore as the double of $LC$ is to $CA$, so is the double of $MF$ to $FA$

But as the double of $MF$ is to $FA$, so is $MF$ to the half of $FA$,
therefore also as the double of $LC$ is to $CA$, so is $MF$ to the half of $FA$

And the halves of the consequents may be taken,
therefore as the double of $LC$ is to the half of $CA$, so is $MF$ to the fourth of $FA$

And $DC$ is double of $LC$ $CM$ is half of $CA$, and $FK$ a fourth part of $FA$,

therefore as $DC$ is to $CM$ so is $MF$ to $FK$

*Componendo* also as the sum of $DC$ $CM$ is to $CM$, so is $MK$ to $KF$, [V 18]
therefore also as the square on the sum of $DC$, $CM$ is to the square on $CM$, so is the square on $MK$ to the square on $KF$

And since when the straight line subtending two sides of the pentagon as $AC$ is cut in extreme and mean ratio the greater segment is equal to the side of the pentagon that is to $DC$ [XIII 8]
while the square on the greater segment added to the half of the whole is five times the square on the half of the whole [XIII 1]

and $CM$ is half of the whole $AC$,

therefore the square on $DC$ $CM$ taken as one straight line is five times the square on $CM$

But it was proved that as the square on $DC$ $CM$ taken as one straight line is to the square on $CM$ so is the square on $MK$ to the square on $KF$

therefore the square on $MK$ is five times the square on $KF$

But the square on $KF$ is rational

for the diameter is rational

therefore the square on $MK$ is also rational

therefore $MK$ is rational

And since $BF$ is quadruple of $FK$

therefore $BK$ is five times $KF$

therefore the square on $BK$ is twenty five times the square on $KF$

But the square on $MK$ is five times the square on $KF$

therefore the square on $BK$ is five times the square on $KM$
therefore the square on $BK$ has not to the square on $KM$ the ratio which a square number has to a square number

therefore $BK$ is incommensurable in length with $KM$ [X 9]

And each of them is rational

Therefore $BK$ $KM$ are rational straight lines commensurable in square only

But if from a rational straight line there be subtracted a rational straight line which is commensurable with the whole in square only the remainder is irrational namely an apotome

therefore $MB$ is an apotome and $MK$ the annex to it [X 73]

I say next that $MB$ is also a fourth apotome

Let the square on $N$ be equal to that by which the square on $BK$ is greater than the square on $KM$
therefore the square on $BK$ is greater than the square on $KM$ by the square on $N$

And, since $KF$ is commensurable with $FB$,

*componendo* also $KB$ is commensurable with $FB$ [x 15]

But $BF$ is commensurable with $BH$,

therefore $BK$ is also commensurable with $BH$ [x 12]

And, since the square on $BK$ is five times the square on $KM$,

therefore the square on $BK$ has to the square on $KM$ the ratio which 5 has to 1

Therefore, *convertendo* the square on $BK$ has to the square on $N$ the ratio which 5 has to 4 [v 19 Por ], and this is not the ratio which a square number has to a square number

therefore $BK$ is incommensurable with $N$, [x 9]

therefore the square on $BK$ is greater than the square on $KM$ by the square on a straight line incommensurable with $BK$

Since then the square on the whole $BK$ is greater than the square on the annex $KM$ by the square on a straight line incommensurable with $BK$,

and the whole $BK$ is commensurable with the rational straight line, $BH$, set out

therefore $MB$ is a fourth apotome [x Deff III 4]

But the rectangle contained by a rational straight line and a fourth apotome is irrational

and its square root is irrational and is called minor [x 94]

But the square on $AB$ is equal to the rectangle $HB$ $BM$,

because when $AH$ is joined the triangle $ABH$ is equiangular with the triangle $ABM$ and as $HB$ is to $BA$ so is $AB$ to $BM$

Therefore the side $AB$ of the pentagon is the irrational straight line called minor

Q E D

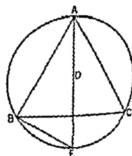
## Proposition 12

*If an equilateral triangle be inscribed in a circle the square on the side of the triangle is triple of the square on the radius of the circle*

Let $ABC$ be a circle

and let the equilateral triangle $ABC$ be inscribed in it

I say that the square on one side of the triangle $ABC$ is triple of the square on the radius of the circle




For let the centre $D$ of the circle $ABC$ be taken

let $AD$ be joined and carried through to $E$

and let $BE$ be joined

Then since the triangle $ABC$ is equilateral

therefore the circumference $BEC$ is a third part of the circumference of the circle $ABC$

Therefore the circumference $BE$ is a sixth part of the circumference of the circle

therefore the straight line $BE$ belongs to a hexagon

therefore it is equal to the radius $DE$ [IV 15 Por ]

And since $AE$ is double of $DE$

the square on $AE$ is quadruple of the square on $ED$ that is of the square on $BE$

But the square on $AE$ is equal to the squares on $AB$ $BE$ [III 31 I 47]

therefore the squares on $AB$ $BE$ are quadruple of the square on $BE$

Therefore *separando* the square on $AB$ is triple of the square on $BE$

But $BE$ is equal to $DE$,

therefore the square on $AB$ is triple of the square on $DE$

Therefore the square on the side of the triangle is triple of the square on the radius Q E D

PROPOSITION 13

*To construct a pyramid, to comprehend it in a given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid*

Let the diameter $AB$ of the given sphere be set out,

and let it be cut at the point $C$ so that $AC$ is double of $CB$,

let the semicircle $ADB$ be described on $AB$,

let $CD$ be drawn from the point $C$ at right angles to $AB$,

and let $DA$ be joined,

let the circle $EFG$ which has its radius equal to $DC$ be set out,

let the equilateral triangle $EFG$ be inscribed in the circle $EFG$, [IV 2]

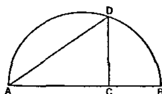
let the centre of the circle the point $H$, be taken [III 1]

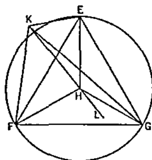
let $EH$ $HF$, $HG$ be joined,

from the point $H$ let $HK$ be set up at right angles to the plane of the circle $EFG$, [XI 12]

let $HK$ equal to the straight line $AC$ be cut off from $HK$,

and let $KE$, $KF$, $KG$ be joined




Now since $KH$ is at right angles to the plane of the circle $EFG$ therefore it will also make right angles with all the straight lines which meet it and are in the plane of the circle $EFG$ [XI Def 3]

But each of the straight lines $HE$ $HF$ $HG$ meets it

therefore $HK$ is at right angles to each of the straight lines $HE$, $HF$ $HG$

And, since $AC$ is equal to $HK$, and $CD$ to $HE$

and they contain right angles

therefore the base $DA$ is equal to the base $KE$ [I 4]

For the same reason

each of the straight lines $KF$ $KG$ is also equal to $DA$

therefore the three straight lines $KE$ $KF$, $KG$ are equal to one another

And since $AC$ is double of $CB$,

therefore $AB$ is triple of $BC$

But, as $AB$ is to $BC$ so is the square on $AD$ to the square on $DC$, as will be proved afterwards

Therefore the square on $AD$ is triple of the square on $DC$

But the square on $FE$ is also triple of the square on $EH$, [XIII 12]

and $DC$ is equal to $EH$,

therefore $DA$ is also equal to $EF$

But $DA$ was proved equal to each of the straight lines $KE$, $KF$, $KG$

therefore each of the straight lines $EF$, $FG$, $GE$ is also equal to each of the straight lines $KE$, $KF$, $KG$,

therefore the four triangles $EFG$, $KEF$, $KFG$, $KEG$ are equilateral

Therefore a pyramid has been constructed out of four equilateral triangles the triangle $EFG$ being its base and the point $K$ its vertex

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For let the straight line $HL$ be produced in a straight line with $KH$,

and let $HL$ be made equal to $CB$

Now, since as $AC$ is to $CD$, so is $CD$ to $CB$ [VI 8 Por]

while $AC$ is equal to $KH$, $CD$ to $HE$, and $CB$ to $HL$,

therefore, as $KH$ is to $HE$, so is $EH$ to $HL$

therefore the rectangle $KH$, $HL$ is equal to the square on $EH$ [VI 17]

And each of the angles $KHE$ $EHL$ is right,

therefore the semicircle described on $KL$ will pass through $E$ also [cf VI 8, III 31]

If then $KL$ remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points $F$ $G$,

since if $FL$ $LG$ be joined the angles at $F$ $G$ similarly become right angles,

and the pyramid will be comprehended in the given sphere

For $KL$ the diameter of the sphere is equal to the diameter $AB$ of the given sphere, inasmuch as $KH$ was made equal to $AC$ and $HL$ to $CB$

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For since $AC$ is double of $CB$,

therefore $AB$ is triple of $BC$,

and *convertendo* $BA$ is one and a half times $AC$

But as $BA$ is to $AC$ so is the square on $BA$ to the square on $AD$

Therefore the square on $BA$ is also one and a half times the square on $AD$

And $BA$ is the diameter of the given sphere and $AD$ is equal to the side of the pyramid

Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid Q E D

## LEMMA

It is to be proved that as $AB$ is to $BC$, so is the square on $AD$ to the square on $DC$

For let the figure of the semicircle be set out

let $DB$ be joined

let the square $EC$ be described on $AC$,

and let the parallelogram $FB$ be completed.

Since then, because the triangle $DAB$ is equiangular with the triangle $DAC$,

as $BA$ is to $AD$, so is $DA$ to $AC$; [VI. 8, VI. 4]

therefore the rectangle $BA$, $AC$ is equal to the square on $AD$. [VI. 17]

And since, as $AB$ is to $BC$, so is $EB$ to $BF$, [VI. 1]

and $EB$ is the rectangle $BA$, $AC$, for $EA$ is equal to $AC$,

and $BF$ is the rectangle $AC$, $CB$,

therefore, as $AB$ is to $BC$, so is the rectangle $BA$, $AC$ to the rectangle $AC$, $CB$.

And the rectangle $BA$, $AC$ is equal to the square on $AD$, and the rectangle $AC$, $CB$ to the square on $DC$,

for the perpendicular $DC$ is a mean proportional between the segments $AC$, $CB$ of the base, because the angle $ADB$ is right. [VI. 8, Por.]

Therefore, as $AB$ is to $BC$, so is the square on $AD$ to the square on $DC$.

Q. E. D.

## Proposition 14

*To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.*

Let the diameter $AB$ of the given sphere be set out,

and let it be bisected at $C$;

let the semicircle $ADB$ be described on $AB$,

let $CD$ be drawn from $C$ at right angles to $AB$,

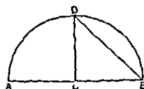
let $DB$ be joined;

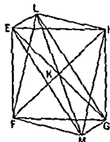
let the square $EFGH$, having each of its sides equal to $DB$, be set out,

let $HF$, $EG$ be joined,

from the point $K$ let the straight line $KL$ be set up at right angles to the plane of the square $EFGH$ [XI. 12], and let it be carried through to the other side of the plane, as $KM$;

from the straight lines $KL$, $KM$ let $KL$, $KM$ be respectively cut off equal to one of the straight lines $EK$, $FK$, $GK$, $HK$,




and let $LE$, $LF$, $LG$, $LH$, $ME$, $MF$, $MG$, $MH$ be joined.

Then, since $KE$ is equal to $KH$,

and the angle $EKH$ is right,

therefore the square on $HE$ is double of the square on $EK$. [I. 47]

Again, since $LK$ is equal to $KE$,

and the angle $LKE$ is right,

therefore the square on $EL$ is double of the square on $EK$ [id]

But the square on $HE$ was also proved double of the square on $EK$,

therefore the square on $LE$ is equal to the square on $EH$,

therefore $LE$ is equal to $EH$

For the same reason

$LH$ is also equal to $HE$,

therefore the triangle $LEH$ is equilateral

Similarly we can prove that each of the remaining triangles of which the sides of the square $FGH$ are the bases, and the points $L$, $M$ the vertices, is equilateral,

therefore an octahedron has been constructed which is contained by eight equilateral triangles

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron

For since the three straight lines $LK$ $KM$ $KE$ are equal to one another,

therefore the semicircle described on $LM$ will also pass through $E$

And for the same reason,

if $LM$ remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved

it will also pass through the points $F$ $G$, $H$,

and the octahedron will have been comprehended in a sphere

I say next that it is also comprehended in the given sphere

For since $LK$ is equal to $KM$,

while $KE$ is common

and they contain right angles

therefore the base $LE$ is equal to the base $EM$ [I 4]

And since the angle $LEM$ is right for it is in a semicircle [III 31]

therefore the square on $LM$ is double of the square on $LE$ [I 47]

Again since $AC$ is equal to $CB$,

$AB$ is double of $BC$

But as $AB$ is to $BC$ so is the square on $AB$ to the square on $BD$,

therefore the square on $AB$ is double of the square on $BD$

But the square on $LM$ was also proved double of the square on $LE$

And the square on $DB$ is equal to the square on $LE$ for $EH$ was made equal to $DB$

Therefore the square on $AB$ is also equal to the square on $LM$,

therefore $AB$ is equal to $LM$

And $AB$ is the diameter of the given sphere

therefore $LM$ is equal to the diameter of the given sphere

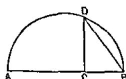
Therefore the octahedron has been comprehended in the given sphere and it has been demonstrated at the same time that the square on the diameter of the sphere is double of the square on the side of the octahedron Q E D

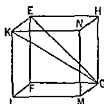
## Proposition 15

*To construct a cube and comprehend it in a sphere like the pyramid and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube*

Let the diameter $AB$ of the given sphere be set out,

and let it be cut at $C$ so that $AC$ is double of $CB$,
let the semicircle $ADB$ be described on $AB$
let $CD$ be drawn from $C$ at right angles to $AB$,
and let $DB$ be joined,
let the square $EFGH$ having its side equal to $DB$ be set out
from $E$ $F$ $G$ $H$ let $EK$ $FL$, $GM$, $HN$ be drawn at right angles to the plane of the square $EFGH$
from $EK$, $FL$, $GM$, $HN$ let $EK$, $FL$ $GM$ $HN$ respectively be cut off equal to one of the straight lines $EF$ $FG$, $GH$, $HE$,
and let $KL$, $LM$, $MN$, $NK$ be joined
therefore the cube $FN$ has been constructed which is contained by six equal squares




It is then required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube

For let $KG$ $EG$ be joined

Then since the angle $KEG$ is right, because $KE$ is also at right angles to the plane $EG$ and of course to the straight line $EG$ also, [XI Def 3]
therefore the semicircle described on $KG$ will also pass through the point $E$

Again since $GF$ is at right angles to each of the straight lines $FL$, $FE$
$GF$ is also at right angles to the plane $FK$
hence also if we join $FK$ $GF$ will be at right angles to $FK$,
and for this reason again the semicircle described on $GK$ will also pass through $F$

Similarly it will also pass through the remaining angular points of the cube

If then $KG$ remaining fixed the semicircle be carried round and restored to the same position from which it began to be moved,
the cube will be comprehended in a sphere

I say next that it is also comprehended in the given sphere

For since $GF$ is equal to $FE$,
and the angle at $F$ is right
therefore the square on $EG$ is double of the square on $EF$

But $EF$ is equal to $EK$,
therefore the square on $EG$ is double of the square on $EK$
hence the squares on $GE$ $EK$ that is the square on $GK$ [I 47] is triple of the square on $EK$

And since $AB$ is triple of $BC$
while as $AB$ is to $BC$ so is the square on $AB$ to the square on $BD$
therefore the square on $AB$ is triple of the square on $BD$

But the square on $GK$ was also proved triple of the square on $KE$

And $KE$ was made equal to $DB$
therefore $KG$ is also equal to $AB$

And $AB$ is the diameter of the given sphere
therefore $KG$ is also equal to the diameter of the given sphere

Therefore the cube has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is triple of the square on the side of the cube. Q. E. D.

### Proposition 16

*To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the side of the icosahedron is the irrational straight line called minor.*

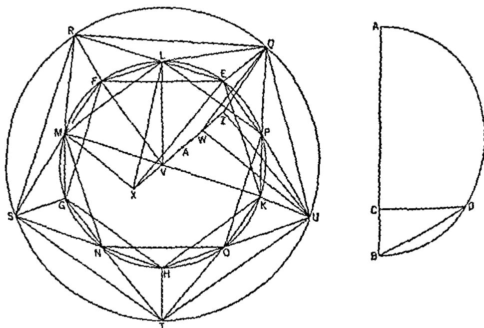
Let the diameter $AB$ of the given sphere be set out,

and let it be cut at $C$ so that $AC$ is quadruple of $CB$,

let the semicircle $ADB$ be described on $AB$,

let the straight line $CD$ be drawn from $C$ at right angles to $AB$,

and let $DB$ be joined;




let the circle $EFGHK$ be set out and let its radius be equal to $DB$,

let the equilateral and equiangular pentagon $EFGHK$ be inscribed in the circle $EFGHK$,

let the circumferences $EF$, $FG$, $GH$, $HK$, $KE$ be bisected at the points $L$, $M$, $N$, $O$, $P$,

and let $LM$, $MN$, $NO$, $OP$, $PL$, $EP$ be joined.

Therefore the pentagon $LMNOP$ is also equilateral,

and the straight line $EP$ belongs to a decagon.

Now from the points $E$, $F$, $G$, $H$, $K$ let the straight lines $EQ$, $FR$, $GS$, $HT$, $KU$ be set up at right angles to the plane of the circle, and let them be equal to the radius of the circle $EFGHK$,

let $QR$, $RS$, $ST$, $TU$, $UQ$, $QL$, $LR$, $RM$, $MS$, $SN$, $NT$, $TO$, $OU$, $UP$, $PQ$ be joined;

Now, since each of the straight lines $EQ$, $KU$ is at right angles to the same plane,

therefore $EQ$ is parallel to $KU$. [XI. 6]

But it is also equal to it,
and the straight lines joining those extremities of equal and parallel straight lines which are in the same direction are equal and parallel. [I. 33]

Therefore $QU$ is equal and parallel to $EK$.

But $EK$ belongs to an equilateral pentagon;
therefore $QU$ also belongs to the equilateral pentagon inscribed in the circle $EFGHK$.

For the same reason
each of the straight lines $QR$, $RS$, $ST$, $TU$ also belongs to the equilateral pentagon inscribed in the circle $EFGHK$;

therefore the pentagon $QRSTU$ is equilateral.

And, since $QE$ belongs to a hexagon,
and $EP$ to a decagon,
and the angle $QEP$ is right,
therefore $QP$ belongs to a pentagon;
for the square on the side of the pentagon is equal to the square on the side of the hexagon and the square on the side of the decagon inscribed in the same circle. [XIII. 10]

For the same reason
$PU$ is also a side of a pentagon.

But $QU$ also belongs to a pentagon;
therefore the triangle $QPU$ is equilateral.

For the same reason
each of the triangles $QLR$, $RMS$, $SNT$, $TOU$ is also equilateral.

And, since each of the straight lines $QL$, $QP$ was proved to belong to a pentagon,
and $LP$ also belongs to a pentagon,
therefore the triangle $QLP$ is equilateral.

For the same reason
each of the triangles $LRM$, $MSN$, $NTO$, $OUP$ is also equilateral.

Let the centre of the circle $EFGHK$, the point $V$, be taken;
from $V$ let $VZ$ be set up at right angles to the plane of the circle;
let it be produced in the other direction, as $VX$,
let there be cut off $VW$, the side of a hexagon, and each of the straight lines $VX$, $WZ$ being sides of a decagon,
and let $QZ$, $QW$, $UZ$, $EV$, $LV$, $LX$, $XM$ be joined.

Now, since each of the straight lines $VW$, $QE$ is at right angles to the plane of the circle,

therefore $VW$ is parallel to $QE$. [XI. 6]

But they are also equal;
therefore $EV$, $QW$ are also equal and parallel. [I. 33]

But $EV$ belongs to a hexagon;
therefore $QW$ also belongs to a hexagon.

And, since $QW$ belongs to a hexagon,
and $WZ$ to a decagon,
and the angle $QWZ$ is right,

therefore $QZ$ belongs to a pentagon [XIII. 10]

For the same reason

$UZ$ also belongs to a pentagon,

inasmuch as, if we join $VK$, $WU$, they will be equal and opposite, and $VK$, being a radius, belongs to a hexagon, [IV. 15 Por.]

therefore $WU$ also belongs to a hexagon.

But $WZ$ belongs to a decagon,

and the angle $UWZ$ is right;

therefore $UZ$ belongs to a pentagon. [XIII. 10]

But $QU$ also belongs to a pentagon;

therefore the triangle $QUZ$ is equilateral.

For the same reason

each of the remaining triangles of which the straight lines $QR$, $RS$, $ST$, $TU$ are the bases and the point $Z$ the vertex is also equilateral.

Again, since $VL$ belongs to a hexagon,

and $VX$ to a decagon,

and the angle $LVX$ is right,

therefore $LX$ belongs to a pentagon. [XIII. 10]

For the same reason

if we join $MV$, which belongs to a hexagon,

$MX$ is also inferred to belong to a pentagon.

But $LM$ also belongs to a pentagon;

therefore the triangle $LMX$ is equilateral.

Similarly it can be proved that each of the remaining triangles of which $MN$, $NO$, $OP$, $PL$ are the bases and the point $X$ the vertex is also equilateral.

Therefore an icosahedron has been constructed which is contained by twenty equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the side of the icosahedron is the irrational straight line called minor.

For, since $VW$ belongs to a hexagon,

and $WZ$ to a decagon,

therefore $VZ$ has been cut in extreme and mean ratio at $W$,

and $VW$ is its greater segment; [XIII. 9]

therefore, as $ZV$ is to $VW$, so is $VW$ to $WZ$.

But $VW$ is equal to $VE$, and $WZ$ to $VX$;

therefore, as $ZV$ is to $VE$, so is $EV$ to $VX$.

And the angles $ZVE$, $EVX$ are right;

therefore, if we join the straight line $EZ$, the angle $XEZ$ will be right because of the similarity of the triangles $XEZ$, $VEZ$.

For the same reason

since, as $ZV$ is to $VW$, so is $VW$ to $WZ$,

and $ZV$ is equal to $XW$, and $VW$ to $WQ$,

therefore, as $XW$ is to $WQ$, so is $QW$ to $WZ$.

And for this reason again,

if we join $QX$, the angle at $Q$ will be right; [VI. 8]

therefore the semicircle described on $XZ$ will also pass through $Q$. [III. 31]

And if, $XZ$ remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through $Q$ and the remaining angular points of the icosahedron,

and the icosahedron will have been comprehended in a sphere

I say next that it is also comprehended in the given sphere

For let $VW$ be bisected at $A'$

Then, since the straight line $VZ$ has been cut in extreme and mean ratio at $W$

and $ZW$ is its lesser segment

therefore the square on $ZW$ added to the half of the greater segment that is $WA'$, is five times the square on the half of the greater segment, [XIII 3]

therefore the square on $ZA'$ is five times the square on $A'W$

And $ZX$ is double of $ZA'$ and $VW$ double of $A'W$,

therefore the square on $ZX$ is five times the square on $WV$

And, since $AC$ is quadruple of $CB$,

therefore $AB$ is five times $BC$

But, as $AB$ is to $BC$, so is the square on $AB$ to the square on $BD$, [VI 8 V Def 9]

therefore the square on $AB$ is five times the square on $BD$

But the square on $ZX$ was also proved to be five times the square on $VW$

And $DB$ is equal to $VW$,

for each of them is equal to the radius of the circle $EFGHK$,

therefore $AB$ is also equal to $XZ$

And $AB$ is the diameter of the given sphere,

therefore $XZ$ is also equal to the diameter of the given sphere

Therefore the icosahedron has been comprehended in the given sphere

I say next that the side of the icosahedron is the irrational straight line called minor

For, since the diameter of the sphere is rational

and the square on it is five times the square on the radius of the circle $EFGHK$,

therefore the radius of the circle $EFGHK$ is also rational

hence its diameter is also rational

But if an equilateral pentagon be inscribed in a circle which has its diameter rational the side of the pentagon is the irrational straight line called minor [XIII 11]

And the side of the pentagon $EFGHK$ is the side of the icosahedron

Therefore the side of the icosahedron is the irrational straight line called minor

PORISM From this it is manifest that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described and that the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle Q E D

### PROPOSITION 17

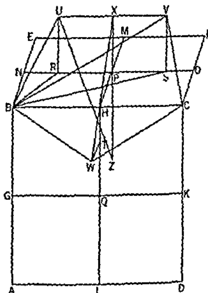
*To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures and to prove that the side of the dodecahedron is the irrational straight line called apotome*

Let $ABCD$ $CBEF$, two planes of the aforesaid cube at right angles to one another, be set out

let the sides $AB$ $BC$, $CD$ $DA$, $EF$, $EB$ $FC$ be bisected at $G$ $H$, $K$, $L$ $M$ $N$, $O$ respectively

let $GK$, $HL$, $MH$, $NO$ be joined;
let the straight lines $NP$, $PO$, $HQ$ be cut in extreme and mean ratio at the points $R$, $S$, $T$ respectively,
and let $RP$, $PS$, $TQ$ be their greater segments;
from the points $R$, $S$, $T$ let $RU$, $SV$, $TW$ be set up at right angles to the planes of the cube towards the outside of the cube,
let them be made equal to $RP$, $PS$, $TQ$,
and let $UB$, $BW$, $WC$, $CV$, $VU$ be joined.

I say that the pentagon $UBWCV$ is equilateral, and in one plane, and is further equiangular.




For let $RB$, $SB$, $VB$ be joined.

Then since the straight line $NP$ has been cut in extreme and mean ratio at $R$,
and $RP$ is the greater segment,
therefore the squares on $PN$, $NR$ are triple of the square on $RP$. [XIII. 4]

But $PN$ is equal to $NB$, and $PR$ to $RU$;
therefore the squares on $BN$, $NR$ are triple of the square on $RU$.

But the square on $BR$ is equal to the squares on $BN$, $NR$; [I. 47]
therefore the square on $BR$ is triple of the square on $RU$;
hence the squares on $BR$, $RU$ are quadruple of the square on $RU$.

But the square on $BU$ is equal to the squares on $BR$, $RU$;
therefore the square on $BU$ is quadruple of the square on $RU$;
therefore $BU$ is double of $RU$.

But $VU$ is also double of $UR$,
inasmuch as $SR$ is also double of $PR$, that is, of $RU$;
therefore $BU$ is equal to $UV$.

Similarly it can be proved that each of the straight lines $BW$, $WC$, $CV$ is also equal to each of the straight lines $BU$, $UV$.

Therefore the pentagon $BUVCW$ is equilateral.

I say next that it is also in one plane.

For let $PX$ be drawn from $P$ parallel to each of the straight lines $RU$, $SV$ and towards the outside of the cube, and let $XH$, $HW$ be joined;
I say that $XHW$ is a straight line.

For, since $HQ$ has been cut in extreme and mean ratio at $T$, and $QT$ is its greater segment,
therefore, as $HQ$ is to $QT$, so is $QT$ to $TH$.

But $HQ$ is equal to $HP$, and $QT$ to each of the straight lines $TW$, $PX$;
therefore, as $HP$ is to $PX$, so is $WT$ to $TH$.

And $HP$ is parallel to $TW$,
for each of them is at right angles to the plane $BD$; [XI. 6]
and $TH$ is parallel to $PX$,

for each of them is at right angles to the plane $BF$. [id.]

But if two triangles, as $NPH$, $HTW$, which have two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel,

the remaining straight lines will be in a straight line; [VI. 32]

therefore $NH$ is in a straight line with $HW$.

But every straight line is in one plane; [XI. 1]

therefore the pentagon $UBWCV$ is in one plane.

I say next that it is also equiangular.

For, since the straight line $NP$ has been cut in extreme and mean ratio at $R$, and $PR$ is the greater segment,

while $PR$ is equal to $PS$,

therefore $NS$ has also been cut in extreme and mean ratio at $P$,

and $NP$ is the greater segment; [XIII. 5]

therefore the squares on $NS$, $SP$ are triple of the square on $NP$. [XIII. 4]

But $NP$ is equal to $NB$, and $PS$ to $SV$;

therefore the squares on $NS$, $SV$ are triple of the square on $NB$;

hence the squares on $VS$, $SN$, $NB$ are quadruple of the square on $NB$.

But the square on $SB$ is equal to the squares on $SN$, $NB$;

therefore the squares on $BS$, $SV$, that is, the square on $BV$—for the angle $VSB$ is right—is quadruple of the square on $NB$;

therefore $VB$ is double of $BN$.

But $BC$ is also double of $BN$;

therefore $BV$ is equal to $BC$.

And, since the two sides $BU$, $UV$ are equal to the two sides $BW$, $WC$,

and the base $BV$ is equal to the base $BC$,

therefore the angle $BUV$ is equal to the angle $BWC$. [I. 8]

Similarly we can prove that the angle $UVC$ is also equal to the angle $BWC$;

therefore the three angles $BWC$, $BUV$, $UVC$ are equal to one another.

But if in an equilateral pentagon three angles are equal to one another, the pentagon will be equiangular, [XIII. 7]

therefore the pentagon $BUVCW$ is equiangular.

And it was also proved equilateral;

therefore the pentagon $BUVCW$ is equilateral and equiangular, and it is on one side $BC$ of the cube.

Therefore, if we make the same construction in the case of each of the twelve sides of the cube,

a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron.

It is then required to comprehend it in the given sphere, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

For let $XP$ be produced, and let the produced straight line be $XZ$;

therefore $PZ$ meets the diameter of the cube, and they bisect one another,

for this has been proved in the last theorem but one of the eleventh book. [XI. 38]

Let them cut at $Z$;

therefore $Z$ is the centre of the sphere which comprehends the cube,

and $ZP$ is half of the side of the cube.

Let $UZ$ be joined.

Now, since the straight line $NS$ has been cut in extreme and mean ratio at $P$,

and $NP$ is its greater segment

therefore the squares on $NS$, $SP$ are triple of the square on $NP$ [XIII 4]

But $NS$ is equal to $XZ$

inasmuch as $NP$ is also equal to $PZ$, and $XP$ to $PS$

But further, $PS$ is also equal to $XU$,

since it is also equal to $RP$,

therefore the squares on $ZX$ $XU$ are triple of the square on $NP$

But the square on $UZ$ is equal to the squares on $ZX$, $XU$,

therefore the square on $UZ$ is triple of the square on $NP$

But the square on the radius of the sphere which comprehends the cube is also triple of the square on the half of the side of the cube

for it has previously been shown how to construct a cube and comprehend it in a sphere and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube [XIII 15]

But if whole is so related to whole so is half to half also,

and $NP$ is half of the side of the cube,

therefore $UZ$ is equal to the radius of the sphere which comprehends the cube

And $Z$ is the centre of the sphere which comprehends the cube,

therefore the point $U$ is on the surface of the sphere

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere,

therefore the dodecahedron has been comprehended in the given sphere

I say next that the side of the dodecahedron is the irrational straight line called apotome

For since when $NP$ has been cut in extreme and mean ratio $RP$ is the greater segment

and when $PO$ has been cut in extreme and mean ratio $PS$ is the greater segment

therefore when the whole $NO$ is cut in extreme and mean ratio $RS$ is the greater segment

[Thus since as $NP$ is to $PR$ so is $PR$ to $RN$

the same is true of the doubles also

for parts have the same ratio as their equimultiples, [V 15]

therefore as $NO$ is to $RS$ so is $RS$ to the sum of $NR$ $SO$

But $NO$ is greater than $RS$

therefore $RS$ is also greater than the sum of $NR$ $SO$,

therefore $NO$ has been cut in extreme and mean ratio

and $RS$ is its greater segment ]

But $RS$ is equal to $UV$

therefore when $NO$ is cut in extreme and mean ratio, $UV$ is the greater segment

And since the diameter of the sphere is rational

and the square on it is triple of the square on the side of the cube

therefore $NO$ being a side of the cube is rational

[But if a rational line be cut in extreme and mean ratio each of the segments is an irrational apotome ]

Therefore $UV$ being a side of the dodecahedron is an irrational apotome [XIII 6]

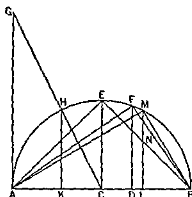
PORISM From this it is manifest that when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron

Q E D

PROPOSITION 18

*To set out the sides of the five figures and to compare them with one another*

Let $AB$, the diameter of the given sphere, be set out,




and let it be cut at $C$ so that $AC$ is equal to $CB$ and at $D$ so that $AD$ is double of $DB$, let the semicircle $AEB$ be described on $AB$, from $C$, $D$ let $CE$, $DF$ be drawn at right angles to $AB$

and let $AF$ $FB$, $EB$ be joined

Then since $AD$ is double of $DB$

therefore $AB$ is triple of $BD$

*Convertendo*, therefore $BA$ is one and a half times $AD$

But as $BA$ is to $AD$ so is the square on $BA$ to the square on $AF$, [V Def 9 VI 8] for the triangle $AFB$ is equiangular with the triangle $AFD$,

therefore the square on $BA$ is one and a half times the square on $AF$

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid [XIII 13]

And $AB$ is the diameter of the sphere,

therefore $AF$ is equal to the side of the pyramid

Again since $AD$ is double of $DB$

therefore $AB$ is triple of $BD$

But as $AB$ is to $BD$ so is the square on $AB$ to the square on $BF$ [VI 8 V Def 9]

therefore the square on $AB$ is triple of the square on $BF$

But the square on the diameter of the sphere is also triple of the square on the side of the cube [XIII 15]

And $AB$ is the diameter of the sphere,

therefore $BF$ is the side of the cube

And since $AC$ is equal to $CB$,

therefore $AB$ is double of $BC$

But as $AB$ is to $BC$ so is the square on $AB$ to the square on $BE$

therefore the square on $AB$ is double of the square on $BE$

But the square on the diameter of the sphere is also double of the square on the side of the octahedron [XIII 14]

And $AB$ is the diameter of the given sphere

therefore $BE$ is the side of the octahedron

Next let $AG$ be drawn from the point $A$ at right angles to the straight line $AB$,

let $AG$ be made equal to $AB$

let $GC$ be joined

and from $H$ let $HK$ be drawn perpendicular to $AB$

Then, since $GA$ is double of $AC$,

for $GA$ is equal to $AB$,
and, as $GA$ is to $AC$, so is $HK$ to $KC$,
therefore $HK$ is also double of $KC$

Therefore the square on $HK$ is quadruple of the square on $KC$
therefore the squares on $HK$, $KC$, that is the square on $HC$, is five times the square on $KC$

But $HC$ is equal to $CB$,
therefore the square on $BC$ is five times the square on $CK$

And, since $AB$ is double of $CB$,
and in them $AD$ is double of $DB$,
therefore the remainder $BD$ is double of the remainder $DC$

Therefore $BC$ is triple of $CD$,
therefore the square on $BC$ is nine times the square on $CD$

But the square on $BC$ is five times the square on $CK$,
therefore the square on $CK$ is greater than the square on $CD$,
therefore $CK$ is greater than $CD$

Let $CL$ be made equal to $CK$
from $L$ let $LM$ be drawn at right angles to $AB$,
and let $MB$ be joined

Now since the square on $BC$ is five times the square on $CK$,
and $AB$ is double of $BC$, and $KL$ double of $CK$,
therefore the square on $AB$ is five times the square on $KL$

But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described [XIII 16 Por]

And $AB$ is the diameter of the sphere
therefore $KL$ is the radius of the circle from which the icosahedron has been described
therefore $KL$ is a side of the hexagon in the said circle [IV 15 Por]

And since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle [XIII 16 Por]

and $AB$ is the diameter of the sphere
while $KL$ is a side of the hexagon
and $AK$ is equal to $LB$

therefore each of the straight lines $AK$ $LB$ is a side of the decagon inscribed in the circle from which the icosahedron has been described

And since $LB$ belongs to a decagon and $ML$ to a hexagon
for $ML$ is equal to $KL$ since it is also equal to $HK$ being the same distance from the centre and each of the straight lines $HK$ $KL$ is double of $KC$
therefore $MB$ belongs to a pentagon [XIII 10]

But the side of the pentagon is the side of the icosahedron, [XIII 16]
therefore $MB$ belongs to the icosahedron

Now, since $FB$ is a side of the cube
let it be cut in extreme and mean ratio at $N$,
and let $NB$ be the greater segment
therefore $NB$ is a side of the dodecahedron [XIII 17 Por]

And since the square on the diameter of the sphere was proved to be one and a half times the square on the side $AF$ of the pyramid double of the square on

the side $BE$ of the octahedron and triple of the side $FB$ of the cube therefore of parts of which the square on the diameter of the sphere contains six the square on the side of the pyramid contains four, the square on the side of the octahedron three and the square on the side of the cube two

Therefore the square on the side of the pyramid is four thirds of the square on the side of the octahedron and double of the square on the side of the cube and the square on the side of the octahedron is one and a half times the square on the side of the cube

The said sides therefore of the three figures I mean the pyramid, the octahedron and the cube, are to one another in rational ratios

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron are not in rational ratios either to one another or to the aforesaid sides

for they are irrational, the one being minor [XIII 16] and the other an apotome [XIII 17]

That the side $MB$ of the icosahedron is greater than the side $NB$ of the dodecahedron we can prove thus

For, since the triangle $FDB$ is equiangular with the triangle $FAB$ [VI 8] proportionally, as $DB$ is to $BF$, so is $BF$ to $BA$ [VI 4]

And since the three straight lines are proportional,

as the first is to the third so is the square on the first to the square on the second [V Def 9 VI 20, Por]

therefore as $DB$ is to $BA$ so is the square on $DB$ to the square on $BF$,

therefore inversely as $AB$ is to $BD$, so is the square on $FB$ to the square on $BD$

But $AB$ is triple of $BD$

therefore the square on $FB$ is triple of the square on $BD$

But the square on $AD$ is also quadruple of the square on $DB$

for $AD$ is double of $DB$,

therefore the square on $AD$ is greater than the square on $FB$,

therefore $AD$ is greater than $FB$,

therefore $AL$ is by far greater than $FB$

And when $AL$ is cut in extreme and mean ratio

$KL$ is the greater segment

inasmuch as $LK$ belongs to a hexagon and $KA$ to a decagon [XIII 9]

and when $FB$ is cut in extreme and mean ratio $NB$ is the greater segment

therefore $KL$ is greater than $NB$

But $KL$ is equal to $LM$,

therefore $LM$ is greater than $NB$

Therefore $MB$ which is a side of the icosahedron is by far greater than $NB$ which is a side of the dodecahedron Q E D

---

I say next that *no other figure besides the said five figures can be constructed which is contained by equilateral and equiangular figures equal to one another*

For a solid angle cannot be constructed with two triangles or indeed planes

With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron and with five the angle of the icosahedron

but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point,

for the angle of the equilateral triangle being two-thirds of a right angle, the six will be equal to four right angles

which is impossible, for any solid angle is contained by angles less than four right angles [XI 21]

For the same reason, neither can a solid angle be constructed by more than six plane angles

By three squares the angle of the cube is contained but by four it is impossible for a solid angle to be contained,

for they will again be four right angles

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained,

but by four such it is impossible for any solid angle to be contained

for the angle of the equilateral pentagon being a right angle and a fifth the four angles will be greater than four right angles

which is impossible

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity

Therefore etc Q E D

## LEMMA

But that *the angle of the equilateral and equiangular pentagon is a right angle and a fifth* we must prove thus

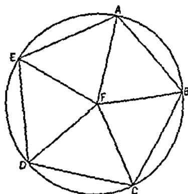
Let $ABCDF$ be an equilateral and equiangular pentagon,

let the circle $ABCDE$ be circumscribed about it,

let its centre $F$ be taken

and let $FA$, $FB$, $FC$, $FD$ $FE$ be joined

Therefore they bisect the angles of the pentagon at $A$ $B$ $C$ $D$, $E$




And since the angles at $F$ are equal to four right angles and are equal

therefore one of them as the angle $AFB$ is one right angle less a fifth,

therefore the remaining angles $FAB$ $ABF$ consist of one right angle and a fifth

But the angle $FAB$ is equal to the angle $FBC$,

therefore the whole angle $ABC$ of the pentagon consists of one right angle and a fifth Q E D

THE WORKS OF ARCHIMEDES
INCLUDING THE METHOD

# BIOGRAPHICAL NOTE

## ARCHIMEDES, c 287–212 B C

ARCHIMEDES was a citizen of Syracuse in Sicily, where he was born around the year 287 B C He was intimate with Hiero King of Syracuse and with his son, Gelo and Plutarch says that he was related to them In his *Sand Reckoner*, which was dedicated to Gelo Archimedes speaks of his father Pheidias as an astronomer who investigated the sizes and distances of the sun and moon

As a young man Archimedes seems to have spent some time in Egypt where he invented the water-screw as a means of drawing water out of the Nile for irrigating the fields though it is also said that he invented this machine to drain bilge water from a huge ship built for King Hiero He may have studied with the pupils of Euclid in Alexandria It was probably there that he made the friendship of Conon of Samos and Eratosthenes To Conon he was in the habit of communicating his discoveries before their publication and it was for Eratosthenes that he wrote the *Method* and through him that he addressed the famous *Cattle Problem* to the mathematicians of Alexandria—if the tradition is to be credited that associates Archimedes with this problem After the death of Conon Archimedes sent his discoveries to Conon s friend and pupil, Dositheus of Pelusium to whom four of the extant treatises are dedicated

His mechanical inventions won great fame for Archimedes and figure largely in the traditions about him After discovering the solution of the problem *To move a given weight by a given force* he boasted to King Hiero Give me a place to stand on and I can move the earth ' Asked for a practical demonstration he contrived a machine by which with the use of only one arm he drew out of the dock a large ship laden with passengers and goods which the combined strength of the Syracusans could scarcely move From that day Hiero ordered that Archimedes was to be believed in everything he might say At the king s request Archimedes then made for him catapults, battering rams cranes and many other engines of war which were later used with such success in the defense of Syracuse against the Romans that they were unable to take the city except by treachery There is also a story in Lucian that Archimedes set fire to the Roman ships by an arrangement of burning glasses

Although Archimedes acquired by his mechanical inventions ' the renown of more than human sagacity according to Plutarch he would not deign to leave behind him any commentary or writing on such subjects since he considered them "sordid and ignoble He did however write a description now lost of an apparatus composed of concentric glass spheres moved by water power representing the Eudoxian system of the world This astronomical machine which survived to be seen and described by Cicero in his *Republic* was sufficiently accurate to show the eclipses of the sun and the moon Except for this lost work *On Sphere making* Archimedes wrote only on strictly mathematical subjects He took all the mathematical sciences for his province arithmetic geometry astronomy mechanics and hydrostatics Unlike Euclid and Apol-

lonius he wrote no textbooks Of his writings, although some have been lost the most important have survived

The absorption of Archimedes in his mathematical investigations was so great that he forgot his food and neglected his person, and when carried by force to the bath Plutarch records "he used to trace geometrical figures in the ashes of the fire and diagrams in the oil on his body Asked by Hiero to discover whether a goldsmith had alloyed with silver the gold of his crown Archimedes found the answer while bathing by considering the water displaced by his body whereupon he is reported to have run home in his excitement without his clothes, shouting, Eureka (I have found it)

Archimedes preoccupation with mathematics is even said to have been the cause of his death In the general massacre which followed the capture of Syracuse by Marcellus in 212 B C, Archimedes was so intent upon a mathematical diagram that he took no notice, and when ordered by a soldier to attend the victorious general he refused until he should have solved his problem whereupon he was slain by the enraged soldier No blame attaches to the Roman general, Marcellus since he had given orders to spare the house and person of the mathematician and in the midst of his triumph he lamented the death of Archimedes provided him with an honorable burial and befriended his surviving relatives In accordance with the expressed desire of Archimedes, his family and friends inscribed on his tomb the figure of his favorite theorem on the sphere and the circumscribed cylinder and the ratio of the containing solid to the contained When Cicero was in Sicily as quaestor in 75 B C, he discovered the neglected and forgotten tomb of Archimedes near the Agrigentine Gate and piously restored it

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# ON THE SPHERE AND CYLINDER

## BOOK ONE

ARCHIMEDES to DOSITHEUS greeting

"On a former occasion I sent you the investigations which I had up to that time completed including the proofs showing that any segment bounded by a straight line and a section of a right angled cone [a parabola] is four thirds of the triangle which has the same base with the segment and equal height Since then certain theorems not hitherto demonstrated have occurred to me, and I have worked out the proofs of them They are these first that the surface of any sphere is four times its greatest circle, next that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment, and, further that any cylinder having its base equal to the greatest circle of those in the sphere, and height equal to the diameter of the sphere is itself [*i e* in content] half as large again as the sphere and its surface also [including its bases] is half as large again as the surface of the sphere Now these properties were all along naturally inherent in the figures referred to but remained unknown to those who were before my time engaged in the study of geometry Having however, now discovered that the properties are true of these figures I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height For though these properties also were naturally inherent in the figures all along yet they were in fact unknown to all the many able geometers who lived before Eudoxus and had not been observed by any one Now however it will be open to those who possess the requisite ability to examine these discoveries of mine They ought to have been published while Conon was still alive for I should conceive that he would best have been able to grasp them and to pronounce upon them the appropriate verdict but as I judge it well to communicate them to those who are conversant with mathematics I send them to you with the proofs written out which it will be open to mathematicians to examine Farewell

"I first set out the axioms and the assumptions which I have used for the proofs of my propositions

## DEFINITIONS

1 "There are in a plane certain terminated bent lines which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side

2 "I apply the term *concave in the same direction* to a line such that, if any two points on it are taken either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself but none on the other side"

3 'Similarly *also there are certain terminated surfaces not themselves being in a plane but having their extremities in a plane and such that they will* either be wholly on the same side of the plane containing their extremities or have no part of them on the other side

4 "I apply the term *concave in the same direction* to surfaces such that, if any two points on them are taken the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it but none on the other side

5 'I use the term *solid sector* when a cone cuts a sphere, and has its apex at the centre of the sphere to denote the figure comprehended by the surface of the cone and the surface of the sphere included within the cone'

6 I apply the term *solid rhombus* when two cones with the same base have their apices on opposite sides of the plane of the base in such a position that their axes lie in a straight line to denote the solid figure made up of both the cones'

## ASSUMPTIONS

1 *Of all lines which have the same extremities the straight line is the least*'

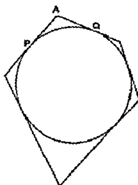
2 'Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of them is either wholly included between the other and the straight line which has the same extremities with it or is partly included by, and is partly common with, the other and that [line] which is included is the lesser [of the two]

3 *Similarly of surfaces which have the same extremities if those extremities are in a plane the plane is the least* [in area]

4 Of other surfaces with the same extremities the extremities being in a plane [any two] such are unequal whenever both are concave in the same direction and one surface is either wholly included between the other and the plane which has the same extremities with it or is partly included by and partly common with the other and that [surface] which is included is the lesser [of the two in area]

5 Further of unequal lines unequal surfaces and unequal solids the greater exceed the less by such a magnitude as when added to itself can be made to exceed any assigned magnitude among those which are comparable with [it and with one another

'These things being premised *if a polygon be inscribed in a circle it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle for* each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it




PROPOSITION 1

*If a polygon be circumscribed about a circle the perimeter of the circumscribed polygon is greater than the perimeter of the circle*

Let any two adjacent sides meeting in $A$ touch the circle at $P$, $Q$ respectively

Then [*Assumptions* 2]

$$PA+AQ>(\text{arc } PQ)$$

A similar inequality holds for each angle of the polygon and by addition, the required result follows

PROPOSITION 2

*Given two unequal magnitudes it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less*

Let $AB$, $D$ represent the two unequal magnitudes $AB$ being the greater

Suppose $BC$ measured along $BA$ equal to $D$ and let $GH$ be any straight line

Then if $CA$ be added to itself a sufficient number of times the sum will exceed $D$ Let $AF$ be this sum and take $E$ on $GH$ produced such that $GH$ is the same multiple of $HE$ that $AF$ is of $AC$

Thus $$EH \quad HG=AC \quad AF$$

But since $AF>D$ (or $CB$)

$$AC \quad AF<AC \quad CB$$

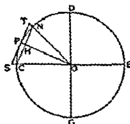
Therefore *componendo*

$$EG \quad GH<AB \quad D$$

Hence $EG$, $GH$ are two lines satisfying the given condition

PROPOSITION 3

*Given two unequal magnitudes and a circle it is possible to inscribe a polygon in the circle and to describe another about it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less*










Let $A$ $B$ represent the given magnitudes $A$ being the greater

Find [Prop 2] two straight lines $F$ $KL$ of which $F$ is the greater such that

$$F \quad KL<A \quad B \qquad (1)$$

Draw $LM$ perpendicular to $LK$ and of such length that $KM = F$

In the given circle let $CE$ $DG$ be two diameters at right angles Then, bisecting the angle $DOC$, bisecting the half again and so on we shall arrive ultimately at an angle (as $NOC$) less than twice the angle $LKM$

Join $NC$ which (by the construction) will be the side of a regular polygon inscribed in the circle Let $OP$ be the radius of the circle bisecting the angle $NOC$ (and therefore bisecting $NC$ at right angles in $H$, say), and let the tangent at $P$ meet $OC$, $ON$ produced in $S$ $T$ respectively

Now since $\angle CON < 2\angle LKM$,

$\angle HOC < \angle LKM$,

and the angles at $H$, $L$ are right

therefore $MK : LK > OC : OH$

$> OP : OH$

Hence $ST : CN < MK : LK$

$< F : LK$,

therefore *a fortiori*, by (1)

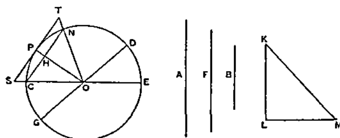
$ST : CN < A : B$

Thus two polygons are found satisfying the given condition

PROPOSITION 4

*Again, given two unequal magnitudes and a sector it is possible to describe a polygon about the sector and to inscribe another in it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than the greater magnitude has to the less*

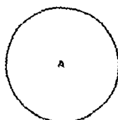
[The inscribed polygon" found in this proposition is one which has for two sides the two radii bounding the sector, while the remaining sides (the number of which is by construction some power of 2) subtend equal parts of the arc of the sector the circumscribed polygon is formed by the tangents parallel to the sides of the inscribed polygon and by the two bounding radii produced ]

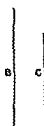


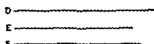

In this case we make the same construction as in the last proposition except that we bisect the angle $COD$ of the sector, instead of the right angle between two diameters then bisect the half again, and so on The proof is exactly similar to the preceding one

PROPOSITION 5

*Given a circle and two unequal magnitudes, to describe a polygon about the circle and inscribe another in it so that the circumscribed polygon may have to the inscribed a ratio less than the greater magnitude has to the less*




Let $A$ be the given circle and $B$ $C$ the given magnitudes $B$ being the greater

Take two unequal straight lines $D$ $E$ of which $D$ is the greater, such that $D$ $E < B$ $C$ [Prop 2] and let $F$ be a mean proportional between $D$ $E$ so that $D$ is also greater than $F$

Describe (in the manner of Prop 3) one polygon about the circle and inscribe another in it, so that *the side of the former has to the side of the latter a ratio less than the ratio* $D$ $F$

Thus the duplicate ratio of the side of the former polygon to the side of the latter is less than the ratio $D^2$ $F^2$

But the said duplicate ratio of the sides is equal to the ratio of the areas of the polygons since they are similar,

therefore the area of the circumscribed polygon has to the area of the inscribed polygon a ratio less than the ratio $D^2$ $F^2$ or $D$ $E$, and *a fortiori* less than the ratio $B$ $C$

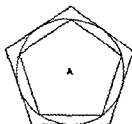
## Proposition 6

Similarly we can show that *given two unequal magnitudes and a sector it is possible to circumscribe a polygon about the sector and inscribe in it another similar one so that the circumscribed may have to the inscribed a ratio less than the greater magnitude has to the less*

And it is likewise clear that *if a circle or a sector as well as a certain area be given it is possible by inscribing regular polygons in the circle or sector and by continually inscribing such in the remaining segments to leave segments of the circle or sector which are [together] less than the given area* For this is proved in the *Elements* [Eucl XII 2]

But it is yet to be proved that *given a circle or sector and an area it is possible to describe a polygon about the circle or sector such that the area remaining between the circumference and the circumscribed figure is less than the given area*

The proof for the circle (which as Archimedes says can be equally applied to a sector) is as follows







Let $A$ be the given circle and $B$ the given area

Now there being two unequal magnitudes $A+B$ and $A$ let a polygon ($C$) be circumscribed about the circle and a polygon ($I$) inscribed in it [as in Prop 5]

so that

$$C \quad I < A+B \quad A \tag{1}$$

The circumscribed polygon ($C$) shall be that required

For the circle ($A$) is greater than the inscribed polygon ($I$)

Therefore, from (1), *a fortiori*

$$C : A < A + B : A,$$

whence $$C < A + B,$$

or $$C - A < B$$

PROPOSITION 7

*If in an isosceles cone [i e a right circular cone] a pyramid be inscribed having an equilateral base, the surface of the pyramid excluding the base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the perpendicular drawn from the apex on one side of the base*

Since the sides of the base of the pyramid are equal it follows that the perpendiculars from the apex to all the sides of the base are equal, and the proof of the proposition is obvious

PROPOSITION 8

*If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding its base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the side [i e a generator] of the cone*

The base of the pyramid is a polygon circumscribed about the circular base of the cone and the line joining the apex of the cone or pyramid to the point of contact of any side of the polygon is perpendicular to that side Also all these perpendiculars being generators of the cone, are equal whence the proposition follows immediately

PROPOSITION 9

*If in the circular base of an isosceles cone a chord be placed and from its extremities straight lines be drawn to the apex of the cone the triangle so formed will be less than the portion of the surface of the cone intercepted between the lines drawn to the apex*

Let $ABC$ be the circular base of the cone, and $O$ its apex

Draw a chord $AB$ in the circle and join $OA$, $OB$ Bisect the arc $ACB$ in $C$ and join $AC$, $BC$, $OC$

Then $$\triangle OAC + \triangle OBC > \triangle OAB$$

Let the excess of the sum of the first two triangles over the third be equal to the area $D$

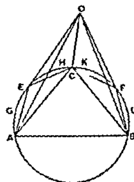
Then $D$ is either less than the sum of the segments $AEC$, $CFB$ or not less

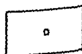
I Let $D$ be not less than the sum of the segments referred to

We have now two surfaces

(1) that consisting of the portion $OAEC$ of the surface of the cone together with the segment $AEC$ and

(2) the triangle $OAC$



and, since the two surfaces have the same extremities (the perimeter of the triangle $OAC$), the former surface is greater than the latter, which is *included* by it [*Assumptions* 3 or 4]

Hence (surface $OAEC$)+(segment $AEC$) $> \triangle OAC$

Similarly (surface $OCFB$)+(segment $CFB$) $> \triangle OBC$

Therefore, since $D$ is not less than the sum of the segments we have, by addition,

(surface $OAECFB$)$+D > \triangle OAC + \triangle OBC$

$> \triangle OAB + D$, by hypothesis

Taking away the common part $D$, we have the required result

II Let $D$ be less than the sum of the segments $AEC$ $CFB$

If now we bisect the arcs $AC$, $CB$, then bisect the halves and so on we shall ultimately leave segments which are together less than $D$ [Prop 6]

Let $AGE$, $EHC$ $CKF$, $FLB$ be those segments, and join $OE$ $OF$

Then as before,

(surface $OAGE$)+(segment $AGE$) $> \triangle OAE$

and (surface $OEHC$)+(segment $EHC$) $> \triangle OEC$

Therefore (surface $OAGHC$)+(segments $AGE$, $EHC$)

$> \triangle OAE + \triangle OEC$

$> \triangle OAC$, *a fortiori*

Similarly for the part of the surface of the cone bounded by $OC$ $OB$ and the arc $CFB$

Hence by addition

(surface $OAGEHCKFLB$)+(segments $AGE$, $EHC$, $CKF$, $FLB$)

$> \triangle OAC + \triangle OBC$

$> \triangle OAB + D$, by hypothesis

But the sum of the segments is less than $D$ and the required result follows

PROPOSITION 10

*If in the plane of the circular base of an isosceles cone two tangents be drawn to the circle meeting in a point and the points of contact and the point of concourse of the tangents be respectively joined to the apex of the cone the sum of the two triangles formed by the joining lines and the two tangents are together greater than the included portion of the surface of the cone*

Let $ABC$ be the circular base of the cone, $O$ its apex $AD$ $BD$ the two tangents to the circle meeting in $D$ Join $OA$ $OB$, $OD$

Let $ECF$ be drawn touching the circle at $C$, the middle point of the arc $ACB$, and therefore parallel to $AB$ Join $OE$ $OF$

Then $ED + DF > EF$,

and, adding $AE + FB$ to each side

$AD + DB > AE + EF + FB$

Now $OA$ $OC$, $OB$ being generators of the cone, are equal and they are respectively perpendicular to the tangents at $A$, $C$, $B$

It follows that

$\triangle OAD + \triangle ODB > \triangle OAE + \triangle OEF + \triangle OFB$

Let the area $G$ be equal to the excess of the first sum over the second

$G$ is then either less or not less than the sum of the spaces $EAHC$ $FCKB$ remaining between the circle and the tangents which sum we will call $L$

I Let $G$ be not less than $L$

We have now two surfaces

(1) that of the pyramid with apex $O$ and base $AEFB$, excluding the face $OAB$,

(2) that consisting of the part $OACB$ of the surface of the cone together with the segment $ACB$

These two surfaces have the same extremities, viz the perimeter of the triangle $OAB$, and, since the former *includes* the latter, the former is the greater [*Assumptions*, 4]

That is the surface of the pyramid exclusive of the face $OAB$ is greater than the sum of the surface $OACB$ and the segment $ACB$

Taking away the segment from each sum, we have

$$\triangle OAE + \triangle OEF + \triangle OFB + L > \text{the surface } OAHCKB$$

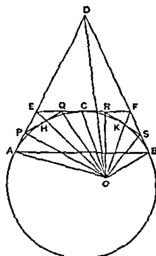
And $G$ is not less than $L$

It follows that

$$\triangle OAE + \triangle OEF + \triangle OFB + G,$$

which is by hypothesis equal to $\triangle OAD + \triangle ODB$ is greater than the same surface

II Let $G$ be less than $L$




If we bisect the arcs $AC$, $CB$ and draw tangents at their middle points, then bisect the halves and draw tangents, and so on, we shall lastly arrive at a polygon such that the sum of the parts remaining between the sides of the polygon and the circumference of the segment is less than $G$

Let the remainders be those between the segment and the polygon $APQRSB$ and let their sum be $M$ Join $OP$ $OQ$ etc

Then as before

$$\triangle OAE + \triangle OEF + \triangle OFB > \triangle OAP + \triangle OPQ + \quad + \triangle OSB$$

Also as before

(surface of pyramid $OAPQRSB$ excluding the face $OAB$) > the part $OACB$ of the surface of the cone together with the segment $ACB$

Taking away the segment from each sum

$$\triangle OAP + \triangle OPQ + \quad + M > \text{the part } OACB \text{ of the surface of the cone}$$

Hence *a fortiori*

$$\triangle OAE + \triangle OEF + \triangle OFB + G$$

which is by hypothesis equal to

$$\triangle OAD + \triangle ODB$$

is greater than the part $OACB$ of the surface of the cone

## Proposition 11

*If a plane parallel to the axis of a right cylinder cut the cylinder the part of the surface of the cylinder cut off by the plane is greater than the area of the parallel ogram in which the plane cuts it*

PROPOSITION 12

*If at the extremities of two generators of any right cylinder tangents be drawn to the circular bases in the planes of those bases respectively and if the pairs of tangents meet, the parallelograms formed by each generator and the two corresponding tangents respectively are together greater than the included portion of the surface of the cylinder between the two generators*

[The proofs of these two propositions follow exactly the methods of Props 9 10 respectively, and it is therefore unnecessary to reproduce them ]

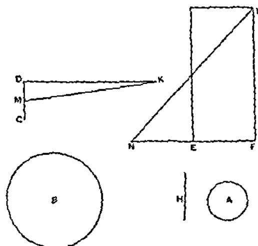
"From the properties thus proved it is clear (1) that, *if a pyramid be inscribed in an isosceles cone the surface of the pyramid excluding the base is less than the surface of the cone* [*excluding the base*], and (2) that, *if a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding the base is greater than the surface of the cone excluding the base*

It is also clear from what has been proved both (1) that *if a prism be inscribed in a right cylinder the surface of the prism made up of its parallelograms* [i e *excluding its bases*] *is less than the surface of the cylinder excluding its bases* and (2) that *if a prism be circumscribed about a right cylinder the surface of the prism made up of its parallelograms is greater than the surface of the cylinder excluding its bases* "

PROPOSITION 13

*The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side* [i e *a generator*] *of the cylinder and the diameter of its base*

Let the base of the cylinder be the circle $A$ and make $CD$ equal to the diameter of this circle and $EF$ equal to the height of the cylinder




Let $H$ be a mean proportional between $CD$ $EF$, and $B$ a circle with radius equal to $H$

Then the circle $B$ shall be equal to the surface of the cylinder (excluding the bases), which we will call $S$

For, if not $B$ must be either greater or less than $S$

I Suppose $B<S$

Then it is possible to circumscribe a regular polygon about $B$ and to inscribe another in it such that the ratio of the former to the latter is less than the ratio $S$ $B$

Suppose this done and circumscribe about $A$ a polygon similar to that described about $B$ then erect on the polygon about $A$ a prism of the same height as the cylinder The prism will therefore be circumscribed to the cylinder

Let $KD$ perpendicular to $CD$ and $FL$ perpendicular to $EF$ be each equal to the perimeter of the polygon about $A$ Bisect $CD$ in $M$, and join $MK$

Then $\triangle KDM$ = the polygon about $A$

Also $\square EL$ = surface of prism (excluding bases)

Produce $FE$ to $N$ so that $FE = EN$, and join $NL$

Now the polygons about $A$, $B$, being similar, are in the duplicate ratio of the radii of $A$, $B$

Thus

$$\triangle KDM \quad (\text{polygon about } B) = MD \quad H^2$$
$$= MD^2 \quad CD \quad EF$$
$$= MD \quad NF$$
$$= \triangle KDM \quad \triangle LFN$$

(since $DK = FL$)

Therefore (polygon about $B$) = $\triangle LFN$
= $\square EL$
= (surface of prism about $A$),

from above

But (polygon about $B$) (polygon in $B$) $< S \quad B$

Therefore

(surface of prism about $A$) (polygon in $B$) $< S \quad B$,

and alternately,

(surface of prism about $A$) $S <$ (polygon in $B$) $B$,

which is impossible since the surface of the prism is greater than $S$, while the polygon inscribed in $B$ is less than $B$

Therefore $B \not< S$

II Suppose $B > S$

Let a regular polygon be circumscribed about $B$ and another inscribed in it so that

(polygon about $B$) (polygon in $B$) $< B \quad S$

Inscribe in $A$ a polygon similar to that inscribed in $B$ and erect a prism on the polygon inscribed in $A$ of the same height as the cylinder

Again let $DK$ $FL$ drawn as before be each equal to the perimeter of the polygon inscribed in $A$

Then, in this case

$\triangle KDM >$ (polygon inscribed in $A$)

(since the perpendicular from the centre on a side of the polygon is less than the radius of $A$)

Also $\triangle LFN = \square EL$ = surface of prism (excluding bases)

Now

(polygon in $A$) (polygon in $B$) = $MD^2 \quad H^2$,
= $\triangle KDM \quad \triangle LFN$, as before

And $\triangle KDM >$ (polygon in $A$)

Therefore

$\triangle LFN$ or (surface of prism) > (polygon in $B$)

But this is impossible because

(polygon about $B$) (polygon in $B$) $< B \quad S$
< (polygon about $B$) $S$ *a fortiori*,

so that (polygon in $B$) $> S$
> (surface of prism) *a fortiori*

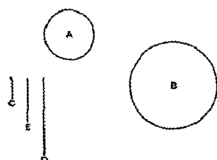
Hence $B$ is neither greater nor less than $S$, and therefore

$$B = S$$

PROPOSITION 14

*The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone [a generator] and the radius of the circle which is the base of the cone*

Let the circle $A$ be the base of the cone draw $C$ equal to the radius of the circle, and $D$ equal to the side of the cone and let $E$ be a mean proportional between $C$, $D$




Draw a circle $B$ with radius equal to $E$

Then shall $B$ be equal to the surface of the cone (excluding the base) which we will call $S$

If not, $B$ must be either greater or less than $S$

I *Suppose* $B<S$

Let a regular polygon be described about $B$ and a similar one inscribed in it such that the former has to the latter a ratio less than the ratio $S$ $B$

Describe about $A$ another similar polygon and on it set up a pyramid with apex the same as that of the cone

Then (polygon about $A$) (polygon about $B$)

$=C^2\ \ E^2$

$=C\ \ D$

= (polygon about $A$) (surface of pyramid excluding base)

*Therefore*

(surface of pyramid = (polygon about $B$)

Now (polygon about $B$) (polygon in $B$) $<S\ \ B$

*Therefore*

(surface of pyramid) (polygon in $B$) $<S\ \ B$

which is impossible (because the surface of the pyramid is greater than $S$, while the polygon in $B$ is less than $B$)

Hence $B \nless S$

II Suppose $B>S$

Take regular polygons circumscribed and inscribed to $B$ such that the ratio of the former to the latter is less than the ratio $B$ $S$

Inscribe in $A$ a similar polygon to that inscribed in $B$ and erect a pyramid on the polygon inscribed in $A$ with apex the same as that of the cone

In this case

(polygon in $A$) (polygon in $B$) $=C^2\ \ E$

$=C\ \ D$

$>$ (polygon in $A$) (surface of pyramid excluding base)

This is clear because the ratio of $C$ to $D$ is greater than the ratio of the perpendicular from the centre of $A$ on a side of the polygon to the perpendicular from the apex of the cone on the same side

*Therefore*

(surface of pyramid) $>$ (polygon in $B$)

But (polygon about $B$) (polygon in $B$) $<B\ \ S$

Therefore, *a fortiori*,

(polygon about $B$) : (surface of pyramid) $< B : S$,

which is impossible.

Since therefore $B$ is neither greater nor less than $S$,

$$B = S.$$

PROPOSITION 15

*The surface of any isosceles cone has the same ratio to its base as the side of the cone has to the radius of the base.*

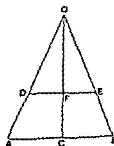
By Prop. 14, the surface of the cone is equal to a circle whose radius is a mean proportional between the side of the cone and the radius of the base.

Hence, since circles are to one another as the squares of their radii, the proposition follows.

PROPOSITION 16

*If an isosceles cone be cut by a plane parallel to the base, the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is a mean proportional between (1) the portion of the side of the cone intercepted by the parallel planes and (2) the line which is equal to the sum of the radii of the circles in the parallel planes.*

Let $OAB$ be a triangle through the axis of a cone, $DE$ its intersection with the plane cutting off the frustum, and $OFC$ the axis of the cone.




Then the surface of the cone $OAB$ is equal to a circle whose radius is equal to $\sqrt{OA \cdot AC}$. [Prop. 14.]

Similarly the surface of the cone $ODE$ is equal to a circle whose radius is equal to $\sqrt{OD \cdot DF}$.

And the surface of the frustum is equal to the difference between the two circles.

Now

$$OA \cdot AC - OD \cdot DF = DA \cdot AC + OD \cdot AC - OD \cdot DF.$$

But

$$OD \cdot AC = OA \cdot DF,$$

since

$$OA : AC = OD : DF.$$

Hence

$$OA \cdot AC - OD \cdot DF = DA \cdot AC + DA \cdot DF$$
$$= DA \cdot (AC + DF).$$

And, since circles are to one another as the squares of their radii, it follows that the difference between the circles whose radii are $\sqrt{OA \cdot AC}$, $\sqrt{OD \cdot DF}$ respectively is equal to a circle whose radius is $\sqrt{DA \cdot (AC + DF)}$.

Therefore the surface of the frustum is equal to this circle.

## LEMMAS

"1. *Cones having equal height have the same ratio as their bases; and those having equal bases have the same ratio as their heights.*[1]

2. *If a cylinder be cut by a plane parallel to the base, then, as the cylinder is to the cylinder, so is the axis to the axis.*[2]

[1] Euclid XII. 11: "Cones and cylinders of equal height are to one another as their bases." Euclid XII. 14: "Cones and cylinders on equal bases are to one another as their heights."

[2] Euclid XII. 13: "If a cylinder be cut by a plane parallel to the opposite planes [the bases], then, as the cylinder is to the cylinder, so will the axis be to the axis."

3 *The cones which have the same bases as the cylinders [and equal height] are in the same ratio as the cylinders*

4 *Also the bases of equal cones are reciprocally proportional to their heights and those cones whose bases are reciprocally proportional to their heights are equal*[1]

5 *Also the cones the diameters of whose bases have the same ratio as their axes are to one another in the triplicate ratio of the diameters of the bases*[2]

And all these propositions have been proved by earlier geometers '

### Proposition 17

*If there be two isosceles cones, and the surface of one cone be equal to the base of the other while the perpendicular from the centre of the base [of the first cone] on the side of that cone is equal to the height [of the second] the cones will be equal*

Let $OAB$, $DEF$ be triangles through the axes of two cones respectively, $C$ $G$ the centres of the respective bases $GH$ the perpendicular from $G$ on $FD$ and suppose that the base of the cone $OAB$ is equal to the surface of the cone $DEF$, and that $OC = GH$

Then since the base of $OAB$ is equal to the surface of $DEF$

(base of cone $OAB$) (base of cone $DEF$)
= (surface of $DEF$) (base of $DEF$)
$= DF$ $FG$ [Prop 15]
$= DG$ $GH$ by similar triangles,
$= DG$ $OC$

Therefore the bases of the cones are reciprocally proportional to their heights whence the cones are equal [*Lemma* 4]

### Proposition 18

*Any solid rhombus consisting of isosceles cones is equal to the cone which has its base equal to the surface of one of the cones composing the rhombus and its height equal to the perpendicular drawn from the apex of the second cone to one side of the first cone*

Let the rhombus be $OABD$ consisting of two cones with apices $O$ $D$ and with a common base (the circle about $AB$ as diameter)

Let $FHK$ be another cone with base equal to the surface of the cone $OAB$ and height $FG$ equal to $DE$ the perpendicular from $D$ on $OB$

Then shall the cone $FHK$ be equal to the rhombus

Construct a third cone $LMN$ with base (the circle about $MN$) equal to the base of $OAB$ and height $LP$ equal to $OD$

[1]Euclid xii 15 The bases of equal cones and cylinders are reciprocally proportional to their heights and those cones and cylinders whose bases are reciprocally proportional to their heights are equal.

[2]Euclid xii 12 Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases

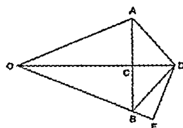
Then, since $LP = OD$,

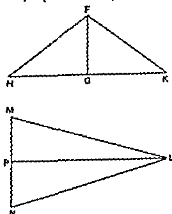
$$LP \cdot CD = OD \cdot CD$$

But [*Lemma* 1] $OD \cdot CD =$ (rhombus $OADB$) : (cone $DAB$),

and $LP \cdot CD =$ (cone $LMN$) : (cone $DAB$)

It follows that




$$\text{(rhombus } OADB) = \text{(cone } LMN) \qquad (1)$$

Again, since $AB = MN$, and

(surface of $OAB$) = (base of $FHK$),

(base of $FHK$) : (base of $LMN$) = (surface of $OAB$) : (base of $OAB$)

$= OB : BC$ [Prop 15]

$= OD : DE$, by similar triangles,

$= LP : FG$, by hypothesis

Thus, in the cones $FHK$, $LMN$ the bases are reciprocally proportional to the heights

Therefore the cones $FHK$, $LMN$ are equal

and hence, by (1) the cone $FHK$ is equal to the given solid rhombus

PROPOSITION 19

*If an isosceles cone be cut by a plane parallel to the base and on the resulting circular section a cone be described having as its apex the centre of the base [of the first cone] and if the rhombus so formed be taken away from the whole cone, the part remaining will be equal to the cone with base equal to the surface of the portion of the first cone between the parallel planes and with height equal to the perpendicular drawn from the centre of the base of the first cone on one side of that cone*

Let the cone $OAB$ be cut by a plane parallel to the base in the circle on $DE$ as diameter Let $C$ be the centre of the base of the cone and with $C$ as apex and the circle about $DE$ as base describe a cone making with the cone $ODE$ the rhombus $ODCE$

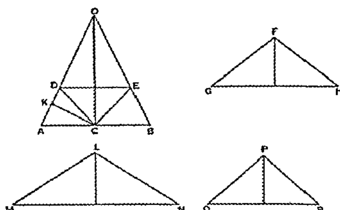
Take a cone $FGH$ with base equal to the surface of the frustum $DABE$ and height equal to the perpendicular ($CK$) from $C$ on $AO$

Then shall the cone $FGH$ be equal to the difference between the cone $OAB$ and the rhombus $ODCE$

Take (1) a cone $LMN$ with base equal to the surface of the cone $OAB$ and height equal to $CK$

(2) a cone $PQR$ with base equal to the surface of the cone $ODE$ and height equal to $CK$

Now, since the surface of the cone $OAB$ is equal to the surface of the cone $ODE$ together with that of the frustum $DABE$, we have, by the construction,




$$(\text{base of } LMN) = (\text{base of } FGH) + (\text{base of } PQR)$$

and since the heights of the three cones are equal

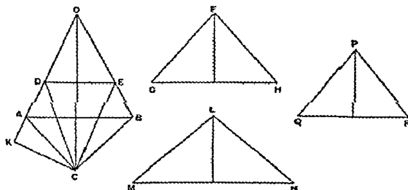
$$(\text{cone } LMN) = (\text{cone } FGH) + (\text{cone } PQR)$$

But the cone $LMN$ is equal to the cone $OAB$ [Prop 17] and the cone $PQR$ is equal to the rhombus $ODCE$ [Prop 18]

Therefore (cone $OAB$) = (cone $FGH$) + (rhombus $ODCE$) and the proposition is proved

PROPOSITION 20

*If one of the two isosceles cones forming a rhombus be cut by a plane parallel to the base and on the resulting circular section a cone be described having the same apex as the second cone and if the resulting rhombus be taken from the whole rhombus the remainder will be equal to the cone with base equal to the surface of the portion of the cone between the parallel planes and with height equal to the perpendicular drawn from the apex of the second cone to the side of the first cone*




Let the rhombus be $OACB$ and let the cone $OAB$ be cut by a plane parallel to its base in the circle about $DE$ as diameter With this circle as base and $C$

as apex describe a cone which therefore with $ODE$ forms the rhombus $ODCE$

Take a cone $FGH$ with base equal to the surface of the frustum $DABE$ and height equal to the perpendicular ($CK$) from $C$ on $OA$

The cone $FGH$ shall be equal to the difference between the rhombi $OACB$, $ODCE$

For take (1) a cone $LMN$ with base equal to the surface of $OAB$ and height equal to $CK$,

(2) a cone $PQR$, with base equal to the surface of $ODE$, and height equal to $CK$

Then, since the surface of $OAB$ is equal to the surface of $ODE$ together with that of the frustum $DABE$ we have, by construction

$$(\text{base of } LMN) = (\text{base of } PQR) + (\text{base of } FGH),$$

and the three cones are of equal height

therefore $$(\text{cone } LMN) = (\text{cone } PQR) + (\text{cone } FGH)$$

But the cone $LMN$ is equal to the rhombus $OACB$, and the cone $PQR$ is equal to the rhombus $ODCE$ [Prop 18]

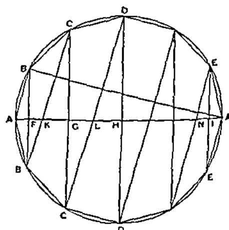
Hence the cone $FGH$ is equal to the difference between the two rhombi $OACB$ $ODCE$

## Proposition 21

*A regular polygon of an even number of sides being inscribed in a circle, as $ABC$ $A'$ $C'B$ $A$ so that $AA'$ is a diameter if two angular points next but one to each other as $B$, $B'$ be joined, and the other lines parallel to $BB'$ and joining pairs of angular points be drawn, as $CC'$, $DD'$ , then*

$(BB'+CC + \quad)\ AA' = A'B\ BA$

Let $BB$ $CC'$ $DD'$, meet $AA'$ in $F$, $G$, $H$ , and let $CB'$ $DC'$, be joined meeting $AA'$ in $K$ $L$, respectively




Then clearly $CB$ , $DC'$ are parallel to one another and to $AB$

Hence by similar triangles,

$$BF\ FA = B\ F\ FK$$
$$= CG\ GK$$
$$= C\ G\ GL$$

$$= E'I\ IA'$$

and summing the antecedents and consequents respectively we have

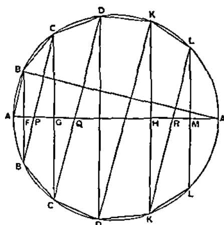
$$(BB + CC' + \quad)\ AA' = BF\ FA$$
$$= A'B\ BA$$

## Proposition 22

*If a polygon be inscribed in a segment of a circle $LAL'$ so that all its sides excluding the base are equal and their number even as $LK$ $A$ $K'L'$, $A$ being the middle point of the segment and if the lines $BB$ $CC'$, parallel to the base $LL'$ and joining pairs of angular points be drawn then*

$$(BB + CC + \quad + LM)\ AM = A'B\ BA$$

*where $M$ is the middle point of $LL$ and $AA$ is the diameter through $M$*




Joining $CB$, $DC'$, $LK'$, as in the last proposition, and supposing that they meet $AM$ in $P$, $Q$ $R$ while $BB'$, $CC'$, , $KK'$ meet $AM$ in $F$ $G$, $H$ we have, by similar triangles

$$BF : FA = B'F : FP$$
$$= CG : PG$$
$$= C'G : GQ$$

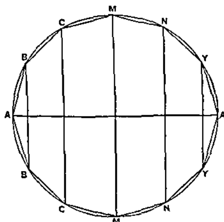
$$= LM : RM,$$

and, summing the antecedents and consequents we obtain

$$(BB' + CC' + \ldots + LM) : AM$$
$$= BF : FA$$
$$= A'B : BA$$

PROPOSITION 23

Take a great circle $ABC$ of a sphere and inscribe in it a regular polygon whose sides are a multiple of four in number Let $AA'$, $MM'$ be diameters at right angles and joining opposite angular points of the polygon




Then, if the polygon and great circle revolve together about the diameter $AA'$, the angular points of the polygon, except $A$, $A'$ will describe circles on the surface of the sphere at right angles to the diameter $AA'$ Also the sides of the polygon will describe portions of conical surfaces e g $BC$ will describe a surface forming part of a cone whose base is a circle about $CC'$ as diameter and whose apex is the point in which $CB$ $C'B'$ produced meet each other and the diameter $AA'$

Comparing the hemisphere $MAM'$ and that half of the figure described by the revolution of the polygon which is included in the hemisphere we see that the surface of the hemisphere and the surface of the inscribed figure have the same boundaries in one plane (viz the circle on $MM'$ as diameter) the former surface entirely includes the latter and they are both concave in the same direction

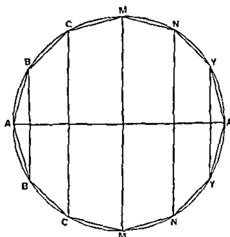
Therefore [*Assumptions* 4] the surface of the hemisphere is greater than that of the inscribed figure and the same is true of the other halves of the figures

Hence *the surface of the sphere is greater than the surface described by the revolution of the polygon inscribed in the great circle about the diameter of the great circle*

## Proposition 24

*If a regular polygon $AB \quad A' \quad B'A$, the number of whose sides is a multiple of four, be inscribed in a great circle of a sphere, and if $BB$ subtending two sides be joined, and all the other lines parallel to $BB'$ and joining pairs of angular points be drawn, then the surface of the figure inscribed in the sphere by the revolution of the polygon about the diameter $AA'$ is equal to a circle the square of whose radius is equal to the rectangle*

$$BA(BB + CC' + \quad )$$




The surface of the figure is made up of the surfaces of parts of different cones

Now the surface of the cone $ABB'$ is equal to a circle whose radius is $\sqrt{BA \ \frac{1}{2}BB}$ [Prop 14]

The surface of the frustum $BB \ C'C$ is equal to a circle of radius $\sqrt{BC \ \frac{1}{2}(BB + CC')}$, [Prop 16]

and so on

It follows since $BA = BC =$ , that the whole surface is equal to a circle whose radius is equal to

$$\sqrt{BA(BB + CC + \quad + MM + \quad + YY')}$$

## Proposition 25

*The surface of the figure inscribed in a sphere as in the last propositions, consisting of portions of conical surfaces, is less than four times the greatest circle in the sphere*

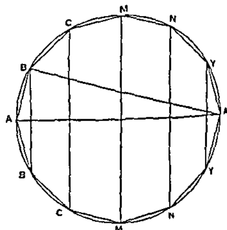
Let $AB \quad A' \quad B'A$ be a regular polygon inscribed in a great circle, the number of its sides being a multiple of four

As before let $BB'$ be drawn subtending two sides, and $CC', \quad YY'$ parallel to $BB$

Let $R$ be a circle such that the square of its radius is equal to

$$AB(BB + CC + \quad + YY'),$$

so that the surface of the figure inscribed in the sphere is equal to $R$ [Prop 24]




Now

$$(BB + CC + \quad + YY') \ AA' = A'B \ AB \quad \text{[Prop 21]}$$

whence

$$AB(BB + CC + \quad + YY') = AA' \ A'B$$

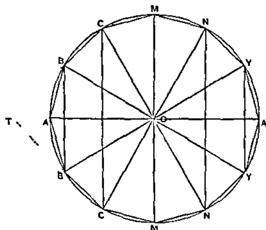
Hence (radius of $R$)$^2 = AA' \ A'B$
$< AA'^2$

Therefore the surface of the inscribed figure or the circle $R$ is less than four times the circle $AMA'M'$

PROPOSITION 26

*The figure inscribed as above in a sphere is equal [in volume] to a cone whose base is a circle equal to the surface of the figure inscribed in the sphere and whose height is equal to the perpendicular drawn from the centre of the sphere to one side of the polygon*

Suppose, as before that $AB \ldots A' \ldots B'A$ is the regular polygon inscribed in a great circle, and let $BB'$, $CC'$, ... be joined




With apex $O$ construct cones whose bases are the circles on $BB'$, $CC'$, ... as diameters in planes perpendicular to $AA'$

Then $OBAB'$ is a solid rhombus, and its volume is equal to a cone whose base is equal to the surface of the cone $ABB'$ and whose height is equal to the perpendicular from $O$ on $AB$ [Prop 18] Let the length of the perpendicular be $p$

Again, if $CB$, $C'B'$ produced meet in $T$, the portion of the solid figure which is described by the revolution of the triangle $BOC$ about $AA'$ is equal to the difference between the rhombi $OCTC'$ and $OBTB'$, i e to a cone whose base is equal to the surface of the frustum $BB'C'C$ and whose height is $p$ [Prop 20]

Proceeding in this manner, and adding, we prove that, since cones of equal height are to one another as their bases, the volume of the solid of revolution is equal to a cone with height $p$ and base equal to the sum of the surfaces of the cone $BAB'$, the frustum $BB'C'C$, etc, i e a cone with height $p$ and base equal to the surface of the solid

PROPOSITION 27

*The figure inscribed in the sphere as before is less than four times the cone whose base is equal to a great circle of the sphere and whose height is equal to the radius of the sphere*

By Prop 26 the volume of the solid figure is equal to a cone whose base is equal to the surface of the solid and whose height is $p$, the perpendicular from $O$ on any side of the polygon Let $R$ be such a cone

Take also a cone $S$ with base equal to the great circle, and height equal to the radius of the sphere

Now, since the surface of the inscribed solid is less than four times the great circle [Prop 25], the base of the cone $R$ is less than four times the base of the cone $S$

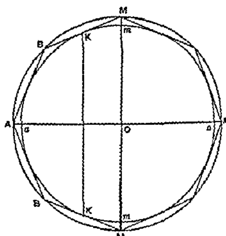
Also the height ($p$) of $R$ is less than the height of $S$

Therefore the volume of $R$ is less than four times that of $S$, and the proposition is proved

PROPOSITION 28

Let a regular polygon, whose sides are a multiple of four in number, be circumscribed about a great circle of a given sphere, as $AB \ldots A' \ldots B'A$, and about the polygon describe another circle, which will therefore have the same centre as the great circle of the sphere Let $AA'$ bisect the polygon and cut the sphere in $a$, $a'$

If the great circle and the circumscribed polygon revolve together about $AA'$ the great circle will describe the surface of a sphere the angular points of the polygon except $A$, $A'$ will move round the surface of a larger sphere, the points of contact of the sides of the polygon with the great circle of the inner sphere will describe circles on that sphere in planes perpendicular to $AA'$, and the sides of the polygon themselves will describe portions of conical surfaces *The circumscribed figure will thus be greater than the sphere itself*




Let any side as $BM$ touch the inner circle in $K$, and let $K'$ be the point of contact of the circle with $B'M'$

Then the circle described by the revolution of $KK'$ about $AA'$ is the boundary in one plane of two surfaces

(1) the surface formed by the revolution of the circular segment $KaK'$, and
(2) the surface formed by the revolution of the part $KB \ldots A \ldots B'K'$ of the polygon

Now the second surface entirely includes the first, and they are both concave in the same direction

therefore [*Assumptions* 4] the second surface is greater than the first

The same is true of the portion of the surface on the opposite side of the circle on $KK'$ as diameter

Hence adding we see that *the surface of the figure circumscribed to the given sphere is greater than that of the sphere itself*

PROPOSITION 29

*In a figure circumscribed to a sphere in the manner shown in the previous proposition the surface is equal to a circle the square on whose radius is equal to*

$$AB(BB' + CC' + \ldots)$$

For the figure circumscribed to the sphere is inscribed in a larger sphere, and the proof of Prop 24 applies

PROPOSITION 30

*The surface of a figure circumscribed as before about a sphere is greater than four times the great circle of the sphere*

Let $AB \ldots A' \ldots B'A$ be the regular polygon of $4n$ sides which by its revolu

tion about $AA'$ describes the figure circumscribing the sphere of which $ama\ m'$ is a great circle Suppose $aa'$, $AA$ to be in one straight line

Let $R$ be a circle equal to the surface of the circumscribed solid

Now

$(BB + CC + \quad) AA' = A'B\ BA$ [as in Prop 21]

so that

$AB(BB' + CC + \quad) = AA\ A B$

Hence (radius of $R$) $= \sqrt{AA\ A B}$ [Prop 29]

$> A B$

But $A B = 2OP$ where $P$ is the point in which $AB$ touches the circle $ama\ m'$

Therefore (radius of $R$) > (diameter of circle $ama\ m$)

whence $R$, and therefore the surface of the circumscribed solid is greater than four times the great circle of the given sphere

PROPOSITION 31

*The solid of revolution circumscribed as before about a sphere is equal to a cone whose base is equal to the surface of the solid and whose height is equal to the radius of the sphere*

The solid is as before a solid inscribed in a larger sphere and, since the perpendicular on any side of the revolving polygon is equal to the radius of the inner sphere the proposition is identical with Prop 26

COR *The solid circumscribed about the smaller sphere is greater than four times the cone whose base is a great circle of the sphere and whose height is equal to the radius of the sphere*

For since the surface of the solid is greater than four times the great circle of the inner sphere [Prop 30] the cone whose base is equal to the surface of the solid and whose height is the radius of the sphere is greater than four times the cone of the same height which has the great circle for base [*Lemma* 1]

Hence by the proposition the volume of the solid is greater than four times the latter cone

PROPOSITION 32

*If a regular polygon with 4n sides be inscribed in a great circle of a sphere as ab a b a and a similar polygon AB A B A be described about the great circle and if the polygons revolve with the great circle about the diameters aa AA respectively so that they describe the surfaces of solid figures inscribed in and circumscribed to the sphere respectively then*

(1) *the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides and*

(2) *the figures themselves [i e their volumes] are in the triplicate ratio of their sides*

(1) Let $AA'$, $aa'$ be in the same straight line, and let $MmOm'M'$ be a diameter at right angles to them

Join $BB'$, $CC'$, and $bb'$, $cc'$, which will all be parallel to one another and $MM'$

Suppose $R$, $S$ to be circles such that

$$R = \text{(surface of circumscribed solid)},$$
$$S = \text{(surface of inscribed solid)}$$

Then $(\text{radius of } R)^2 = AB(BB' + CC' + \quad)$ [Prop 29]

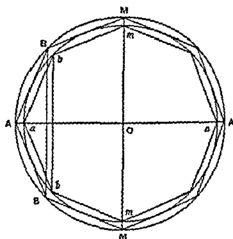
$(\text{radius of } S)^2 = ab(bb' + cc' + \quad)$ [Prop 24]

And, since the polygons are similar the rectangles in these two equations are similar and are therefore in the ratio of

$$AB^2 \quad ab^2$$

Hence

(surface of circumscribed solid) (surface of inscribed solid) $= AB^2 \quad ab^2$




(2) Take a cone $V$ whose base is the circle $R$ and whose height is equal to $Oa$ and a cone $W$ whose base is the circle $S$ and whose height is equal to the perpendicular from $O$ on $ab$ which we will call $p$

Then $V$, $W$ are respectively equal to the volumes of the circumscribed and inscribed figures [Props 31, 26]

Now since the polygons are similar

$$AB \quad ab = Oa \quad p$$
$$= (\text{height of cone } V) \quad (\text{height of cone } W),$$

and, as shown above the bases of the cones (the circles $R$, $S$) are in the ratio of $AB^2$ to $ab^2$

Therefore $V \quad W = AB^3 \quad ab^3$

## Proposition 33

*The surface of any sphere is equal to four times the greatest circle in it*

Let $C$ be a circle equal to four times the great circle

Then if $C$ is not equal to the surface of the sphere it must either be less or greater

I Suppose $C$ less than the surface of the sphere

It is then possible to find two lines $\beta$ $\gamma$ of which $\beta$ is the greater, such that

$$\beta \quad \gamma < (\text{surface of sphere}) \quad C$$ [Prop 2]

Take such lines and let $\delta$ be a mean proportional between them

Suppose similar regular polygons with $4n$ sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the ratio $\beta$ $\delta$ [Prop 3]

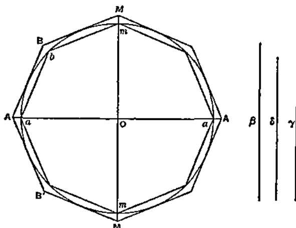
Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolution as before

Then (surface of outer solid) (surface of inner solid)

$= (\text{side of outer})^2 \ (\text{side of inner})^2$ [Prop 32]

$< \beta^2 \ \delta^2$, or $\beta \ \gamma$

$<$ (surface of sphere) $C$, *a fortiori*




But this is impossible since the surface of the circumscribed solid is greater than that of the sphere [Prop 28], while the surface of the inscribed solid is less than $C$ [Prop 25]

Therefore $C$ is not less than the surface of the sphere

II Suppose $C$ greater than the surface of the sphere

Take lines $\beta$, $\gamma$, of which $\beta$ is the greater, such that

$$\beta \ \gamma < C \ \text{(surface of sphere)}$$

Circumscribe and inscribe to the great circle similar regular polygons, as before such that their sides are in a ratio less than that of $\beta$ to $\delta$, and suppose solids of revolution generated in the usual manner

Then, in this case,

(surface of circumscribed solid) (surface of inscribed solid)

$< C$ (surface of sphere)

But this is impossible, because the surface of the circumscribed solid is greater than $C$ [Prop 30] while the surface of the inscribed solid is less than that of the sphere [Prop 23]

Thus $C$ is not greater than the surface of the sphere

Therefore since it is neither greater nor less $C$ is equal to the surface of the sphere

## Proposition 34

*Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere*

Let the sphere be that of which $ama \ m$ is a great circle

If now the sphere is not equal to four times the cone described it is either greater or less

I If possible let the sphere be greater than four times the cone

Suppose $V$ to be a cone whose base is equal to four times the great circle and whose height is equal to the radius of the sphere

*Then by hypothesis the sphere is greater than $V$, and two lines $\beta$, $\gamma$ can be* found (of which $\beta$ is the greater) such that

$$\beta \quad \gamma < (\text{volume of sphere}) \quad V$$

Between $\beta$ and $\gamma$ place two arithmetic means $\delta$, $\epsilon$

As before let similar regular polygons with sides $4n$ in number be circumscribed about and inscribed in the great circle, such that their sides are in a ratio less than $\beta \quad \delta$

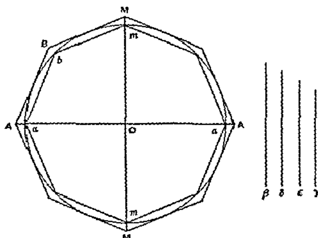
Imagine the diameter $aa'$ of the circle to be in the same straight line with a diameter of both polygons and imagine the latter to revolve with the circle about $aa'$ describing the surfaces of two solids of revolution The volumes of these solids are therefore in the triplicate ratio of their sides [Prop 32]

Thus (vol of outer solid) (vol of inscribed solid)

$< \beta^3 \quad \delta^3$, by hypothesis,

$< \beta \quad \gamma$, *a fortiori* (since $\beta \quad \gamma > \beta^3 \quad \delta^3$),

$<$(volume of sphere) $V$, *a fortiori*




But this is impossible since the volume of the circumscribed solid is greater than that of the sphere [Prop 28] while the volume of the inscribed solid is less than $V$ [Prop 27]

Hence the sphere is not greater than $V$ or four times the cone described in the enunciation

II If possible let the sphere be less than $V$

In this case we take $\beta \quad \gamma$ ($\beta$ being the greater) such that

$$\beta \quad \gamma < V \quad (\text{volume of sphere})$$

The rest of the construction and proof proceeding as before we have finally

(volume of outer solid) (volume of inscribed solid)

$< V$ (volume of sphere)

But this is impossible because the volume of the outer solid is greater than $V$ [Prop 31 Cor ] and the volume of the inscribed solid is less than the volume of the sphere

Hence the sphere is not less than $V$

Since then the sphere is neither less nor greater than $V$, it is equal to $V$, or to four times the cone described in the enunciation

COR From what has been proved it follows that *every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is* $\frac{3}{2}$ *of the sphere, and its surface together with its bases is* $\frac{3}{2}$ *of the surface of the sphere*

For the cylinder is three times the cone with the same base and height [Eucl XII 10] i e six times the cone with the same base and with height equal to the *radius of* the sphere

But the sphere is four times the latter cone [Prop 34] Therefore the cylinder is $\frac{3}{2}$ of the sphere

Again the surface of a cylinder (excluding the bases) is equal to a circle whose radius is a mean proportional between the height of the cylinder and the diameter of its base [Prop 13]

In this case the height is equal to the diameter of the base and therefore the *circle is* that whose radius is the diameter *of the* sphere or a circle equal to four times the great circle of the sphere

Therefore the surface of the cylinder with the bases is equal to six times the great circle

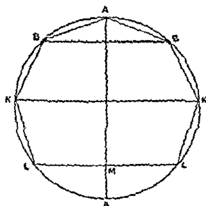
And the surface of the sphere is four times the great circle [Prop 33] whence

(surface of cylinder with bases) $= \frac{3}{2}$ (surface of sphere)

## PROPOSITION 35

*If in a segment of a circle LAL (where A is the middle point of the arc) a polygon LK A K'L' be inscribed of which LL is one side while the other sides are 2n in number and all equal, and if the polygon revolve with the segment about the diameter AM generating a solid figure inscribed in a segment of a sphere then the surface of the inscribed solid is equal to a circle the square on whose radius is equal to the rectangle*

$$AB\left(BB + CC + \quad + KK + \frac{LL'}{2}\right)$$




The surface of the inscribed figure is made up of portions of surfaces of cones

If we take these successively the surface of the cone $BAB$ is equal to a circle whose radius is

$$\sqrt{AB \cdot \tfrac{1}{2}BB'} \qquad \text{[Prop 14]}$$

The surface of the frustum of a cone $BCC B$ is equal to a circle whose radius is

$$\sqrt{AB \cdot \frac{BB + CC}{2}}, \quad \text{[Prop 16]}$$

and so on

Proceeding in this way and adding we find *since circles are to one another as the squares of their radii* that the surface of the inscribed figure is equal to a circle whose radius is

$$\sqrt{AB\left(BB' + CC + \quad + KK + \frac{LL}{2}\right)}$$

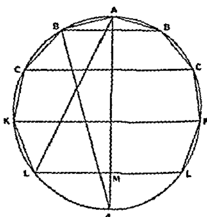
### PROPOSITION 36

*The surface of the figure inscribed as before in the segment of a sphere is less than that of the segment of the sphere*

This is clear because the circular base of the segment is a common boundary of each of two surfaces, of which one, the segment, includes the other, the solid, while both are concave in the same direction [*Assumptions*, 4]

### PROPOSITION 37

*The surface of the solid figure inscribed in the segment of the sphere by the revolution of LK ... A ... K'L' about AM is less than a circle with radius equal to AL*




Let the diameter $AM$ meet the circle of which $LAL'$ is a segment again in $A'$ Join $A'B$

As in Prop 35, the surface of the inscribed solid is equal to a circle the square on whose radius is

$AB(BB'+CC'+ \dots +KK'+LM)$

But this rectangle

$=A'B \cdot AM$ [Prop 22]

$<A'A \cdot AM$

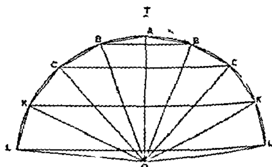
$<AL^2$

Hence the surface of the inscribed solid is less than the circle whose radius is $AL$

### PROPOSITION 38

*The solid figure described as before in a segment of a sphere less than a hemisphere, together with the cone whose base is the base of the segment and whose apex is the centre of the sphere is equal to a cone whose base is equal to the surface of the inscribed solid and whose height is equal to the perpendicular from the centre of the sphere on any side of the polygon*

Let $O$ be the centre of the sphere, and $p$ the length of the perpendicular from $O$ on $AB$




Suppose cones described with $O$ as apex and with the circles on $BB'$, $CC'$ ... as diameters as bases

Then the rhombus $OBAB'$ is equal to a cone whose base is equal to the surface of the cone $BAB'$ and whose height is $p$ [Prop 18]

Again if $CB$, $C'B'$ meet in $T$ the solid described by the triangle $BOC$ as the polygon revolves about $AO$ is the difference between the rhombi $OCTC'$ and $OBTB'$ and is therefore equal to a cone whose base is equal to the surface of

the frustum $BCC'B'$ and whose height is $p$ [Prop 20]

Similarly for the part of the solid described by the triangle $COD$ as the polygon revolves and so on

Hence, by addition the solid figure inscribed in the segment together with the cone $OLL'$ is equal to a cone whose base is the surface of the inscribed solid and whose height is $p$

COR *The cone whose base is a circle with radius equal to $AL$ and whose height is equal to the radius of the sphere is greater than the sum of the inscribed solid and the cone $OLL'$*

For by the proposition, the inscribed solid together with the cone $OLL'$ is equal to a cone with base equal to the surface of the solid and with height $p$

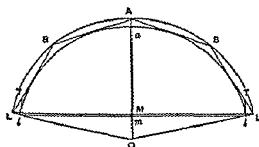
This latter cone is less than a cone with height equal to $OA$ and with base equal to the circle whose radius is $AL$ because the height $p$ is less than $OA$ while the surface of the solid is less than a circle with radius $AL$ [Prop 37]

PROPOSITION 39

Let $lal'$ be a segment of a great circle of a sphere being less than a semicircle Let $O$ be the centre of the sphere and join $Ol$, $Ol'$ Suppose a polygon circumscribed about the sector $Olal'$ such that its sides excluding the two radii are $2n$ in number and all equal as $LK$ … $BA$, $AB'$, … $K'L'$ and let $OA$ be that radius of the great circle which bisects the segment $lal'$

The circle circumscribing the polygon will then have the same centre $O$ as the given great circle

Now suppose the polygon and the two circles to revolve together about $OA$ The two circles will describe spheres the angular points except $A$ will describe circles on the outer sphere with diameters $BB'$ etc the points of contact of the sides with the inner segment will describe circles on the inner sphere the sides themselves will describe the surfaces of cones or frusta of cones and the whole figure circumscribed to the segment of the inner sphere by the revolution of the equal sides of the polygon will have for its base the circle on $LL'$ as diameter




*The surface of the solid figure so circumscribed about the sector of the sphere [excluding its base] will be greater than that of the segment of the sphere whose base is the circle on $ll'$ as diameter*

For draw the tangents $lT$, $l'T'$ to the inner segment at $l$, $l'$ These with the sides of the polygon will describe by their revolution a solid whose surface is greater than that of the segment [*Assumptions* 4]

But the surface described by the revolution of $lT$ is less than that described by the revolution of $LT$ since the angle $TlL$ is a right angle and therefore $LT > lT$

Hence *a fortiori* the surface described by $LK$ … $A$ … $K'L'$ is greater than that of the segment

Cor *The surface of the figure so described about the sector of the sphere is equal to a circle the square on whose radius is equal to the rectangle*

$$AB\,(BB'+CC'+\quad +KK'+\tfrac{1}{2}LL')$$

For the circumscribed figure is inscribed in the outer sphere, and the proof of Prop 35 therefore applies

PROPOSITION 40

*The surface of the figure circumscribed to the sector as before is greater than a circle whose radius is equal to al*

Let the diameter $AaO$ meet the great circle and the circle circumscribing the revolving polygon again in $a'$, $A'$ Join $A'B$, and let $ON$ be drawn to $N$, the point of contact of $AB$ with the inner circle

Now by Prop 39 Cor, the surface of the solid figure circumscribed to the sector $OlAl$ is equal to a circle the square on whose radius is equal to the rectangle

$$AB\left(BB'+CC'+\quad +KK'+\frac{LL'}{2}\right)$$

But this rectangle is equal to $A'B\ AM$ [as in Prop 22]

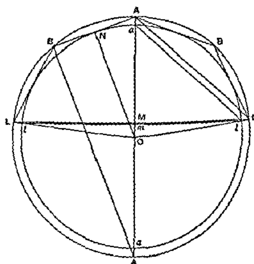
Next since $AL'$ $al$ are parallel the triangles $AML'$ $aml$ are similar And $AL'>al'$ therefore $AM>am$

Also $A'B=2ON=aa'$

Therefore $A'B\ AM>am\ aa'$

$>al^2$

Hence the surface of the solid figure circumscribed to the sector is greater than a circle whose radius is equal to $al$ or $al$




Cor 1 *The volume of the figure circumscribed about the sector together with the cone whose apex is O and base the circle on LL' as diameter is equal to the volume of a cone whose base is equal to the surface of the circumscribed figure and whose height is ON*

For the figure is inscribed in the outer sphere which has the same centre as the inner Hence the proof of Prop 38 applies

Cor 2 *The volume of the circumscribed figure with the cone OLL' is greater than the cone whose base is a circle with radius equal to al and whose height is equal to the radius (Oa) of the inner sphere*

For the volume of the figure with the cone $OLL'$ is equal to a cone whose base is equal to the surface of the figure and whose height is equal to $ON$

And the surface of the figure is greater than a circle with radius equal to $al$ [Prop 40] while the heights $Oa$ $ON$ are equal

## PROPOSITION 41

Let $lal$ be a segment of a great circle of a sphere which is less than a semicircle

Suppose a polygon inscribed in the sector $Olal$ such that the sides $lk \quad ba$, $ab \quad kl$ are $2n$ in number and all equal Let a similar polygon be circumscribed about the sector so that its sides are parallel to those of the first polygon and draw the circle circumscribing the outer polygon

Now let the polygons and circles revolve together about $OaA$ the radius bisecting the segment $lal$

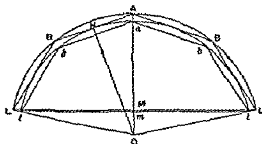
Then (1) *the surfaces of the outer and inner solids of revolution so described are in the ratio of* $AB^2$ *to* $ab^2$, *and* (2) *their volumes together with the corresponding cones with the same base and with apex* $O$ *in each case are as* $AB^3$ *to* $ab^3$

(1) For the surfaces are equal to circles the squares on whose radii are equal respectively to

$$AB\left(BB + CC' + \quad + KK' + \frac{LL'}{2}\right), \qquad \text{[Prop 39 Cor]}$$

and

$$ab\left(bb + cc + \quad + kk + \frac{ll}{2}\right) \qquad \text{[Prop 35]}$$




But these rectangles are in the ratio of $AB^2$ to $ab^2$ Therefore so are the surfaces

(2) Let $OnN$ be drawn perpendicular to $ab$ and $AB$, and suppose the circles which are equal to the surfaces of the outer and inner solids of revolution to be denoted by $S$, $s$ respectively

Now the volume of the circumscribed solid together with the cone $OLL$ is equal to a cone whose base is $S$ and whose height is $ON$ [*Prop* 40 *Cor* 1]

And the volume of the inscribed figure with the cone $Oll$ is equal to a cone with base $s$ and height $On$ [*Prop* 38]

But $\qquad S \quad s = AB \quad ab^2$

and $\qquad ON \quad On = AB \quad ab$

Therefore the volume of the circumscribed solid together with the cone $OLL$ is to the volume of the inscribed solid together with the cone $Oll$ as $AB^3$ is to $ab^3$ [*Lemma* 5]

## PROPOSITION 42

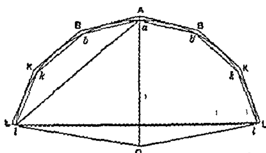
*If* $lal$ *be a segment of a sphere less than a hemisphere and* $Oa$ *the radius perpendicular to the base of the segment the surface of the segment is equal to a circle whose radius is equal to* $al$

Let $R$ be a circle whose radius is equal to $al$ Then the surface of the segment which we will call $S$ must if it be not equal to $R$ be either greater or less than $R$

I Suppose, if possible, $S > R$

Let $lal'$ be a segment of a great circle which is less than a semicircle Join $Ol$, $Ol'$, and let similar polygons with $2n$ equal sides be circumscribed and inscribed to the sector, as in the previous propositions, but such that

(circumscribed polygon) (inscribed polygon) $< S$ $R$ [Prop 6]




Let the polygons now revolve with the segment about $OaA$, generating solids of revolution circumscribed and inscribed to the segment of the sphere

Then

$$\text{(surface of outer solid)} \quad \text{(surface of inner solid)}$$
$$= AB^2 \quad ab^2 \qquad \text{[Prop 41]}$$
$$= \text{(circumscribed polygon)} \quad \text{(inscribed polygon)}$$
$$< S \quad R \text{ by hypothesis}$$

But the surface of the outer solid is greater than $S$ [Prop 39]

Therefore the surface of the inner solid is greater than $R$, which is impossible, by Prop 37

II Suppose if possible $S < R$

In this case we circumscribe and inscribe polygons such that their ratio is less than $R$ $S$ and we arrive at the result that

$$\text{(surface of outer solid)} \quad \text{(surface of inner solid)}$$
$$< R \quad S$$

But the surface of the outer solid is greater than $R$ [Prop 40] Therefore the surface of the inner solid is greater than $S$ which is impossible [Prop 36]

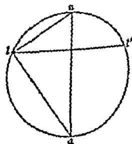
Hence since $S$ is neither greater nor less than $R$,

$$S = R$$

## Proposition 43

*Even if the segment of the sphere is greater than a hemisphere, its surface is still equal to a circle whose radius is equal to al*

For let $lal'a'$ be a great circle of the sphere $aa'$ being the diameter perpendicular to $ll'$ and let $la'l'$ be a segment less than a semicircle




Then by Prop 42 the surface of the segment $la'l'$ of the sphere is equal to a circle with radius equal to $a'l$

Also the surface of the whole sphere is equal to a circle with radius equal to $aa'$ [Prop 33]

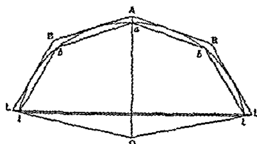
But $aa'^2 - a'l^2 = al^2$ and circles are to one another as the squares on their radii

Therefore the surface of the segment $lal'$ being the difference between the surfaces of the sphere and of $la'l'$ is equal to a circle with radius equal to $al$

PROPOSITION 44

*The volume of any sector of a sphere is equal to a cone whose base is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere*

Let $R$ be a cone whose base is equal to the surface of the segment $lal$ of a sphere and whose height is equal to the radius of the sphere, and let $S$ be the volume of the sector $Olal$




β
δ
ε
γ

Then if $S$ is not equal to $R$ it must be either greater or less

I Suppose, if possible, that $S>R$

Find two straight lines $\beta$ $\gamma$, of which $\beta$ is the greater, such that

$$\beta\ \ \gamma<S\ \ R$$

and let $\delta$ $\epsilon$ be two arithmetic means between $\beta$ $\gamma$

Let $lal$ be a segment of a great circle of the sphere Join $Ol$ $Ol$ and let similar polygons with $2n$ equal sides be circumscribed and inscribed to the sector of the circle as before but such that their sides are in a ratio less than $\beta$ $\delta$ [Prop 4]

Then let the two polygons revolve with the segment about $OaA$ generating two solids of revolution

Denoting the volumes of these solids by $V$ $v$ respectively we have

$$(V+\text{cone } OLL')\ \ (v+\text{cone } Oll')=AB^3\ \ ab^3 \qquad \text{[Prop 41]}$$
$$<\beta^3\ \ \delta^3$$
$$<\beta\ \ \gamma, \textit{ a fortiori},$$
$$<S\ \ R \text{ by hypothesis}$$

Now $\qquad (V+\text{cone } OLL')>S$

Therefore also $\qquad (v+\text{cone } Oll')>R$

But this is impossible by Prop 38 Cor combined with Props 42 43

Hence $\qquad S \not> R$

II Suppose if possible that $S<R$

In this case we take $\beta$ $\gamma$ such that

$$\beta\ \ \gamma<R\ \ S$$

and the rest of the construction proceeds as before

We thus obtain the relation

$$(V+\text{cone } OLL')\ \ (v+\text{cone } Oll')<R\ \ S$$

Now $\qquad (v+\text{cone } Oll')<S$

Therefore $\qquad (V+\text{cone } OLL')<R$

which is impossible by Prop 40 Cor 2 combined with Props 42 43

Since then $S$ is neither greater nor less than $R$

$$S=R$$

# ON THE SPHERE AND CYLINDER

## BOOK TWO

ARCHIMEDES to Dositheus greeting

'On a former occasion you asked me to write out the proofs of the problems the enunciations of which I had myself sent to Conon In point of fact they depend for the most part on the theorems of which I have already sent you the demonstrations, namely (1) that the surface of any sphere is four times the greatest circle in the sphere (2) that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of its base (3) that the cylinder whose base is the greatest circle in any sphere and whose height is equal to the diameter of the sphere is itself in magnitude half as large again as the sphere while its surface [including the two bases] is half as large again as the surface of the sphere, and (4) that any solid sector is equal to a cone whose base is the circle which is equal to the surface of the segment of the sphere included in the sector and whose height is equal to the radius of the sphere Such then of the theorems and problems as depend on these theorems I have written out in the book which I send herewith, those which are discovered by means of a different sort of investigation, those namely which relate to spirals and the conoids I will endeavour to send you soon

The first of the problems was as follows *Given a sphere to find a plane area equal to the surface of the sphere*

The solution of this is obvious from the theorems aforesaid For four times the greatest circle in the sphere is both a plane area and equal to the surface of the sphere

The second problem was the following

PROPOSITION 1 (PROBLEM)

*Given a cone or a cylinder to find a sphere equal to the cone or to the cylinder*

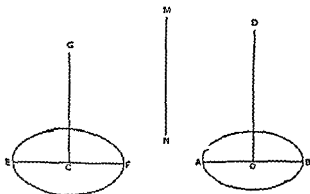
If $V$ be the given cone or cylinder we can make a cylinder equal to $\frac{3}{2}V$ Let this cylinder be the cylinder whose base is the circle on $AB$ as diameter and whose height is $OD$

Now if we could make another cylinder equal to the cylinder ($OD$) but such that its height is equal to the diameter of its base the problem would be solved because this latter cylinder would be equal to $\frac{3}{2}V$ and the sphere whose diameter is equal to the height (or to the diameter of the base) of the same cylinder would then be the sphere required [I 34 Cor]

Suppose the problem solved and let the cylinder ($CG$) be equal to the cylinder ($OD$), while $EF$ the diameter of the base is equal to the height $CG$

Then since in equal cylinders the heights and bases are reciprocally proportional,

$$AB^2\ \ EF^2 = CG\ \ OD$$
$$= EF\ \ OD \qquad (1)$$




Suppose $MN$ to be such a line that

$$EF = AB\ \ MN \qquad (2)$$

Hence $$AB\ \ EF = EF\ \ MN,$$

and combining (1) and (2), we have

$$AB\ \ MN = EF\ \ OD$$

or $$AB\ \ EF = MN\ \ OD$$

Therefore $$AB\ \ EF = EF\ \ MN = MN\ \ OD$$

and $EF$ $MN$ *are two mean proportionals between* $AB$ $OD$

The synthesis of the problem is therefore as follows Take two mean proportionals $EF$, $MN$ between $AB$ and $OD$ and describe a cylinder whose base is a circle on $EF$ as diameter and whose height $CG$ is equal to $EF$

Then since

$$AB\ \ EF = EF\ \ MN = MN\ \ OD$$
$$EF^2 = AB\ \ MN$$

and therefore $$AB^2\ \ EF^2 = AB\ \ MN$$
$$= EF\ \ OD$$
$$= CG\ \ OD$$

whence the bases of the two cylinders $(OD)$ $(CG)$ are reciprocally proportional to their heights

Therefore the cylinders are equal and it follows that

$$\text{cylinder } (CG) = \tfrac{3}{2}V$$

The sphere on $EF$ as diameter is therefore the sphere required being equal to $V$

PROPOSITION 2

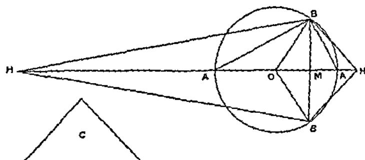
*If $BAB$ be a segment of a sphere $BB$ a diameter of the base of the segment and $O$ the centre of the sphere and if $AA$ be the diameter of the sphere bisecting $BB$ in $M$ then the volume of the segment is equal to that of a cone whose base is the same as that of the segment and whose height is $h$ where*

$$h\ \ A'M = OA' + A'M\ \ A'M$$

Measure $MH$ along $MA$ equal to $h$ and $MH'$ along $MA'$ equal to $h'$ where

$$h'\ \ AM = OA + AM\ \ AM$$

Suppose the three cones constructed which have $O$, $H$, $H'$ for their apices and the base ($BB'$) of the segment for their common base. Join $AB$, $A'B$.




Let $C$ be a cone whose base is equal to the surface of the segment $BAB'$ of the sphere, i.e. to a circle with radius equal to $AB$ [I. 42], and whose height is equal to $OA$.

Then the cone $C$ is equal to the solid sector $OBAB'$ [I. 44].

Now, since $HM : MA = OA' + A'M : A'M$,

*dividendo* $HA : AM = OA : A'M$,

and, alternately $HA : AO = AM : MA'$,

so that

$$HO : OA = AA' : A'M$$
$$= AB^2 : BM^2$$
$$= \text{(base of cone } C) : \text{(circle on } BB' \text{ as diameter)}$$

But $OA$ is equal to the height of the cone $C$; therefore, since cones are equal if their bases and heights are reciprocally proportional, it follows that the cone $C$ (or the solid sector $OBAB'$) is equal to a cone whose base is the circle on $BB'$ as diameter and whose height is equal to $OH$.

And this latter cone is equal to the sum of two others having the same base and with heights $OM$, $MH$, i.e. to the solid rhombus $OBHB'$.

Hence the sector $OBAB'$ is equal to the rhombus $OBHB'$.

Taking away the common part, the cone $OBB'$,

the segment $BAB'$ = the cone $HBB'$.

Similarly, by the same method, we can prove that

the segment $BA'B'$ = the cone $H'BB'$.

*Alternative proof of the latter property.*

Suppose $D$ to be a cone whose base is equal to the surface of the whole sphere and whose height is equal to $OA$.

Thus $D$ is equal to the volume of the sphere. [I. 33, 34]

Now, since $OA + A'M : A'M = H'M : MA$,

*dividendo* and *alternando*, as before,

$$OA : AH' = A'M : MA.$$

Again, since $HM : MA' = OA + AM : AM$,

$$HA' : OA = AM : MA'$$
$$= OA : AH', \text{ from above} \quad (1)$$

*Componendo*, $HO : OA = OH' : H'A$ (2)

Alternately $H'O : OH = OA : AH$

and *componendo*, $HH' \; HO = OH \; HA$,

$= H'O \; OA$ from (1),

whence $HH \; OA = H O \; OH$ (3)

Next since $H O \; OH = OA \; AH$ by (2)

$= A'M \; MA$

$(H'O+OH)^2 \; H O \; OH = (A'M+MA)^2 \; A'M \; MA$,

whence, by means of (3),

$HH \; HH \; OA = AA' \; A'M \; MA$

or $HH \; OA = AA'^2 \; BM^2$

Now the cone $D$ which is equal to the sphere has for its base a circle whose radius is equal to $AA'$, and for its height a line equal to $OA$

Hence this cone $D$ is equal to a cone whose base is the circle on $BB$ as diameter and whose height is equal to $HH$

therefore the cone $D$ = the rhombus $HBH B$,

or the rhombus $HBH B$ = the sphere

But the segment $BAB$ = the cone $HBB$

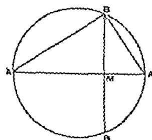
therefore the remaining segment $BA B$ = the cone $H BB$

COR *The segment* $BAB$ *is to a cone with the same base and equal height in the ratio of* $OA + A'M$ *to* $A'M$

PROPOSITION 3 (PROBLEM)

*To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio*

Suppose the problem solved Let $AA'$ be a diameter of a great circle of the sphere and suppose that a plane perpendicular to $AA$ cuts the plane of the great circle in the straight line $BB$ and $AA$ in $M$, and that it divides the sphere so that the surface of the segment $BAB$ has to the surface of the segment $BA B$ the given ratio




Now these surfaces are respectively equal to circles with radii equal to $AB$, $A B$ [I 42 43]

Hence the ratio $AB^2 \; A B$ is equal to the given ratio i e $AM$ is to $MA$ in the given ratio

Accordingly the synthesis proceeds as follows

If $H \; K$ be the given ratio divide $A A$ in $M$ so that

$AM \; MA = H \; K$

Then $AM \; MA = AB \; A B^2$

= (circle with radius $AB$) (circle with radius $A'B$)

= (surface of segment $BAB$) (surface of segment $BA B$)

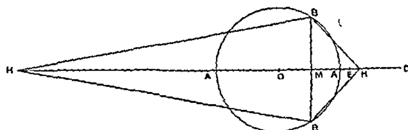
Thus the ratio of the surfaces of the segments is equal to the ratio $H \; K$

PROPOSITION 4 (PROBLEM)

*To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio*

Suppose the problem solved and let the required plane cut the great circle

$ABA'$ at right angles to the line $BB'$. Let $AA'$ be that diameter of the great circle which bisects $BB'$ at right angles (in $M$), and let $O$ be the centre of the sphere.



Take $H$ on $OA$ produced, and $H'$ on $OA'$ produced, such that

$$OA+A'M : A'M = HM : MA, \qquad (1)$$

and

$$OA+AM : AM = H'M : MA'. \qquad (2)$$

Join $BH$, $B'H$, $BH'$, $B'H'$.

Then the cones $HBB'$, $H'BB'$ are respectively equal to the segments $BAB'$, $BA'B'$ of the sphere [Prop. 2].

Hence the ratio of the cones, and therefore of their altitudes, is given, i.e.

$$HM : H'M = \text{the given ratio}. \qquad (3)$$

We have now three equations (1), (2), (3) in which there appear three as yet undetermined points $M$, $H$, $H'$, and it is first necessary to find, by means of them, another equation in which only one of these points ($M$) appears, i.e. we have, so to speak, to *eliminate* $H$, $H'$.

Now, from (3), it is clear that $HH' : H'M$ is also a given ratio, and Archimedes' method of elimination is, *first*, to find values for each of the ratios $A'H' : H'M$ and $HH' : H'A'$ which are alike independent of $H$, $H'$, and then, *secondly*, to equate the ratio compounded of these two ratios to the known value of the ratio $HH' : H'M$.

(a) To find such a value for $A'H' : H'M$.

It is at once clear from equation (2) above that

$$A'H' : H'M = OA : OA+AM. \qquad (4)$$

(b) To find such a value for $HH' : A'H'$.

From (1) we derive

$$A'M : MA = OA'+A'M : HM$$
$$= OA' : AH, \qquad (5)$$

and from (2)

$$A'M : MA = H'M : OA+AM$$
$$= A'H' : OA. \qquad (6)$$

Thus $HA : AO = OA' : A'H'$,

whence $OH : OA' = OH' : A'H'$,

or $OH : OH' = OA' : A'H'$.

It follows that

$$HH' : OH' = OH' : A'H',$$

or $HH' \cdot H'A' = OH'^2$.

Therefore $HH' : H'A' = OH'^2 : H'A'^2$

$$= AA'^2 : A'M^2, \text{ by means of (6).}$$

(c) To express the ratios $A'H' : H'M$ and $HH' : H'M$ more simply we make

the following construction Produce $OA$ to $D$ so that $OA = AD$ ($D$ will lie beyond $H$, for $A'M > MA$ and therefore by (5) $OA > AH$)

Then $$A H \quad H'M = OA \quad OA + AM$$
$$= AD \quad DM \qquad (7)$$

Now divide $AD$ at $E$ so that
$$HH \quad H'M = AD \quad DE \qquad (8)$$

Thus using equations (8) (7) and the value of $HH \quad H'A'$ above found we have
$$AD \quad DE = HH \quad H'M$$
$$= (HH \quad H A')\ (A H \quad H M)$$
$$= (AA^2 \quad A M^2)\ (AD \quad DM)$$

But $$AD \quad DE = (DM \quad DE)\ (AD \quad DM)$$

Therefore $$MD \quad DE = AA^2 \quad A'M^2 \qquad (9)$$

And $D$ is given since $AD = OA$ Also $AD \quad DE$ (being equal to $HH \quad H'M$) is a given ratio Therefore $DE$ is given

*Hence the problem reduces itself to the problem of dividing $A'D$ into two parts at $M$ so that*
$$MD \quad (\text{a given length}) = (\text{a given area}) \quad A'M$$

Archimedes adds 'If the problem is propounded in this general form it requires a διορισμός [i e it is necessary to investigate the limits of possibility] but if there be added the conditions subsisting in the present case it does not require a διορισμός

In the present case the problem is

*Given a straight line $A'A$ produced to $D$ so that $A A = 2AD$ and given a point $E$ on $AD$ to cut $AA$ in a point $M$ so that*
$$AA^2 \quad A'M^2 = MD \quad DE$$

And the analysis and synthesis of both problems will be given at the end [1]

The synthesis of the main problem will be as follows Let $R \quad S$ be the given ratio $R$ being less than $S$ $AA$ being a diameter of a great circle and $O$ the centre produce $OA$ to $D$ so that $OA = AD$ and divide $AD$ in $E$ so that
$$AE \quad ED = R \quad S$$

Then cut $AA$ in $M$ so that
$$MD \quad DE = AA \quad A M^2$$

Through $M$ erect a plane perpendicular to $AA$ this plane will then divide the sphere into segments which will be to one another as $R$ to $S$

Take $H$ on $A A$ produced and $H$ on $AA$ produced so that
$$OA + A M \quad A M = HM \quad MA \qquad (1)$$
$$OA + AM \quad AM = H M \quad MA \qquad (2)$$

We have then to show that
$$HM \quad MH = R \quad S \text{ or } AE \quad ED$$

(a) We first find the value of $HH \quad H A$ as follows

As was shown in the analysis (b)
$$HH \quad H A = OH^2$$

or $$HH \quad H A = OH^2 \quad H A^2$$
$$= AA^2 \quad A M^2$$
$$= MD \quad DE \text{ by construction}$$

[1] As Archimedes commentator Eutocius notes we do not find the promise kept in any of the copies Sir Thomas Heath s translation of Eutocius note on the matter along with the solutions of Dionysodorus and Diocles is omitted from this edition.—Ed

(β) Next we have

$$H'A' : H'M = OA : OA + AM$$
$$= AD : DM.$$

Therefore $$HH' : H'M = (HH' : H'A') \cdot (H'A' : H'M)$$
$$= (MD : DE) \cdot (AD : DM)$$
$$= AD : DE,$$

whence $$HM : MH' = AE : ED$$
$$= R : S.$$

Q. E. D.

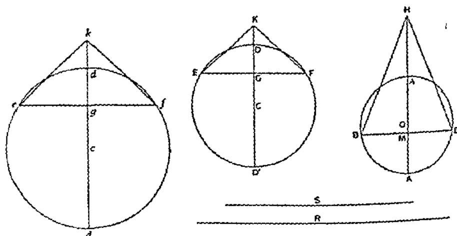
PROPOSITION 5. (PROBLEM.)

*To construct a segment of a sphere similar to one segment and equal in volume to another.*

Let $ABB'$ be one segment whose vertex is $A$ and whose base is the circle on $BB'$ as diameter, and let $DEF$ be another segment whose vertex is $D$ and whose base is the circle on $EF$ as diameter. Let $AA'$, $DD'$ be diameters of the great circles passing through $BB'$, $EF$ respectively, and let $O$, $C$ be the respective centres of the spheres.

Suppose it required to draw a segment similar to $DEF$ and equal in volume to $ABB'$.

*Analysis.* Suppose the problem solved, and let $def$ be the required segment, $d$ being the vertex and $ef$ the diameter of the base. Let $dd'$ be the diameter of the sphere which bisects $ef$ at right angles, $c$ the centre of the sphere.




Let $M$, $G$, $g$ be the points where $BB'$, $EF$, $ef$ are bisected at right angles by $AA'$, $DD'$, $dd'$ respectively, and produce $OA$, $CD$, $cd$ respectively to $H$, $K$, $k$ so that

$$\left.\begin{aligned} OA + A'M : A'M &= HM : MA \\ CD + DG : DG &= KG : GD \\ cd + dg : dg &= kg : gd \end{aligned}\right\},$$

and suppose cones formed with vertices $H$, $K$, $k$ and with the same bases as the respective segments. The cones will then be equal to the segments respectively. [Prop. 2]

Therefore, by hypothesis,

the cone $HBB$ = the cone $kef$

Hence

(circle on diameter $BB$) (circle on diameter $ef$) $= kg$ $HM$,

so that $BB^2$ $ef^2 = kg$ $HM$ (1)

But since the segments $DEF$ $def$ are similar so are the cones $KEF$ $kef$

Therefore $KG$ $EF = kg$ $ef$

And the ratio $KG$ $EF$ is given Therefore the ratio $kg$ $ef$ is given

Suppose a length $R$ taken such that

$$kg \quad ef = HM \quad R$$ (2)

Thus $R$ is given

Again since $kg$ $HM = BB'^2$ $ef^2 = ef$ $R$, by (1) and (2), suppose a length $S$ taken such that

$$ef^2 = BB' \quad S$$

or

$$BB'^2 \quad ef^2 = BB \quad S$$

Thus

$$BB' \quad ef = ef \quad S = S \quad R,$$

and $ef$, $S$ *are two mean proportionals in continued proportion between* $BB$, $R$

*Synthesis* Let $ABB$ $DEF$ be great circles $AA'$, $DD$ the diameters bisecting $BB'$, $EF$ at right angles in $M$, $G$ respectively, and $O$, $C$ the centres

Take $H$, $K$ in the same way as before and construct the cones $HBB$ $KEF$, which are therefore equal to the respective segments $ABB'$, $DEF$

Let $R$ be a straight line such that

$$KG \quad EF = HM \quad R$$

and between $BB'$ $R$ take two mean proportionals $ef$, $S$

On $ef$ as base describe a segment of a circle with vertex $d$ and similar to the segment of a circle $DEF$ Complete the circle, and let $dd'$ be the diameter through $d$ and $c$ the centre Conceive a sphere constructed of which $def$ is a great circle and through $ef$ draw a plane at right angles to $dd$

Then shall $def$ be the required segment of a sphere

For the segments $DEF$ $def$ of the spheres are similar, like the circular segments $DEF$ $def$

Produce $cd$ to $k$ so that

$$cd + d'g \quad d'g = kg \quad gd$$

The cones $KEF$, $kef$ are then similar

Therefore $kg$ $ef = KG$ $EF = HM$ $R$,

whence $kg$ $HM = ef$ $R$

But since $BB$ $ef$ $S$ $R$ are in continued proportion

$$\begin{aligned} BB \quad ef^2 &= BB \quad S \\ &= ef \quad R \\ &= kg \quad HM \end{aligned}$$

Thus the bases of the cones $HBB$, $kef$ are reciprocally proportional to their heights The cones are therefore equal and $def$ is the segment required being equal in volume to the cone $kef$ [Prop 2]

### Proposition 6 (Problem)

*Given two segments of spheres to find a third segment of a sphere similar to one of the given segments and having its surface equal to that of the other*

Let $ABB$ be the segment to whose surface the surface of the required segment is to be equal $ABA'B$ the great circle whose plane cuts the plane of the

base of the segment $ABB'$ at right angles in $BB'$ Let $AA'$ be the diameter which bisects $BB'$ at right angles

Let $DEF$ be the segment to which the required segment is to be similar, *$DED'F$ the great circle cutting the base of the segment at right angles in $EF$* Let $DD'$ be the diameter bisecting $EF$ at right angles in $G$

Suppose the problem solved *def* being a segment similar to $DEF$ and having its surface equal to that of $ABB'$ and complete the figure for *def* as for $DEF$, corresponding points being denoted by small and capital letters respectively

Join $AB$, $DF$ $df$

Now, since the surfaces of the segments *def*, $ABB'$ are equal, so are the circles on $df$, $AB$ as diameters [I 42 43]

that is $df = AB$

From the similarity of the segments $DEF$, *def* we obtain

$dd \quad dg = DD \quad DG$,

and $dg \quad df = DG \quad DF$,

whence $d'd \quad df = D'D \quad DF$,

or $d'd \quad AB = D'D \quad DF$

But $AB$, $D'D$, $DF$ are all given

therefore $d'd$ is given

Accordingly the synthesis is as follows

Take $dd$ such that

$$dd \quad AB = DD \quad DF \qquad (1)$$

Describe a circle on $dd$ as diameter, and conceive a sphere constructed of which this circle is a great circle

Divide $dd$ at $g$ so that

$$d'g \quad gd = DG \quad GD$$

and draw through $g$ a plane perpendicular to $dd$ cutting off the segment *def* of the sphere and intersecting the plane of the great circle in $ef$ The segments *def* $DEF$ are thus similar and

$$dg \quad df = DG \quad DF$$

But from above *componendo*

$$d'd \quad dg = D'D \quad DG$$

Therefore *ex aequali* $\quad d'd \quad df = DD \quad DF$

whence by (1) $df = AB$

Therefore the segment *def* has its surface equal to the surface of the segment $ABB$ [I 42 43], while it is also similar to the segment $DEF$

PROPOSITION 7 (PROBLEM)

*From a given sphere to cut off a segment by a plane so that the segment may have a given ratio to the cone which has the same base as the segment and equal height*

Let $AA'$ be the diameter of a great circle of the sphere It is required to draw a plane at right angles to $AA$ cutting off a segment as $ABB$, such that the segment $ABB$ has to the cone $ABB$ a given ratio

*Analysis*

Suppose the problem solved, and let the plane of section cut the plane of the great circle in $BB'$ and the diameter $AA'$ in $M$. Let $O$ be the centre of the sphere.

Produce $OA$ to $H$ so that

$$OA' + A'M : A'M = HM : MA. \quad (1)$$

Thus the cone $HBB'$ is equal to the segment $ABB'$. [Prop. 2]

Therefore the given ratio must be equal to the ratio of the cone $HBB'$ to the cone $A'BB'$, i.e. to the ratio $HM : MA'$.

Hence the ratio $OA' + A'M : A'M$ is given; and therefore $A'M$ is given.

*Διορισμός*

Now $\quad OA' : A'M > OA' : A'A$,

so that $\quad OA' + A'M : A'M > OA' + A'A : A'A$

$$> 3 : 2.$$

Thus, *in order that a solution may be possible, it is a necessary condition that the given ratio must be greater than* $3 : 2$.

The *synthesis* proceeds thus.

Let $AA'$ be a diameter of a great circle of the sphere, $O$ the centre.

Take a line $DE$, and a point $F$ on it, such that $DE : EF$ is equal to the given ratio, being greater than $3 : 2$.

Now, since $\quad OA' + A'A : A'A = 3 : 2$,

$$DE : EF > OA' + A'A : A'A,$$

so that $\quad DF : FE > OA : AA'$.

Hence a point $M$ can be found on $AA'$ such that

$$DF : FE = OA' : A'M. \quad (2)$$

Through $M$ draw a plane at right angles to $AA'$ intersecting the plane of the great circle in $BB'$, and cutting off from the sphere the segment $ABB'$.

As before, take $H$ on $OA$ produced such that

$$OA' + A'M : A'M = HM : MA.$$

Therefore $HM : MA' = DE : EF$, by means of (2).

It follows that the cone $HBB'$, or the segment $ABB'$, is to the cone $A'BB'$ in the given ratio $DE : EF$.

## Proposition 8

*If a sphere be cut by a plane not passing through the centre into two segments $A'BB'$, $ABB'$, of which $A'BB'$ is the greater, then the ratio*

*(segmt. $A'BB'$) : (segmt. $ABB'$)*

$$< (\text{surface of } A'BB')^2 : (\text{surface of } ABB')^2,$$

$$\text{but} > (\text{surface of } A'BB')^{\frac{3}{2}} : (\text{surface of } ABB')^{\frac{3}{2}}.$$

Let the plane of section cut a great circle $A'BAB'$ at right angles in $BB'$, and let $AA'$ be the diameter bisecting $BB'$ at right angles in $M$.

Let $O$ be the centre of the sphere.

$R$ will fall between $O$ and $M$

Also, since $AB^2 = DE$ , $AR = CD$

Produce $OA'$ to $K$ so that $OA' = A'K$, and produce $A'A$ to $H$ so that

$$A'K \quad A'M = HA \quad AM,$$

or *componendo* $\quad A'K + A'M \quad A'M = HM \quad MA$ (1)

Thus the cone $HBB'$ is equal to the segment $ABB'$ [Prop 2]

Again, produce $CD$ to $F$ so that $CD = DF$, and the cone $FEE'$ will be equal to the hemisphere $DEE'$ [Prop 2]

Now $\quad AR \quad RA' > AM \quad MA'$,

and $\quad AR^2 = \frac{1}{2}AB^2 = \frac{1}{2}AM \quad AA' = AM \quad A'K$

Hence

$$AR \quad RA' + RA^2 > AM \quad MA' + AM \quad A'K,$$

or $\quad AA' \quad AR > AM \quad MK$

$> HM \quad A'M$, by (1)

Therefore $\quad AA' \quad A'M > HM \quad AR$,

or $\quad AB^2 \quad BM^2 > HM \quad AR$,

i e $\quad AR^2 \quad BM > HM \quad 2AR$ since $AB^2 = 2AR$ ,

$> HM \quad CF$

Thus since $AR = CD$, or $CE$,

(circle on diam $EE$ ) (circle on diam $BB'$) $> HM \quad CF$

It follows that

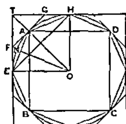
(the cone $FEE$ ) > (the cone $HBB$ ),

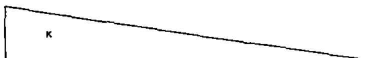
and therefore the hemisphere $DEE'$ is greater in volume than the segment $ABB$

# MEASUREMENT OF A CIRCLE

Proposition 1

*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle*




Let $ABCD$ be the given circle $K$ the triangle described

Then if the circle is not equal to $K$ it must be either greater or less

I If possible let the circle be greater than $K$

Inscribe a square $ABCD$ bisect the arcs $AB$ $BC$ $CD$ $DA$ then bisect (if necessary) the halves and so on until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over $K$

Thus the area of the polygon is greater than $K$

Let $AE$ be any side of it and $ON$ the perpendicular on $AE$ from the centre $O$

Then $ON$ is less than the radius of the circle and therefore less than one of the sides about the right angle in $K$ Also the perimeter of the polygon is less than the circumference of the circle i e less than the other side about the right angle in $K$

Therefore the area of the polygon is less than $K$ which is inconsistent with the hypothesis

Thus the area of the circle is not greater than $K$

II If possible let the circle be less than $K$

Circumscribe a square and let two adjacent sides touching the circle in $E$ $H$ meet in $T$ Bisect the arcs between adjacent points of contact and draw the

tangents at the points of bisection. Let $A$ be the middle point of the arc $EH$, and $FAG$ the tangent at $A$.

Then the angle $TAG$ is a right angle.

Therefore $$TG > GA$$ $$> GH.$$

It follows that the triangle $FTG$ is greater than half the area $TEAH$.

Similarly, if the arc $AH$ be bisected and the tangent at the point of bisection be drawn, it will cut off from the area $GAH$ more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of $K$ over the area of the circle.

Thus the area of the polygon will be less than $K$.

Now, since the perpendicular from $O$ on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle $K$; which is impossible.

Therefore the area of the circle is not less than $K$.

Since then the area of the circle is neither greater nor less than $K$, it is equal to it.

## Proposition 2

*The area of a circle is to the square on its diameter as 11 to 14.*[1]

## Proposition 3

*The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.*[2]

I. Let $AB$ be the diameter of any circle, $O$ its centre, $AC$ the tangent at $A$; and let the angle $AOC$ be one-third of a right angle.

Then $$OA : AC\ [= \sqrt{3} : 1] > 265 : 153, \qquad (1)$$

and $$OC : CA\ [= 2 : 1] = 306 : 153. \qquad (2)$$

First, draw $OD$ bisecting the angle $AOC$ and meeting $AC$ in $D$.

Now $$CO : OA = CD : DA, \qquad \text{[Eucl. VI. 3]}$$

so that $$[CO + OA : OA = CA : DA, \text{ or}]$$ $$CO + OA : CA = OA : AD.$$

Therefore [by (1) and (2)]

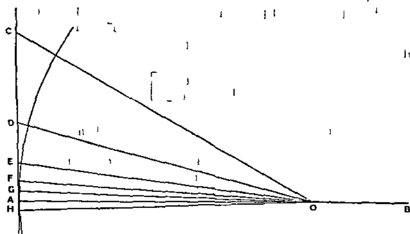
$$OA : AD > 571 : 153. \qquad (3)$$

[1] The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.

[2] In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated.

Hence $OD^2 \; AD^2 [=(OA^2 + AD^2) \; AD^2]$
$> (571^2 + 153^2) \; 153^2]$
$> 349450 \; 23409,$

so that $OD \; DA > 591\frac{1}{8} \; 153$ (4)




*Secondly* let $OE$ bisect the angle $AOD$, meeting $AD$ in $E$

[Then $DO \; OA = DE \; EA$,

so that $DO + OA \; DA = OA \; AE$.]

Therefore $OA \; AE [> (591\frac{1}{8} + 571) \; 153$, by (3) and (4)]
$> 1162\frac{1}{8} \; 153$ (5)

[It follows that

$OE^2 \; EA^2 > \{(1162\frac{1}{8})^2 + 153^2\} \; 153^2$
$> (1350534\frac{33}{64} + 23409) \; 23409$
$> 1373943\frac{33}{64} \; 23409.]$

Thus $OE \; EA > 1172\frac{1}{8} \; 153$ (6)

*Thirdly* let $OF$ bisect the angle $AOE$ and meet $AE$ in $F$

We thus obtain the result [corresponding to (3) and (5) above] that

$OA \; AF [> (1162\frac{1}{8} + 1172\frac{1}{8}) \; 153]$
$> 2334\frac{1}{4} \; 153$ (7)

[Therefore $OF^2 \; FA^2 > \{(2334\frac{1}{4})^2 + 153^2\} \; 153^2$
$> 5472132\frac{1}{16} \; 23409.]$

Thus $OF \; FA > 2339\frac{1}{4} \; 153$ (8)

*Fourthly* let $OG$ bisect the angle $AOF$ meeting $AF$ in $G$

We have then

$OA \; AG [> (2334\frac{1}{4} + 2339\frac{1}{4}) \; 153$ by means of (7) and (8)]
$> 4673\frac{1}{2} \; 153$

Now the angle $AOC$ which is one third of a right angle has been bisected four times and it follows that

$\angle AOG = \frac{1}{48}$ (a right angle)

Make the angle $AOH$ on the other side of $OA$ equal to the angle $AOG$ and let $GA$ produced meet $OH$ in $H$

Then $\angle GOH = \frac{1}{24}$ (a right angle)

Thus $GH$ is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $OA : AG > 4673\frac{1}{2} : 153$,

while $AB = 2OA,\ GH = 2AG,$

it follows that

$AB :$ (perimeter of polygon of 96 sides) $[> 4673\frac{1}{2} : 153 \times 96]$

$> 4673\frac{1}{2} : 14688$.

But
$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}}$$
$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$$
$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter $AB$.

II. Next let $AB$ be the diameter of a circle, and let $AC$, meeting the circle in $C$, make the angle $CAB$ equal to one-third of a right angle. Join $BC$.

Then $AC : CB [= \sqrt{3} : 1] < 1351 : 780$.

*First*, let $AD$ bisect the angle $BAC$ and meet $BC$ in $d$ and the circle in $D$. Join $BD$.

Then
$$\angle BAD = \angle dAC$$
$$= \angle dBD,$$

and the angles at $D$, $C$ are both right angles.

It follows that the triangles $ADB$, $[ACd]$, $BDd$ are similar.

Therefore $AD : DB = BD : Dd$

$[= AC : Cd]$

$= AB : Bd$ [Eucl. VI. 3]

$= AB + AC : Bd + Cd$

$= AB + AC : BC$

or $BA + AC : BC = AD : DB$.

[But $AC : CB < 1351 : 780$, from above,

while $BA : BC = 2 : 1$

$= 1560 : 780$.]

Therefore $AD : DB < 2911 : 780$. (1)

[Hence $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$< 9082321 : 608400$.]

Thus $AB : BD < 3013\frac{3}{4} : 780$. (2)

*Secondly*, let $AE$ bisect the angle $BAD$, meeting the circle in $E$; and let $BE$ be joined.

Then we prove, in the same way as before, that

$AE : EB [= BA + AD : BD$

$< (3013\frac{3}{4} + 2911) : 780$, by (1) and (2)]

$< 5924\frac{3}{4} : 780$

$< 5924\frac{3}{4} \times \frac{4}{13} : 780 \times \frac{4}{13}$

$< 1823 : 240$. (3)

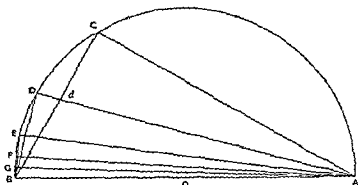
[Hence $AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$

$< 3380929 : 57600$.]

Therefore $AB : BE < 1838\frac{9}{11} : 240$. (4)

*Thirdly*, let $AF$ bisect the angle $BAE$, meeting the circle in $F$.

Thus $AF \ FB [= BA + AE \ BE$
$< 3661\tfrac{9}{11} \ 240$ by (3) and (4)]
$< 3661\tfrac{9}{11} \times \tfrac{11}{40} \ 240 \times \tfrac{11}{40}$
$< 1007 \ 66$ (5)




[It follows that

$$AB^2 \ BF^2 < (1007^2 + 66^2) \ 66^2$$
$$< 1018405 \ 4356 ]$$

Therefore $AB \ BF < 1009\tfrac{1}{6} \ 66$ (6)

*Fourthly*, let the angle $BAF$ be bisected by $AG$ meeting the circle in $G$

Then $AG \ GB [= BA + AF \ BF]$
$< 2016\tfrac{1}{6} \ 66$, by (5) and (6)

[And $AB^2 \ BG^2 < \{(2016\tfrac{1}{6})^2 + 66^2\} \ 66^2$
$< 4069284\tfrac{1}{36} \ 4356 ]$

Therefore $AB \ BG < 2017\tfrac{1}{4} \ 66$

whence $BG \ AB > 66 \ 2017\tfrac{1}{4}$ (7)

[Now the angle $BAG$ which is the result of the fourth bisection of the angle $BAC$ or of one-third of a right angle is equal to one-forty eighth of a right angle

Thus the angle subtended by $BG$ at the centre is
$\tfrac{1}{24}$ (a right angle) ]

Therefore $BG$ is a side of a regular inscribed polygon of 96 sides

It follows from (7) that

(perimeter of polygon) $AB [> 96 \times 66 \ 2017\tfrac{1}{4}]$
$> 6336 \ 2017\tfrac{1}{4}$

And $$\frac{6336}{2017\tfrac{1}{4}} > 3\tfrac{10}{71}$$

Much more then is the circumference of the circle greater than $3\tfrac{10}{71}$ times the diameter

Thus the ratio of the circumference to the diameter
$< 3\tfrac{1}{7}$ but $> 3\tfrac{10}{71}$

# ON CONOIDS AND SPHEROIDS

## INTRODUCTION[1]

"ARCHIMEDES to Dositheus greeting

"In this book I have set forth and send you the proofs of the remaining theorems not included in what I sent you before and also of some others discovered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty And this is why even the propositions themselves were not published with the rest But afterwards, when I had studied them with greater care, I discovered what I had failed in before

"Now the remainder of the earlier theorems were propositions concerning the right angled conoid [paraboloid of revolution] but the discoveries which I have now added relate to an obtuse-angled conoid [hyperboloid of revolution] and to spheroidal figures some of which I call *oblong* and others *flat*"

I Concerning the *right-angled conoid* it was laid down that, if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the right angled cone is called a *right-angled conoid* and the diameter which has remained fixed is called its *axis*, while its *vertex* is the point in which the axis meets the surface of the conoid And if a plane touch the right-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid the *base* of the segment cut off is defined as the portion intercepted by the section of the conoid on the cutting plane the *vertex* [of the segment] as the point in which the first plane touches the conoid and the *axis* [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment parallel to the axis of the conoid

'The questions propounded for consideration were'

(1) why if a segment of the right angled conoid be cut off by a plane at right angles to the axis will the segment so cut off be half as large again as the cone which has the same base as the segment and the same axis, and

(2) 'why if two segments be cut off from the right angled conoid by planes drawn in any manner will the segments so cut off have to one another the duplicate ratio of their axes

II "Respecting the *obtuse-angled conoid* we lay down the following premisses If there be in a plane a section of an obtuse-angled cone [a hyperbola], its

[1]The whole of this introductory matter including the definitions is translated literally from the Greek text in order that the terminology of Archimedes may be faithfully represented When this has once been set out nothing will be lost by returning to modern phraseology and notation These will accordingly be employed as usual, when we come to the actual propositions of the treatise

diameter [axis], and the nearest lines to the section of the obtuse-angled cone [i e the asymptotes of the hyperbola], and if, the diameter [axis] remaining fixed, the plane containing the aforesaid lines be made to revolve about it and return to the position from which it started the nearest lines to the section of the obtuse-angled cone [the asymptotes] will clearly comprehend an isosceles cone whose vertex will be the point of concourse of the nearest lines and whose axis will be the diameter [axis] which has remained fixed The figure comprehended by the section of the obtuse-angled cone is called an *obtuse-angled conoid* [hyperboloid of revolution] its *axis* is the diameter which has remained fixed, and its *vertex* the point in which the axis meets the surface of the conoid The cone comprehended by the nearest lines to the section of the obtuse-angled cone is called [the cone] *enveloping the conoid* and the straight line between the vertex of the conoid and the vertex of the cone enveloping the conoid is called [the line] *adjacent to the axis* And if a plane touch the obtuse-angled conoid and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the *base* of the segment so cut off is defined as the portion intercepted by the section of the conoid on the cutting plane the *vertex* [of the segment] as the point of contact of the plane which touches the conoid the *axis* [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment and the vertex of the cone enveloping the conoid, and the straight line between the said vertices is called *adjacent to the axis*

"Right angled conoids are all similar, but of obtuse-angled conoids let those be called similar in which the cones enveloping the conoids are similar

The following questions are propounded for consideration"

(1) 'why, if a segment be cut off from the obtuse-angled conoid by a plane at right angles to the axis, the segment so cut off has to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis and '

(2) 'why, if a segment of the obtuse-angled conoid be cut off by a plane not at right angles to the axis, the segment so cut off will bear to the figure which has the same base as the segment and the same axis, being a segment of a cone, the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis

III Concerning spheroidal figures we lay down the following premisses If a section of an acute-angled cone [ellipse] be made to revolve about the greater diameter [major axis] which *remains fixed and return to the position from* which it started, the figure comprehended by the section of the acute-angled cone is called an *oblong spheroid* But if the section of the acute-angled cone revolve about the lesser diameter [minor axis] which remains fixed and return to the position from which it started the figure comprehended by the section of the acute-angled cone is called a *flat spheroid* In either of the spheroids the *axis* is defined as the diameter [axis] which has remained fixed the *vertex* as the point in which the axis meets the surface of the spheroid the *centre* as the middle point of the axis and the *diameter* as the line drawn through the centre at right angles to the axis And if parallel planes touch without cutting, either

of the spheroidal figures, and if another plane be drawn parallel to the tangent planes and cutting the spheroid, the *base* of the resulting segments is defined as the portion intercepted by the section of the spheroid on the cutting plane, their *vertices* as the points in which the parallel planes touch the spheroid and their *axes* as the portions cut off within the segments from the straight line joining their vertices And that the planes touching the spheroid meet its surface at one point only and that the straight line joining the points of contact passes through the centre of the spheroid we shall prove Those spheroidal figures are called *similar* in which the axes have the same ratio to the 'diameters' And let segments of spheroidal figures and conoids be called *similar* if they are cut off from similar figures and have their bases similar, while their axes, being either at right angles to the planes of the bases or making equal angles with the corresponding diameters [axes] of the bases have the same ratio to one another as the corresponding diameters [axes] of the bases

"The following questions about spheroids are propounded for consideration

(1) 'why if one of the spheroidal figures be cut by a plane through the centre at right angles to the axis each of the resulting segments will be double of the cone having the same base as the segment and the same axis, while if the plane of section be at right angles to the axis without passing through the centre (*a*) the greater of the resulting segments will bear to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the lesser segment bears to the axis of the lesser segment and (*b*) the lesser segment bears to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the greater segment bears to the axis of the greater segment

(2) why if one of the spheroids be cut by a plane passing through the centre but not at right angles to the axis each of the resulting segments will be double of the figure having the same base as the segment and the same axis and consisting of a segment of a cone

(3) But if the plane cutting the spheroid be neither through the centre nor at right angles to the axis (*a*) the greater of the resulting segments will have to the figure which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the lesser segment bears to the axis of the lesser segment and (*b*) the lesser segment will have to the figure with the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the greater segment bears to the axis of the greater segment And the figure referred to is in these cases also a segment of a cone

'When the aforesaid theorems are proved there are discovered by means of them many theorems and problems

'Such for example are the theorems

(1) 'that similar spheroids and similar segments both of spheroidal figures and conoids have to one another the triplicate ratio of their axes and

(2) 'that in equal spheroidal figures the squares on the 'diameters' are reciprocally proportional to the axes and if in spheroidal figures the squares on

the 'diameters' are reciprocally proportional to the axes, the spheroids are equal

Such also is the problem, From a given spheroidal figure or conoid to cut off a segment by a plane drawn parallel to a given plane so that the segment cut off is equal to a given cone or cylinder or to a given sphere

"After prefixing therefore the theorems and directions which are necessary for the proof of them, I will then proceed to expound the propositions them selves to you Farewell '

## DEFINITIONS

'If a cone be cut by a plane meeting all the sides [generators] of the cone the section will be either a circle or a section of an acute-angled cone [an ellipse] If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone But if the section be a section of an acute-angled cone [an ellipse] let the figure cut off from the cone towards the same parts as the vertex of the cone be called a *segment of a cone* Let the *base* of the segment be defined as the plane compre hended by the section of the acute-angled cone its *vertex* as the point which is also the vertex of the cone, and its *axis* as the straight line joining the vertex of the cone to the centre of the section of the acute-angled cone

'And if a cylinder be cut by two parallel planes meeting all the sides [gen erators] of the cylinder the sections will be either circles or sections of acute-angled cones [ellipses] equal and similar to one another If then the sections be circles it is clear that the figure cut off from the cylinder between the parallel planes will be a cylinder But, if the sections be sections of acute-angled cones [ellipses] let the figure cut off from the cylinder between the parallel planes be called a *frustum of a cylinder* And let the *bases* of the frustum be defined as the planes comprehended by the sections of the acute-angled cones [ellipses] and the *axis* as the straight line joining the centres of the sections of the acute-angled cones so that the axis will be in the same straight line with the axis of the cylinder '

### Lemma

*If in an ascending arithmetical progression consisting of the magnitudes* $A_1$, $A_2$ $A$ *the common difference be equal to the least term* $A_1$, *then*

$$n\ A_n < 2(A_1 + A_2 + \quad + A\ ),$$

*and*

$$> 2(A_1 + A_2 + \quad + A_{\ -1})$$

[The proof of this is given incidentally in the treatise *On Spirals* Prop 11 By placing lines side by side to represent the terms of the progression and then producing each so as to make it equal to the greatest term Archimedes gives the equivalent of the following proof

If $S_n = A_1 + A_2 + \quad + A_{n-1} + A$

we have also $S = A_n + A_{\ -1} + A_{\ -2} + \quad + A_1$

And $A_1 + A_{n-1} = A_2 + A_{n-2} = \quad = A_n$

Therefore $2S = (n+1)A\ ,$

whence $n\ A_n < 2S$

and $n\ A > 2S_{n-1}$

Thus if the progression is $a$ $2a$, $na$

$$S = \frac{n(n+1)}{2}a,$$

and $n^2 a < 2S$

but $> 2S_{n-1}$.]

## Proposition 1

*If $A_1, B_1, C_1, \dots K_1$ and $A_2, B_2, C_2, \dots K_2$ be two series of magnitudes such that*

$$\left.\begin{aligned} A_1 : B_1 &= A_2 : B_2 \\ B_1 : C_1 &= B_2 : C_2, \text{ and so on} \end{aligned}\right\} \quad (\alpha)$$

*and if $A_3, B_3, C_3, \dots K_3$ and $A_4, B_4, C_4, \dots K_4$ be two other series such that*

$$\left.\begin{aligned} A_1 : A_3 &= A_2 : A_4 \\ B_1 : B_3 &= B_2 : B_4, \text{ and so on} \end{aligned}\right\} \quad (\beta)$$

*then* $$(A_1+B_1+C_1+\dots+K_1) : (A_3+B_3+C_3+\dots+K_3)$$
$$=(A_2+B_2+C_2+\dots+K_2) : (A_4+B_4+\dots+K_4).$$

The proof is as follows.

Since $A_3 : A_1 = A_4 : A_2$,

and $A_1 : B_1 = A_2 : B_2$,

while $B_1 : B_3 = B_2 : B_4$,

$$\left.\begin{aligned} &\text{we have, } ex\ aequali, & A_3 : B_3 &= A_4 : B_4 \\ &\text{Similarly} & B_3 : C_3 &= B_4 : C_4, \text{ and so on} \end{aligned}\right\} \quad (\gamma)$$

Again, it follows from equations ($\alpha$) that

$$A_1 : A_2 = B_1 : B_2 = C_1 : C_2 = \dots$$

Therefore

$$A_1 : A_2 = (A_1+B_1+C_1+\dots+K_1) : (A_2+B_2+\dots+K_2),$$

or $$(A_1+B_1+C_1+\dots+K_1) : A_1 = (A_2+B_2+C_2+\dots+K_2) : A_2;$$

and $$A_1 : A_3 = A_2 : A_4;$$

while from equations ($\gamma$) it follows in like manner that

$$A_3 : (A_3+B_3+C_3+\dots+K_3) = A_4 : (A_4+B_4+C_4+\dots+K_4).$$

By the last three equations, *ex aequali*,

$$(A_1+B_1+C_1+\dots+K_1) : (A_3+B_3+C_3+\dots+K_3)$$
$$=(A_2+B_2+C_2+\dots+K_2) : (A_4+B_4+C_4+\dots+K_4).$$

Cor. If any terms in the third and fourth series corresponding to terms in the first and second be left out, the result is the same. For example, if the last terms $K_3$, $K_4$ are absent,

$$(A_1+B_1+C_1+\dots+K_1) : (A_3+B_3+C_3+\dots+I_3)$$
$$=(A_2+B_2+C_2+\dots+K_2) : (A_4+B_4+C_4+\dots+I_4),$$

where $I$ immediately precedes $K$ in each series.

## Lemma to Proposition 2

[*On Spirals*, Prop. 10.]

*If $A_1, A_2, A_3, \dots A_n$ be n lines forming an ascending arithmetical progression in which the common difference is equal to the least term $A_1$, then*

$$(n+1)A_n^2 + A_1(A_1+A_2+A_3+\dots+A_n) = 3(A_1^2+A_2^2+A_3^2+\dots+A_n^2).$$

Let the lines $A_n, A_{n-1}, \dots A_1$ be placed in a row from left to right. Produce $A_{n-1}, A_{n-2}, \dots A_1$ until they are each equal to $A_n$, so that the parts produced are respectively equal to $A_1, A_2, \dots A_{n-1}$.

Taking each line successively, we have

$$2A_n^2 = 2A_n^2,$$
$$(A_1 + A_{n-1})^2 = A_1^2 + A_{n-1}^2 + 2A_1 \cdot A_{n-1},$$
$$(A_2 + A_{n-2})^2 = A_2^2 + A_{n-2}^2 + 2A_2 \cdot A_{n-2},$$

$$(A_{n-1}+A_1)^2 = A_{n-1}^2 + A_1^2 + 2A_{n-1}\cdot A_1.$$

And, by addition

$$(n+1)A_n^2 = 2(A_1^2+A_2^2+\dots+A_n^2)$$
$$+2A_1\cdot A_{n-1}+2A_2\cdot A_{n-2}+\dots+2A_{n-1}\cdot A_1.$$

Therefore in order to obtain the required result, we have to prove that

$$2(A_1\cdot A_{n-1}+A_2\cdot A_{n-2}+\dots+A_{n-1}\cdot A_1)+A_1(A_1+A_2+A_3+\dots+A_n)$$
$$=A_1^2+A_2^2+\dots+A_n^2 \qquad (\alpha)$$

Now

$2A_2\cdot A_{n-2} = A_1\cdot 4A_{n-2}$, because $A_2=2A_1$,

$2A_3\cdot A_{n-3} = A_1\cdot 6A_{n-3}$, because $A_3=3A_1$,

$2A_{n-1}\cdot A_1 = A_1\cdot 2(n-1)A_1$.

It follows that

$$2(A_1\cdot A_{n-1}+A_2\cdot A_{n-2}+\dots+A_{n-1}\cdot A_1)+A_1(A_1+A_2+\dots+A_n)=A_1\{A_n+3A_{n-1}+5A_{n-2}+\dots+(2n-1)A_1\}.$$

And this last expression can be proved to be equal to

$$A_1^2+A_2^2+\dots+A_n^2.$$

For $\quad A_n^2 = A_1(n\cdot A_n)$

$= A_1\{A_n+(n-1)A_n\}$

$= A_1\{A_n+2(A_{n-1}+A_{n-2}+\dots+A_1)\}$,

because $(n-1)A_n = A_{n-1}+A_1$
$+A_{n-2}+A_2$
$+\dots$
$+A_1+A_{n-1}$.

Similarly $\quad A_{n-1}^2 = A_1\{A_{n-1}+2(A_{n-2}+A_{n-3}+\dots+A_1)\}$,

$A_2^2 = A_1(A_2+2A_1)$,

$A_1^2 = A_1\cdot A_1$;

whence, by addition,

$$A_1^2+A_2^2+A_3^2+\dots+A_n^2 = A_1\{A_n+3A_{n-1}+5A_{n-2}+\dots+(2n-1)A_1\}.$$

Thus the equation marked $(\alpha)$ above is true; and it follows that

$$(n+1)A_n^2+A_1(A_1+A_2+A_3+\dots+A_n)=3(A_1^2+A_2^2+\dots+A_n^2).$$

COR. 1. *From this it is evident that*

$$n\cdot A_n^2 < 3(A_1^2+A_2^2+\dots+A_n^2). \qquad (1)$$

Also $\quad A_n^2 = A_1\{A_n+2(A_{n-1}+A_{n-2}+\dots+A_1)\}$, as above,

so that $\quad A_n^2 > A_1(A_n+A_{n-1}+\dots+A_1)$,

and therefore

$$A_n^2 + A_1(A_1+A_2+\dots+A_n) < 2A_n^2.$$

*It follows from the proposition that*

$$n\cdot A_n^2 > 3(A_1^2+A_2^2+\dots+A_{n-1}^2). \qquad (2)$$

COR. 2. All these results will hold if we substitute *similar figures* for squares on all the lines; for similar figures are in the duplicate ratio of their sides.

PROPOSITION 2

*If $A_1, A_2, \ldots A_n$ be any number of areas such that*

$$A_1 = ax + x^2,$$
$$A_2 = a \cdot 2x + (2x)^2,$$
$$A_3 = a \cdot 3x + (3x)^2,$$
$$\ldots\ldots\ldots$$
$$A_n = a \cdot nx + (nx)^2,$$

*then* $$n \cdot A_n : (A_1 + A_2 + \ldots + A_n) < (a + nx) : \left(\frac{a}{2} + \frac{nx}{3}\right),$$

*and* $$n \cdot A_n : (A_1 + A_2 + \ldots + A_{n-1}) > (a + nx) : \left(\frac{a}{2} + \frac{nx}{3}\right).$$

For by the Lemma immediately preceding Prop. 1,

$$n \cdot anx < 2(ax + a \cdot 2x + \ldots + a \cdot nx)$$

and $$> 2(ax + a \cdot 2x + \ldots + a \cdot \overline{n-1}x).$$

Also by the Lemma preceding this proposition

$$n \cdot (nx)^2 < 3\{x^2 + (2x)^2 + (3x)^2 + \ldots + (nx)^2\}$$

and $$> 3\{x^2 + (2x)^2 + \ldots + (\overline{n-1}x)^2\}.$$

Hence

$$\frac{an^2x}{2} + \frac{n(nx)^2}{3} < [(ax + x^2) + \{a \cdot 2x + (2x)^2\} + \ldots + \{a \cdot nx + (nx)^2\}],$$

and

$$> [(ax + x^2) + \{a \cdot 2x + (2x)^2\} + \ldots + \{a \cdot \overline{n-1}x + (\overline{n-1}x)^2\}],$$

or $$\frac{an^2x}{2} + \frac{n(nx)^2}{3} < A_1 + A_2 + \ldots + A_n$$

and $$> A_1 + A_2 + \ldots + A_{n-1}.$$

It follows that

$$n \cdot A_n : (A_1 + A_2 + \ldots + A_n) < n\{a \cdot nx + (nx)^2\} : \left\{\frac{an^2x}{2} + \frac{n(nx)^2}{3}\right\},$$

or $$n \cdot A_n : (A_1 + A_2 + \ldots + A_n) < (a + nx) : \left(\frac{a}{2} + \frac{nx}{3}\right);$$

also $$n \cdot A_n : (A_1 + A_2 + \ldots + A_{n-1}) > (a + nx) : \left(\frac{a}{2} + \frac{nx}{3}\right).$$

PROPOSITION 3

(1) *If $TP$, $TP'$ be two tangents to any conic meeting in $T$, and if $Qq$, $Q'q'$ be any two chords parallel respectively to $TP$, $TP'$ and meeting in $O$, then*

$$QO \cdot Oq : Q'O \cdot Oq' = TP^2 : TP'^2.$$

"And this is proved in the elements of conics."[1]

(2) *If $QQ'$ be a chord of a parabola bisected in $V$ by the diameter $PV$, and if $PV$ be of constant length, then the areas of the triangle $PQQ'$ and of the segment $PQQ'$ are both constant whatever be the direction of $QQ'$.*

Let $ABB'$ be the particular segment of the parabola whose vertex is $A$, so that $BB'$ is bisected perpendicularly by the axis at the point $H$, where $AH = PV$.

Draw $QD$ perpendicular to $PV$.

[1] In the treatises on conics by Aristaeus and Euclid.

Let $p$ be the parameter of the principal ordinates and let $p$ be another line of such length that

$$QV \quad QD = p \quad p ,$$

it will then follow that $p$ is equal to the parameter of the ordinates to the diameter $PV$ i e those which are parallel to $QV$

"For this is proved in the conics [1]

Thus $QV^2 = p \; PV$

And $BH^2 = p \; AH$ while $AH = PV$

Therefore $QV^2 \quad BH = p \quad p$

But $QV^2 \quad QD = p \quad p$ ,

hence $BH = QD$

Thus $BH \; AH = QD \; PV$

and therefore $\triangle ABB = \triangle PQQ$ ,

that is the area of the triangle $PQQ$ is constant so long as $PV$ is of constant length

Hence also the area of the segment $PQQ$ is constant under the same conditions for the segment is equal to $\frac{4}{3}\triangle PQQ$ [*Quadrature of the Parabola*, Prop 17 or 24]

## Proposition 4

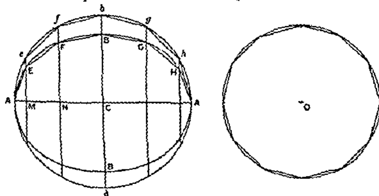
*The area of any ellipse is to that of the auxiliary circle as the minor axis to the major*

Let $AA$ be the major and $BB$ the minor axis of the ellipse and let $BB$ meet the auxiliary circle in $b$ $b$

Suppose $O$ to be such a circle that

$$(\text{circle } AbA\,b) \quad O = CA \quad CB$$

Then shall $O$ be equal to the area of the ellipse




For if not $O$ must be either greater or less than the ellipse

I If possible let $O$ be greater than the ellipse

We can then inscribe in the circle $O$ an equilateral polygon of $4n$ sides such that its area is greater than that of the ellipse [cf *On the Sphere and Cylinder* I 6]

[1] The theorem which is here assumed by Archimedes as known is easily deduced from Apollonius I 49

Let this be done and inscribe in the auxiliary circle of the ellipse the polygon $AefbghA'$ similar to that inscribed in $O$. Let the perpendiculars $eM$, $fN$, on $AA'$ meet the ellipse in $E$, $F$, respectively. Join $AF$, $EF$, $FB$,

Suppose that $P'$ denotes the area of the polygon inscribed in the auxiliary circle and $P$ that of the polygon inscribed in the ellipse.

Then, since all the lines $eM$, $fN$, are cut in the same proportions at $E$, $F$,

i.e. $eM : EM = fN : FN = \ldots = bC : BC$,

the pairs of triangles, as $eAM$, $EAM$, and the pairs of trapeziums, as $eMNf$, $EMNF$, are all in the same ratio to one another as $bC$ to $BC$, or as $CA$ to $CB$.

Therefore, by addition,

$$P' : P = CA : CB$$

Now $P'$ : (polygon inscribed in $O$)

$$= (\text{circle } AbA'b') : O$$
$$= CA : CB, \text{ by hypothesis}$$

Therefore $P$ is equal to the polygon inscribed in $O$.

But this is impossible, because the latter polygon is by hypothesis greater than the ellipse, and *a fortiori* greater than $P$.

Hence $O$ is not greater than the ellipse.

II. If possible, let $O$ be less than the ellipse.

In this case we inscribe in the *ellipse* a polygon $P$ with $4n$ equal sides such that $P > O$.

Let the perpendiculars from the angular points on the axis $AA'$ be produced to meet the auxiliary circle, and let the corresponding polygon ($P'$) in the circle be formed.

Inscribe in $O$ a polygon similar to $P'$.

Then $\quad P' : P = CA : CB$

$$= (\text{circle } AbA'b') : O, \text{ by hypothesis},$$
$$= P' : (\text{polygon inscribed in } O).$$

Therefore the polygon inscribed in $O$ is equal to the polygon $P$, which is impossible, because $P > O$.

Hence $O$, being neither greater nor less than the ellipse, is equal to it, and the required result follows.

### Proposition 5

*If $AA'$, $BB'$ be the major and minor axis of an ellipse respectively, and if $d$ be the diameter of any circle, then*

$$(\textit{area of ellipse}) : (\textit{area of circle}) = AA' \cdot BB' : d^2$$

For

$$(\text{area of ellipse}) : (\text{area of auxiliary circle}) = BB' : AA' \quad \text{[Prop. 4]}$$
$$= AA' \cdot BB' : AA'^2$$

And

$$(\text{area of aux. circle}) : (\text{area of circle with diam. } d) = AA'^2 : d^2$$

Therefore the required result follows *ex aequali*.

### Proposition 6

*The areas of ellipses are as the rectangles under their axes.*

This follows at once from Props. 4, 5.

Cor. *The areas of similar ellipses are as the squares of corresponding axes.*

### Proposition 7

*Given an ellipse with centre C, and a line CO drawn perpendicular to its plane it is possible to find a circular cone with vertex O and such that the given ellipse is a section of it [or, in other words to find the circular sections of the cone with vertex O passing through the circumference of the ellipse]*

Conceive an ellipse with $BB'$ as its minor axis and lying in a plane perpendicular to that of the paper Let $CO$ be drawn perpendicular to the plane of the ellipse and let $O$ be the vertex of the required cone Produce $OB$ $OC$ $OB'$ and in the same plane with them draw $BED$ meeting $OC$ $OB$ produced in $E$, $D$ respectively and in such a direction that

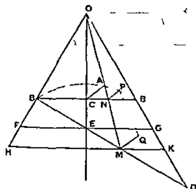
$$BE \quad ED \quad EO^2 = CA^2 \quad CO,$$

where $CA$ is half the major axis of the ellipse

And this is possible, since

$$BE \quad ED \quad EO > BC \cdot CB \quad CO^2$$

[Both the construction and this proposition are assumed as known ]




Now conceive a circle with $BD$ as diameter lying in a plane at right angles to that of the paper, and describe a cone with this circle for its base and with vertex $O$

We have therefore to prove that the given ellipse is a section of the cone or if $P$ be any point on the ellipse, that $P$ lies on the surface of the cone

Draw $PN$ perpendicular to $BB'$ Join $ON$ and produce it to meet $BD$ in $M$ and let $MQ$ be drawn in the plane of the circle on $BD$ as diameter perpendicular to $BD$ and meeting the circle in $Q$ Also let $FG$, $HK$ be drawn through $E$, $M$ respectively parallel to $BB'$

We have then

$$\begin{aligned} QM^2 \quad HM \quad MK &= BM \quad MD \quad HM \quad MK \\ &= BE \quad ED \quad FE \quad EG \\ &= (BE \quad ED \quad EO)\ (EO^2 \quad FE \quad EG) \\ &= (CA^2 \quad CO)\ (CO^2 \quad BC \quad CB) \\ &= CA^2 \quad CB^2 \\ &= PN^2 \quad BN \quad NB \end{aligned}$$

Therefore

$$\begin{aligned} QM^2 \quad PN^2 &= HM \quad MK \quad BN \quad NB \\ &= OM^2 \quad ON^2 \end{aligned}$$

whence since $PN$ $QM$ are parallel $OPQ$ is a straight line

But $Q$ is on the circumference of the circle on $BD$ as diameter, therefore $OQ$ is a generator of the cone and hence $P$ lies on the cone

Thus the cone passes through all points on the ellipse

### Proposition 8

*Given an ellipse, a plane through one of its axes $AA'$ and perpendicular to the plane of the ellipse and a line CO drawn from C the centre in the given plane through $AA'$ but not perpendicular to AA it is possible to find a cone with vertex*

*O such that the given ellipse is a section of it [or, in other words, to find the circular sections of the cone with vertex O whose surface passes through the circumference of the ellipse].*

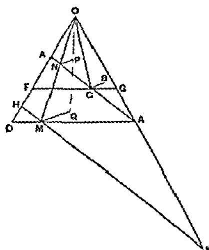
By hypothesis $OA$, $OA'$ are unequal. Produce $OA'$ to $D$ so that $OA = OD$. Join $AD$, and draw $FG$ through $C$ parallel to it.

The given ellipse is to be supposed to lie in a plane perpendicular to the plane of the paper. Let $BB'$ be the other axis of the ellipse.

Conceive a plane through $AD$ perpendicular to the plane of the paper, and in it describe either (a), if $CB^2 = FC \cdot CG$, a circle with diameter $AD$, or (b), if not, an ellipse on $AD$ as axis such that, if $d$ be the other axis,

$$d^2 : AD^2 = CB^2 : FC \cdot CG.$$

Take a cone with vertex $O$ whose surface passes through the circle or ellipse just drawn. This is possible even when the curve is an ellipse, because the line from $O$ to the middle point of $AD$ is perpendicular to the plane of the ellipse, and the construction is effected by means of Prop. 7.




Let $P$ be any point on the given ellipse, and we have only to prove that $P$ lies on the surface of the cone so described.

Draw $PN$ perpendicular to $AA'$. Join $ON$, and produce it to meet $AD$ in $M$. Through $M$ draw $HK$ parallel to $A'A$.

Lastly, draw $MQ$ perpendicular to the plane of the paper (and therefore perpendicular to both $HK$ and $AD$) meeting the ellipse or circle about $AD$ (and therefore the surface of the cone) in $Q$.

Then

$$QM^2 : HM \cdot MK = (QM^2 : DM \cdot MA) \cdot (DM \cdot MA : HM \cdot MK)$$
$$= (d^2 : AD^2) \cdot (FC \cdot CG : A'C \cdot CA)$$
$$= (CB^2 : FC \cdot CG) \cdot (FC \cdot CG : A'C \cdot CA)$$
$$= CB^2 : CA^2$$
$$= PN^2 : A'N \cdot NA.$$

Therefore, alternately,

$$QM^2 : PN^2 = HM \cdot MK : A'N \cdot NA$$
$$= OM^2 : ON^2.$$

Thus, since $PN$, $QM$ are parallel, $OPQ$ is a straight line; and, $Q$ being on the surface of the cone, it follows that $P$ is also on the surface of the cone.

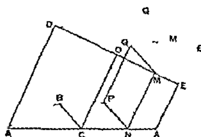
Similarly all points on the ellipse are also on the cone, and the ellipse is therefore a section of the cone.

## Proposition 9

*Given an ellipse, a plane through one of its axes and perpendicular to that of the ellipse, and a straight line CO drawn from the centre C of the ellipse in the given plane through the axis but not perpendicular to that axis, it is possible to find a*

*cylinder with axis OC such that the ellipse is a section of it [or in other words, to find the circular sections of the cylinder with axis OC whose surface passes through the circumference of the given ellipse]*

Let $AA'$ be an axis of the ellipse and suppose the plane of the ellipse to be perpendicular to that of the paper, so that $OC$ lies in the plane of the paper




Draw $AD$ $A'E$ parallel to $CO$, and let $DE$ be the line through $O$ perpendicular to both $AD$ and $A'E$

We have now three different cases according as the other axis $BB'$ of the ellipse is (1) equal to, (2) greater than or (3) less than, $DE$

(1) Suppose $BB' = DE$

Draw a plane through $DE$ at right angles to $OC$ and in this plane describe a circle on $DE$ as diameter Through this circle describe a cylinder with axis $OC$

This cylinder shall be the cylinder required or its surface shall pass through every point $P$ of the ellipse

For if $P$ be any point on the ellipse draw $PN$ perpendicular to $AA'$, through $N$ draw $NM$ parallel to $CO$ meeting $DE$ in $M$ and through $M$, in the plane of the circle on $DE$ as diameter draw $MQ$ perpendicular to $DE$ meeting the circle in $Q$

Then since $$DE = BB'$$

$$PN^2 \quad AN \quad NA' = DO^2 \quad AC \quad CA'$$

And $$DM \quad ME \quad AN \quad NA' = DO^2 \quad AC^2,$$

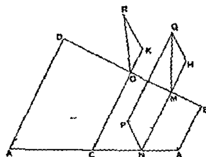
since $AD$ $NM$ $CO$ $A'E$ are parallel

Therefore $$PN^2 = DM \quad ME$$
$$= QM^2$$

by the property of the circle

Hence since $PN$ $QM$ are equal as well as parallel $PQ$ is parallel to $MN$ and therefore to $CO$ It follows that $PQ$ is a generator of the cylinder, whose surface accordingly passes through $P$

(2) If $BB' > DE$ we take $E'$ on $A'E$ such that $DE' = BB'$ and describe a circle on $DE'$ as diameter in a plane perpendicular to that of the paper, and the rest of the construction and proof is exactly similar to those given for case (1)




(3) Suppose $BB' < DE$

Take a point $K$ on $CO$ produced such that
$$DO^2 - CB^2 = OK^2$$

From $K$ draw $KR$ perpendicular to the plane of the paper and equal to $CB$

Thus $OR^2 = OK^2 + CB^2 = OD^2$

In the plane containing $DE$ $OR$ describe a circle on $DE$ as diameter Through this circle (which must pass through $R$) draw a cylinder with axis $OC$

We have then to prove that if $P$ be any point on the given ellipse $P$ lies on the cylinder so described

Draw $PN$ perpendicular to $AA'$, and through $N$ draw $NM$ parallel to $CO$ meeting $DE$ in $M$. In the plane of the circle on $DE$ as diameter draw $MQ$ perpendicular to $DE$ and meeting the circle in $Q$.

Lastly, draw $QH$ perpendicular to $NM$ produced. $QH$ will then be perpendicular to the plane containing $AC$, $DE$, i.e. the plane of the paper.

Now $\quad QH^2 : QM^2 = KR^2 : OR^2$, by similar triangles.

And $\quad QM^2 : AN \cdot NA' = DM \cdot ME : AN \cdot NA'$

$$= OD^2 : CA^2.$$

Hence, *ex aequali*, since $OR = OD$,

$$QH^2 : AN \cdot NA' = KR^2 : CA^2$$
$$= CB^2 : CA^2$$
$$= PN^2 : AN \cdot NA'.$$

Thus $QH = PN$. And $QH$, $PN$ are also parallel. Accordingly $PQ$ is parallel to $MN$, and therefore to $CO$, so that $PQ$ is a generator, and the cylinder passes through $P$.

## Proposition 10

It was proved by the earlier geometers that *any two cones have to one another the ratio compounded of the ratios of their bases and of their heights*.[1] The same method of proof will show that *any segments of cones have to one another the ratio compounded of the ratios of their bases and of their heights*.

The proposition that *any frustum of a cylinder is triple of the conical segment which has the same base as the frustum and equal height* is also proved in the same manner as the proposition that *the cylinder is triple of the cone which has the same base as the cylinder and equal height*.[2]

## Proposition 11

(1) *If a paraboloid of revolution be cut by a plane through, or parallel to, the axis, the section will be a parabola equal to the original parabola which by its revolution generates the paraboloid. And the axis of the section will be the intersection between the cutting plane and the plane through the axis of the paraboloid at right angles to the cutting plane.*

*If the paraboloid be cut by a plane at right angles to its axis, the section will be a circle whose centre is on the axis.*

(2) *If a hyperboloid of revolution be cut by a plane through the axis, parallel to the axis, or through the centre, the section will be a hyperbola, (a) if the section be through the axis, equal, (b) if parallel to the axis, similar, (c) if through the centre, not similar, to the original hyperbola which by its revolution generates the hyperboloid. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the hyperboloid at right angles to the cutting plane.*

*Any section of the hyperboloid by a plane at right angles to the axis will be a circle whose centre is on the axis.*

(3) *If any of the spheroidal figures be cut by a plane through the axis or parallel to the axis, the section will be an ellipse, (a) if the section be through the axis, equal, (b) if parallel to the axis, similar, to the ellipse which by its revolution gen-*

[1] This follows from Eucl. xii. 11 and 14 taken together. Cf. *On the Sphere and Cylinder* i, Lemma 1.

[2] This proposition was proved by Eudoxus, as stated in the preface to *On the Sphere and Cylinder* i. Cf. Eucl. xii. 10.

*erates the figure And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the spheroid at right angles to the cutting plane*

*If the section be by a plane at right angles to the axis of the spheroid, it will be a circle whose centre is on the axis*

(4) *If any of the said figures be cut by a plane through the axis, and if a perpendicular be drawn to the plane of section from any point on the surface of the figure but not on the section that perpendicular will fall within the section*

And the proofs of all these propositions are evident

PROPOSITION 12

*If a paraboloid of revolution be cut by a plane neither parallel nor pendicular to the axis and if the plane through the axis perpendicular to the cutting plane intersect it in a straight line of which the portion intercepted within the paraboloid is $RR'$ the section of the paraboloid will be an ellipse whose major axis is $RR'$ and whose minor axis is equal to the perpendicular distance between the lines through $R$, $R'$ parallel to the axis of the paraboloid*

Suppose the cutting plane to be perpendicular to the plane of the paper and let the latter be the plane through the axis $ANF$ of the paraboloid which intersects the cutting plane at right angles in $RR'$ Let $RH$ be parallel to the axis of the paraboloid, and $R'H$ perpendicular to $RH$

Let $Q$ be any point on the section made by the cutting plane and from $Q$ draw $QM$ perpendicular to $RR'$ $QM$ will therefore be perpendicular to the plane of the paper

Through $M$ draw $DMFE$ perpendicular to the axis $ANF$ meeting the parabolic section made by the plane of the paper in $D$, $E$ Then $QM$ is perpendicular to $DE$ and if a plane be drawn through $DE$, $QM$, it will be perpendicular to the axis and will cut the paraboloid in a circular section

Since $Q$ is on this circle

$$QM^2 = DM \cdot ME$$

Again if $PT$ be that tangent to the parabolic section in the plane of the paper which is parallel to $RR'$ and if the tangent at $A$ meet $PT$ in $O$ then from the property of the parabola

$$DM \cdot ME : RM \cdot MR' = AO^2 : OP^2 \quad \text{[Prop 3 (1)]}$$
$$= AO^2 : OT^2 \text{ since } AN = AT$$

Therefore $\quad QM^2 : RM \cdot MR' = AO^2 : OT^2$
$$= R'H^2 : RR'^2$$

by similar triangles

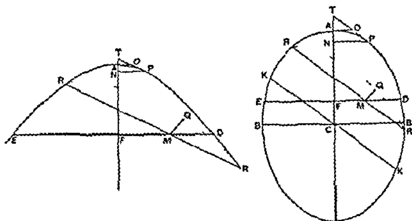
Hence $Q$ lies on an ellipse whose major axis is $RR'$ and whose minor axis is equal to $R'H$

PROPOSITIONS 13, 14

*If a hyperboloid of revolution be cut by a plane meeting all the generators of the enveloping cone or if an oblong spheroid be cut by a plane not perpendicular to*

*the axis*[1] *and if a plane through the axis intersect the cutting plane at right angles in a straight line on which the hyperboloid or spheroid intercepts a length RR, then the section by the cutting plane will be an ellipse whose major axis is RR*

Suppose the cutting plane to be at right angles to the plane of the paper, and suppose the latter plane to be that through the axis $ANF$ which intersects the




cutting plane at right angles in $RR$. The section of the hyperboloid or spheroid by the plane of the paper is thus a hyperbola or ellipse having $ANF$ for its transverse or major axis

Take any point on the section made by the cutting plane as $Q$ and draw $QM$ perpendicular to $RR$. $QM$ will then be perpendicular to the plane of the paper

Through $M$ draw $DFE$ at right angles to the axis $ANF$ meeting the hyperbola or ellipse in $D$, $E$ and through $QM$, $DE$ let a plane be described. This plane will accordingly be perpendicular to the axis and will cut the hyperboloid or spheroid in a circular section

Thus $$QM^2 = DM \cdot ME$$

Let $PT$ be that tangent to the hyperbola or ellipse which is parallel to $RR$, and let the tangent at $A$ meet $PT$ in $O$

Then by the property of the hyperbola or ellipse
$$DM \cdot ME : RM \cdot MR = OA^2 : OP^2$$
or
$$QM^2 : RM \cdot MR = OA^2 : OP^2$$

Now (1) in the hyperbola $OA < OP$ because $AT < AN$ and accordingly
$$OT < OP \text{ while } OA < OT$$

(2) in the ellipse if $KK'$ be the diameter parallel to $RR$, and $BB'$ the minor axis
$$BC \cdot CB' : KC \cdot CK' = OA^2 : OP^2$$
and $BC \cdot CB' < KC \cdot CK'$ so that $OA < OP$

Hence in both cases the locus of $Q$ is an ellipse whose major axis is $RR$

Cor 1 If the spheroid be a flat spheroid the section will be an ellipse, and everything will proceed as before except that $RR$ will in this case be the *minor* axis

[1] Archimedes begins Prop. 14 for the *spheroid* with the remark that when the cutting plane *passes through or is parallel to the axis the case is clear* Cf Prop 11 (3)

COR 2 In all conoids or spheroids parallel sections will be similar, since the ratio $OA^2$ $OP^2$ is the same for all the parallel sections

PROPOSITION 15

(1) *If from any point on the surface of a conoid a line be drawn, in the case of the paraboloid, parallel to the axis and, in the case of the hyperboloid parallel to any line passing through the vertex of the enveloping cone the part of the straight line which is in the same direction as the convexity of the surface will fall without it, and the part which is in the other direction within it*

For, if a plane be drawn, in the case of the paraboloid, through the axis and the point and in the case of the hyperboloid through the given point and through the given straight line drawn through the vertex of the enveloping cone, the section by the plane will be (a) in the paraboloid a parabola whose axis is the axis of the paraboloid, (b) in the hyperboloid a hyperbola in which the given line through the vertex of the enveloping cone is a diameter [1]

[Prop 11]

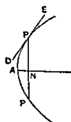
Hence the property follows from the plane properties of the conics

(2) *If a plane touch a conoid without cutting it, it will touch it at one point only, and the plane drawn through the point of contact and the axis of the conoid will be at right angles to the plane which touches it*

For if possible let the plane touch at two points Draw through each point a parallel to the axis The plane passing through both parallels will therefore either pass through or be parallel to the axis Hence the section of the conoid made by this plane will be a conic [Prop 11 (1), (2)], the two points will lie on this conic and the line joining them will lie within the conic and therefore within the conoid But this line will be in the tangent plane since the two points are in it Therefore some portion of the tangent plane will be within the conoid which is impossible since the plane does not cut it

Therefore the tangent plane touches in one point only

That the plane through the point of contact and the axis is perpendicular to the tangent plane is evident in the particular case where the point of contact is the vertex of the conoid For if two planes through the axis cut it in two conics the tangents at the vertex in both conics will be perpendicular to the axis of the conoid And all such tangents will be in the tangent plane which must therefore be perpendicular to the axis and to any plane through the axis




If the point of contact $P$ is not the vertex draw the plane passing through the axis $AN$ and the point $P$ It will cut the conoid in a conic whose axis is $AN$ and the tangent plane in a line $DPE$ touching the conic at $P$ Draw $PNP$ perpendicular to the axis and draw a plane through it also perpendicular to the axis This plane will make a circular section and meet the tangent plane in a tangent to the circle which will therefore be at right angles to $PN$ Hence the tangent to the circle will be at right angles to the plane containing $PN$, $AN$, and it follows that this last plane is perpendicular to the tangent plane

[1] There seems to be some error in the text here which says that the *diameter* (i e axis) of the hyperbola is the straight line drawn in the conoid from the vertex of the cone But this straight line is not in general the *axis* of the section.

### Proposition 16

(1) *If a plane touch any of the spheroidal figures without cutting it, it will touch at one point only, and the plane through the point of contact and the axis will be at right angles to the tangent plane*

This is proved by the same method as the last proposition

(2) *If any conoid or spheroid be cut by a plane through the axis and if through any tangent to the resulting conic a plane be erected at right angles to the plane of section the plane so erected will touch the conoid or spheroid in the same point as that in which the line touches the conic*

For it cannot meet the surface at any other point If it did the perpendicular from the second point on the cutting plane would be perpendicular also to the tangent to the conic and would therefore fall outside the surface But it must fall within it [Prop 11 (4)]

(3) *If two parallel planes touch any of the spheroidal figures the line joining the points of contact will pass through the centre of the spheroid*

If the planes are at right angles to the axis the proposition is obvious If not, the plane through the axis and one point of contact is at right angles to the tangent plane at that point It is therefore at right angles to the parallel tangent plane and therefore passes through the second point of contact Hence both points of contact lie on one plane through the axis and the proposition is reduced to a plane one

### Proposition 17

*If two parallel planes touch any of the spheroidal figures and another plane be drawn parallel to the tangent planes and passing through the centre the line drawn through any point of the circumference of the resulting section parallel to the chord of contact of the tangent planes will fall outside the spheroid*

This is proved at once by reduction to a plane proposition

Archimedes adds that it is evident that if the plane parallel to the tangent planes does not pass through the centre a straight line drawn in the manner described will fall without the spheroid in the direction of the smaller segment but within it in the other direction

### Proposition 18

*Any spheroidal figure which is cut by a plane through the centre is divided both as regards its surface and its volume into two equal parts by that plane*

To prove this Archimedes takes another equal and similar spheroid divides it similarly by a plane through the centre and then uses the method of application

### Propositions 19 20

*Given a segment cut off by a plane from a paraboloid or hyperboloid of revolution or a segment of a spheroid less than half the spheroid also cut off by a plane it is possible to inscribe in the segment one solid figure and to circumscribe about it another solid figure each made up of cylinders or frusta of cylinders of equal height, and such that the circumscribed figure exceeds the inscribed figure by a volume less than that of any given solid*

Let the plane base of the segment be perpendicular to the plane of the paper

and let the plane of the paper be the plane through the axis of the conoid or spheroid which cuts the base of the segment at right angles in $BC$. The section in the plane of the paper is then a conic $BAC$. [Prop. 11]

Let $EAF$ be that tangent to the conic which is parallel to $BC$, and let $A$ be the point of contact. Through $EAF$ draw a plane parallel to the plane through $BC$ bounding the segment. The plane so drawn will then touch the conoid or spheroid at $A$. [Prop. 16]

(1) If the base of the segment is at right angles to the axis of the conoid or spheroid, $A$ will be the vertex of the conoid or spheroid, and its axis $AD$ will bisect $BC$ at right angles.

(2) If the base of the segment is not at right angles to the axis of the conoid or spheroid, we draw $AD$

(a) in the paraboloid, parallel to the axis,

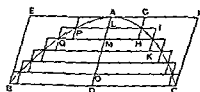
(b) in the hyperboloid, through the centre (or the vertex of the enveloping cone),

(c) in the spheroid, through the centre,

and in all the cases it will follow that $AD$ bisects $BC$ in $D$.

Then $A$ will be the vertex of the segment, and $AD$ will be its axis.

Further, the base of the segment will be a circle or an ellipse with $BC$ as diameter or as an axis respectively, and with centre $D$. We can therefore describe through this circle or ellipse a cylinder or a 'frustum' of a cylinder whose axis is $AD$. [Prop. 9]




Dividing this cylinder or frustum continually into equal parts by planes parallel to the base, we shall at length arrive at a cylinder or frustum less in volume than any given solid.

Let this cylinder or frustum be that whose axis is $OD$, and let $AD$ be divided into parts equal to $OD$ at $L$, $M$, .... Through $L$, $M$, ... draw lines parallel to $BC$ meeting the conic in $P$, $Q$, ..., and through these lines draw planes parallel to the base of the segment. These will cut the conoid or spheroid in circles or similar ellipses. On each of these circles or ellipses describe two cylinders or frusta of cylinders each with axis equal to $OD$, one of them lying in the direction of $A$ and the other in the direction of $D$, as shown in the figure.

Then the cylinders or frusta of cylinders drawn in the direction of $A$ make up a circumscribed figure, and those in the direction of $D$ an inscribed figure in relation to the segment.

Also the cylinder or frustum $PG$ in the circumscribed figure is equal to the cylinder or frustum $PH$ in the inscribed figure, $QI$ in the circumscribed figure is equal to $QK$ in the inscribed figure, and so on.

Therefore, by addition,

(circumscribed fig.) = (inscr. fig.) + (cylinder or frustum whose axis is $OD$).

But the cylinder or frustum whose axis is $OD$ is less than the given solid figure; whence the proposition follows.

'Having set out these preliminary propositions, let us proceed to demonstrate the theorems propounded with reference to the figures.'

PROPOSITIONS 21, 22

*Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis.*

Let the base of the segment be perpendicular to the plane of the paper, and let the plane of the paper be the plane through the axis of the paraboloid which cuts the base of the segment at right angles in $BC$ and makes the parabolic section $BAC$.

Let $EF$ be that tangent to the parabola which is parallel to $BC$, and let $A$ be the point of contact.

Then (1), if the plane of the base of the segment is perpendicular to the axis of the paraboloid, that axis is the line $AD$ bisecting $BC$ at right angles in $D$.

(2) If the plane of the base is not perpendicular to the axis of the paraboloid, draw $AD$ parallel to the axis of the paraboloid. $AD$ will then bisect $BC$, but not at right angles.

Draw through $EF$ a plane parallel to the base of the segment. This will touch the paraboloid at $A$, and $A$ will be the vertex of the segment, $AD$ its axis.

The base of the segment will be a circle with diameter $BC$ or an ellipse with $BC$ as major axis.

Accordingly a cylinder or a frustum of a cylinder can be found passing through the circle or ellipse and having $AD$ for its axis [Prop. 9], and likewise a cone or a segment of a cone can be drawn passing through the circle or ellipse and having $A$ for vertex and $AD$ for axis. [Prop. 8]

Suppose $X$ to be a cone equal to $\frac{3}{2}$ (cone or segment of cone $ABC$). The cone $X$ is therefore equal to half the cylinder or frustum of a cylinder $EC$. [Cf. Prop. 10]

We shall prove that the volume of the segment of the paraboloid is equal to $X$.

If not, the segment must be either greater or less than $X$.

I. If possible, let the segment be greater than $X$.

We can then inscribe and circumscribe, as in the last proposition, figures made up of cylinders or frusta of cylinders with equal height and such that

$$(\text{circumscribed fig.}) - (\text{inscribed fig.}) < (\text{segment}) - X.$$

Let the greatest of the cylinders or frusta forming the circumscribed figure be that whose base is the circle or ellipse about $BC$ and whose axis is $OD$, and let the smallest of them be that whose base is the circle or ellipse about $PP'$ and whose axis is $AL$.

Let the greatest of the cylinders forming the inscribed figure be that whose base is the circle or ellipse about $RR'$ and whose axis is $OD$, and let the smallest be that whose base is the circle or ellipse about $PP'$ and whose axis is $LM$.

Produce all the plane bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum $EC$.

Now, since

$$(\text{circumscribed fig.}) - (\text{inscr. fig.}) < (\text{segment}) - X,$$

it follows that (inscribed figure) $> X$ ($\alpha$)

Next comparing successively the cylinders or frusta with heights equal to $OD$ and respectively forming parts of the complete cylinder or frustum $EC$ and of the inscribed figure, we have

(first cylinder or frustum in $EC$) (first in inscr fig)

$= BD \quad RO$

$= AD \quad AO$

$= BD \quad TO$ where $AB$ meets $OR$ in $T$

And (second cylinder or frustum in $EC$) (second in inscr fig)

$= HO \quad SN$ in like manner,

and so on

Hence [Prop 1] (cylinder or frustum $EC$) (inscribed figure)

$= (BD+HO+ \quad ) \; (TO+SN+ \quad )$,

where $BD$ $HO$, are all equal, and $BD$, $TO$ $SN$, diminish in arithmetical progression

But [Lemma preceding Prop 1]

$BD+HO+ \quad > 2(TO+SN+ \quad )$

Therefore (cylinder or frustum $EC$) $> 2$ (inscribed fig),

or $X >$ (inscribed fig),

which is impossible, by ($\alpha$) above

II If possible, let the segment be less than $X$

In this case we inscribe and circumscribe figures as before but such that

(circumscr fig) $-$ (inscr fig) $< X -$ (segment),

whence it follows that

(circumscribed figure) $< X$ ($\beta$)

And, comparing the cylinders or frusta making up the complete cylinder or frustum $CE$ and the *circumscribed* figure respectively we have

(first cylinder or frustum in $CE$) (first in circumscr fig)

$= BD^2 \quad BD^2$

$= BD \quad BD$

(second in $CE$) (second in circumscr fig)

$= HO^2 \quad RO^2$

$= AD \quad AO$

$= HO \quad TO$,

and so on

Hence [Prop 1]

(cylinder or frustum $CE$) (circumscribed fig)

$= (BD+HO+ \quad ) \; (BD+TO+ \quad )$

$< 2 \quad 1$ [Lemma preceding Prop 1]

and it follows that

$X <$ (circumscribed fig)

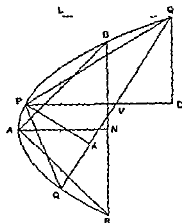
which is impossible, by ($\beta$)

Thus the segment being neither greater nor less than $X$ is equal to it, and therefore to $\frac{3}{2}$ (cone or segment of cone $ABC$)

PROPOSITION 23

*If from a paraboloid of revolution two segments be cut off one by a plane perpendicular to the axis the other by a plane not perpendicular to the axis and if the axes of the segments are equal the segments will be equal in volume*

Let the two planes be supposed perpendicular to the plane of the paper, and let the latter plane be the plane through the axis of the paraboloid cutting the other two planes at right angles in $BB'$, $QQ'$ respectively and the paraboloid itself in the parabola $QPQ'B'$




Let $AN$, $PV$ be the equal axes of the segments and $A$, $P$ their respective vertices

Draw $QL$ parallel to $AN$ or $PV$ and $QL$ perpendicular to $QL$

Now, since the segments of the parabolic section cut off by $BB'$, $QQ'$ have equal axes, the triangles $ABB'$, $PQQ'$ are equal [Prop 3] Also if $QD$ be perpendicular to $PV$, $QD = BN$ (as in the same Prop 3)

Conceive two cones drawn with the same bases as the segments and with $A$, $P$ as vertices respectively The height of the cone $PQQ'$ is then $PK$ where $PK$ is perpendicular to $QQ'$

Now the cones are in the ratio compounded of the ratios of their bases and of their heights i e the ratio compounded of (1) the ratio of the circle about $BB'$ to the ellipse about $QQ'$, and (2) the ratio of $AN$ to $PK$

That is to say, we have by means of Props 5, 12,

$$(\text{cone } ABB') : (\text{cone } PQQ') = (BB'^2 : QQ' \cdot QL) \cdot (AN : PK)$$

And $BB' = 2BN = 2QD = QL$, while $QQ' = 2QV$

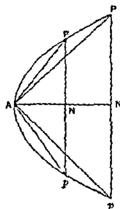
Therefore

$$\begin{aligned}(\text{cone } ABB') : (\text{cone } PQQ') &= (QD : QV) \cdot (AN : PK) \\ &= (PK : PV) \cdot (AN : PK) \\ &= AN : PV\end{aligned}$$

Since $AN = PV$, the ratio of the cones is a ratio of equality, and it follows that the segments being each half as large again as the respective cones [Prop 22], are equal

## Proposition 24

*If from a paraboloid of revolution two segments be cut off by planes drawn in any manner, the segments will be to one another as the squares on their axes*




For let the paraboloid be cut by a plane through the axis in the parabolic section $P'PApp'$, and let the axis of the parabola and paraboloid be $ANN'$

Measure along $ANN'$ the lengths $AN$, $AN'$ equal to the respective axes of the given segments and through $N$, $N'$ draw planes perpendicular to the axis making circular sections on $Pp$, $P'p'$ as diameters respectively With these circles as bases and with the common vertex $A$ let two cones be described

Now the segments of the paraboloid whose bases are the circles about $Pp$, $P'p'$ are equal to the given segments respectively, since their respective axes are

equal [Prop. 23]; and, since the segments $APp$, $AP'p'$ are half as large again as the cones $APp$, $AP'p'$ respectively, we have only to show that the cones are in the ratio of $AV^2$ to $AV'^2$.

But

$$(\text{cone } APp) : (\text{cone } AP'p') = (PV^2 : P'V'^2) \cdot (AV : AV')$$
$$= (AV : AV') \cdot (AV : AV')$$
$$= AV^2 : AV'^2;$$

thus the proposition is proved.

PROPOSITIONS 25, 26.

*In any hyperboloid of revolution, if $A$ be the vertex and $AD$ the axis of any segment cut off by a plane, and if $CA$ be the semidiameter of the hyperboloid through $A$ ($CA$ being of course in the same straight line with $AD$), then*

$$(\textit{segment}) : (\textit{cone with same base and axis}) = (AD+3CA) : (AD+2CA).$$

Let the plane cutting off the segment be perpendicular to the plane of the paper, and let the latter plane be the plane through the axis of the hyperboloid which intersects the cutting plane at right angles in $BB'$, and makes the hyperbolic segment $BAB'$. Let $C$ be the centre of the hyperboloid (or the vertex of the enveloping cone).

Let $EF$ be that tangent to the hyperbolic section which is parallel to $BB'$. Let $EF$ touch at $A$, and join $CA$. Then $CA$ produced will bisect $BB'$ at $D$, $CA$ will be a semi-diameter of the hyperboloid, $A$ will be the vertex of the segment, and $AD$ its axis. Produce $AC$ to $A'$ and $H$, so that $AC=CA'=A'H$.

Through $EF$ draw a plane parallel to the base of the segment. This plane will touch the hyperboloid at $A$.

Then (1) if the base of the segment is at right angles to the axis of the hyperboloid, $A$ will be the vertex and $AD$ the axis of the hyperboloid as well as of the segment, and the base of the segment will be a circle on $BB'$ as diameter.

(2) If the base of the segment is not perpendicular to the axis of the hyperboloid, the base will be an ellipse on $BB'$ as major axis. [Prop. 13]

Then we can draw a cylinder or a frustum of a cylinder $FBB'E$ passing through the circle or ellipse about $BB'$ and having $AD$ for its axis; also we can describe a cone or a segment of a cone through the circle or ellipse and having $A$ for its vertex.

We have to prove that

$$(\text{segment } ABB') : (\text{cone or segment of cone } ABB') = HD : A'D.$$

Let $V$ be a cone such that

$$V : (\text{cone or segment of cone } ABB') = HD : A'D, \qquad (\alpha)$$

and we have to prove that $V$ is equal to the segment.

Now

$$(\text{cylinder or frustum } EB') : (\text{cone or segmt. of cone } ABB') = 3 : 1.$$

Therefore, by means of $(\alpha)$,

$$(\text{cylinder or frustum } EB') : V = A'D : \frac{HD}{3}. \qquad (\beta)$$

If the segment is not equal to $V$, it must either be greater or less.

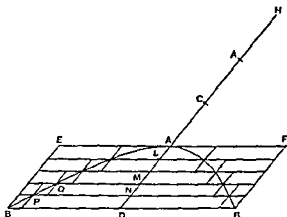
I. If possible, let the segment be greater than $V$.

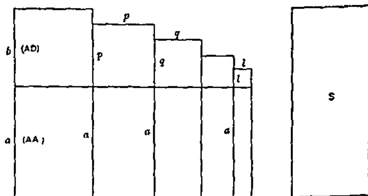
Inscribe and circumscribe to the segment figures made up of cylinders or frusta of cylinders, with axes along $AD$ and all equal to one another, such that

$$(\text{circumscribed fig.}) - (\text{inscr. fig.}) < (\text{segmt.}) - V,$$

whence $$(\text{inscribed figure}) > V \qquad (\gamma)$$

Produce all the planes forming the bases of the cylinders or frusta of cylinders to meet the surface of the complete cylinder or frustum $EB'$.




Then if $AD$ be the axis of the greatest cylinder or frustum in the circumscribed figure, the complete cylinder will be divided into cylinders or frusta each equal to this greatest cylinder or frustum.

Let there be a number of straight lines $a$ equal to $AA'$ and as many in number as the parts into which $AD$ is divided by the bases of the cylinders or frusta. To each line $a$ apply a rectangle which shall overlap it by a square, and let the greatest of the rectangles be equal to the rectangle $AD \cdot A'D$ and the least equal to the rectangle $AL \cdot A'L$; also let the sides of the overlapping squares $b, p, q, \ldots l$ be in descending arithmetical progression. Thus $b, p, q, \ldots l$ will be respectively equal to $AD, AN, AM, \ldots AL$, and the rectangles $(ab+b^2)$, $(ap+p^2)$, $\ldots (al+l^2)$ will be respectively equal to $AD \cdot A'D$, $AN \cdot A'N$, $\ldots AL \cdot A'L$.

Suppose further that we have a series of spaces $S$ each equal to the largest rectangle $AD \cdot A'D$ and as many in number as the diminishing rectangles.

Comparing now the successive cylinders or frusta (1) in the complete cylin-

der or frustum $EB'$ and (2) in the inscribed figure, beginning from the base of the segment, we have

(first cylinder or frustum in $FB$) (first in inscr figure)
$=BD \quad PN^2$
$=AD \; A'D \quad AN \; A'N$, from the hyperbola
$=S \quad (ap+p)$

Again

(second cylinder or frustum in $EB$) (second in inscr fig)
$=BD^2 \quad QM^2$
$=AD \; A'D \quad AM \; A'M$
$=S \quad (aq+q^2)$

and so on

The last cylinder or frustum in the complete cylinder or frustum $EB'$ has no cylinder or frustum corresponding to it in the inscribed figure

Combining the proportions, we have [Prop 1]

(cylinder or frustum $EB'$) (inscribed figure)
$=$(sum of all the spaces $S$) $(ap+p^2)+(aq+q^2)+$

$$>(a+b) \quad \left(\frac{a}{2}+\frac{b}{3}\right) \qquad \text{[Prop 2]}$$

$$>A'D \quad \frac{HD}{3} \quad \text{since } a=AA' \quad b=AD,$$

$>(EB) \quad V$ by ($\beta$) above

Hence (inscribed figure) $<V$

But this is impossible because by ($\gamma$) above the inscribed figure is greater than $V$

II Next suppose if possible, that the segment is less than $V$

In this case we circumscribe and inscribe figures such that

(circumscribed fig) $-$ (inscribed fig) $<V-$(segment),

whence we derive

$$V>\text{(circumscribed figure)} \qquad (\delta)$$

We now compare successive cylinders or frusta in the complete cylinder or frustum and in the *circumscribed* figure and we have

(first cylinder or frustum in $EB'$) (first in circumscribed fig)
$=S \quad S$
$=S \quad (ab+b^2)$

(second in $EB$) (second in circumscribed fig)
$=S \quad (ap+p)$

and so on

Hence [Prop 1]

(cylinder or frustum $EB$) (circumscribed fig)
$=$(sum of all spaces $S$) $(ab+b^2)+(ap+p)+$

$$<(a+b) \quad \left(\frac{a}{2}+\frac{b}{3}\right) \qquad \text{[Prop 2]}$$

$$<A'D \quad \frac{HD}{3}$$

$<(EB) \quad V$ by ($\beta$) above

Hence the circumscribed figure is greater than I which is impossible, by (δ) above

Thus the segment is neither greater nor less than I, and is therefore equal to it

Therefore by (α),

$$(\text{segment } ABB') : (\text{cone or segment of cone } ABB') = (AD+3CA) : (AD+2CA)$$

PROPOSITIONS 27, 28, 29, 30

(1) *In any spheroid whose centre is C, if a plane meeting the axis cut off a segment not greater than half the spheroid and having A for its vertex and AD for its axis, and if A'D be the axis of the remaining segment of the spheroid then*

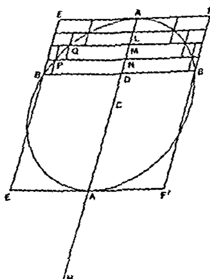
$$(\textit{first segmt}) : (\textit{cone or segmt of cone with same base and axis}) = CA+A'D : A'D$$

$$[=3CA-AD : 2CA-AD]$$

(2) *As a particular case, if the plane passes through the centre so that the segment is half the spheroid, half the spheroid is double of the cone or segment of a cone which has the same vertex and axis*

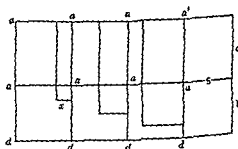
Let the plane cutting off the segment be at right angles to the plane of the paper and let the latter plane be the plane through the axis of the spheroid which intersects the cutting plane in $BB'$ and makes the elliptic section $ABA'B'$

Let $EF$, $E'F'$ be the two tangents to the ellipse which are parallel to $BB'$ let them touch it in $A$, $A'$ and through the tangents draw planes parallel to the base of the segment These planes will touch the spheroid at $A$, $A'$ which will be the vertices of the two segments into which it is divided Also $AA'$ will pass through the centre $C$ and bisect $BB'$ in $D$




Then (1) if the base of the segments be perpendicular to the axis of the spheroid $A$, $A'$ will be the vertices of the spheroid as well as of the segments $AA'$ will be the axis of the spheroid and the base of the segments will be a circle on $BB'$ as diameter

(2) if the base of the segments be not perpendicular to the axis of the spheroid the base of the segments will be an ellipse of which $BB'$ is one axis and $AD$, $A'D$ will be the axes of the segments respectively



We can now draw a cylinder or a frustum of a cylinder $EBB'F$ through the circle or ellipse about $BB'$ and having $AD$ for its axis, and we can also draw a cone or a segment of a cone passing through the circle or ellipse about $BB'$ and having $A$ for its vertex.

We have then to show that, if $CA'$ be produced to $H$ so that $CA'=A'H$,

$$(\text{segment } ABB') : (\text{cone or segment of cone } ABB') = HD : A'D.$$

Let $V$ be such a cone that

$$V : (\text{cone or segment of cone } ABB') = HD : A'D; \qquad (\alpha)$$

and we have to show that the segment $ABB'$ is equal to $V$.

But, since

$$(\text{cylinder or frustum } EB') : (\text{cone or segment of cone } ABB') = 3 : 1,$$

we have, by the aid of $(\alpha)$,

$$(\text{cylinder or frustum } EB') : V = A'D : \frac{HD}{3}. \qquad (\beta)$$

Now, if the segment $ABB'$ is not equal to $V$, it must be either greater or less.

I. Suppose, if possible, that the segment is greater than $V$.

Let figures be inscribed and circumscribed to the segment consisting of cylinders or frusta of cylinders, with axes along $AD$ and all equal to one another, such that

$$(\text{circumscribed fig.}) - (\text{inscribed fig.}) < (\text{segment}) - V,$$

whence it follows that

$$(\text{inscribed fig.}) > V. \qquad (\gamma)$$

Produce all the planes forming the bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum $EB'$. Thus, if $ND$ be the axis of the greatest cylinder or frustum of a cylinder in the circumscribed figure, the complete cylinder or frustum $EB'$ will be divided into cylinders or frusta of cylinders each equal to the greatest of those in the circumscribed figure.

Take straight lines $da'$ each equal to $A'D$ and as many in number as the parts into which $AD$ is divided by the bases of the cylinders or frusta, and measure $da$ along $da'$ equal to $AD$. It follows that $aa'=2CD$.

Apply to each of the lines $a'd$ rectangles with height equal to $ad$, and draw the squares on each of the lines $ad$ as in the figure. Let $S$ denote the area of each complete rectangle.

From the first rectangle take away a gnomon with breadth equal to $AN$ (i.e. with each end of a length equal to $AN$); take away from the second rectangle a gnomon with breadth equal to $AM$, and so on, the last rectangle having no gnomon taken from it.

Then

$$\begin{aligned}\text{the first gnomon} &= A'D \cdot AD - ND \cdot (A'D - AN)\\ &= A'D \cdot AN + ND \cdot AN\\ &= AN \cdot A'N.\end{aligned}$$

Similarly

$$\text{the second gnomon} = AM \cdot A'M,$$

and so on.

And the last gnomon (that in the last rectangle but one) is equal to $AL \cdot A'L$.

Also, after the gnomons are taken away from the successive rectangles, the remainders (which we will call $R_1, R_2, \ldots R_n$, where $n$ is the number of rectangles and accordingly $R_n=S$) are rectangles applied to straight lines each of

length $aa'$ and "exceeding by squares" whose sides are respectively equal to $DN, DM, \quad DA$

For brevity let $DV$ be denoted by $x$, and $aa'$ or $2CD$ by $c$, so that

$$R_1 = cx + x^2 \quad R_2 = c \; 2x + (2x)^2,$$

Then, comparing successively the cylinders or frusta of cylinders (1) in the complete cylinder or frustum $EB$ and (2) in the inscribed figure, we have

(first cylinder or frustum in $EB$ ) (first in inscribed fig )

$$= BD^2 \quad PV^2$$

$$= AD \; A'D \quad AN \; A'N$$

$= S$ (first gnomon),

(second cylinder or frustum in $EB'$) (second in inscribed fig )

$= S$ (second gnomon)

and so on

The last of the cylinders or frusta in the cylinder or frustum $EB$ has none corresponding to it in the inscribed figure, and there is no corresponding gnomon

Combining the proportions, we have [by Prop 1]

(cylinder or frustum $EB$ ) (inscribed fig )

= (sum of all spaces $S$) (sum of gnomons)

Now the differences between $S$ and the successive gnomons are $R_1 \; R \qquad R$ , while

$$R_1 = cx + x^2,$$
$$R_2 = c \; 2x + (2x)^2,$$

$$R_n = cb + b^2 = S,$$

where $b = nx = AD$

Hence [Prop 2]

(sum of all spaces $S$) $(R_1 + R_2 + \quad + R \;) < (c+b) \quad \left(\frac{c}{2} + \frac{b}{3}\right)$

It follows that

(sum of all spaces $S$) (sum of gnomons) $> (c+b) \quad \left(\frac{c}{2} + \frac{2b}{3}\right)$

$$> \text{1}'D \quad \frac{HD}{3}$$

Thus (cylinder or frustum $EB$ ) (inscribed fig )

$$> A \; D \quad \frac{HD}{3}$$

$>$ (cylinder or frustum $EB$ ) $V$,

from ($\beta$) above

Therefore (inscribed fig ) $< V$,

which is impossible by ($\gamma$) above

Hence the segment $ABB'$ is not greater than $V$

II If possible let the segment $ABB$ be less than $V$

We then inscribe and circumscribe figures such that

(circumscribed fig ) − (inscribed fig ) $< V$ − (segment), $\quad$ ($\delta$)

whence $V >$ (circumscribed fig )

In this case we compare the cylinders or frusta in ($EB$ ) with those in the *circumscribed* figure

Thus

(first cylinder or frustum in $EB'$) (first in circumscribed fig )
$=S$ $S$
(second in $EB'$) (second in circumscribed fig )
$=S$ (first gnomon),

and so on

Lastly (last in $EB'$) (last in circumscribed fig )
$=S$ (last gnomon)

Now

$$\{S+(\text{all the gnomons})\}=nS-(R_1+R_2+ \quad +R_{n-1})$$

And $$nS \quad R_1+R_2+ \quad +R_{-1}>(c+b) \quad \left(\frac{c}{2}+\frac{b}{3}\right),$$ [Prop 2]

so that

$$nS \quad \{S+(\text{all the gnomons})\}<(c+b) \quad \left(\frac{c}{2}+\frac{2b}{3}\right)$$

It follows that, if we combine the above proportions as in Prop 1, we obtain
(cylinder or frustum $EB'$) (circumscribed fig )

$$<(c+b) \quad \left(\frac{c}{2}+\frac{2b}{3}\right)$$

$$<A'D \quad \frac{HD}{3}$$

$<(EB\ ) \quad V$, by ($\beta$) above

Hence the circumscribed figure is greater than $V$, which is impossible by ($\delta$) above

Thus ince the segment $ABB$ is neither greater nor less than $V$, it is equal to it and the proposition is proved

(2) The particular case [Props 27, 28] where the segment is half the spheroid differs from the above in that the distance $CD$ or $c/2$ vanishes, and the rectangles $cb+b$ are simply squares ($b$ ) so that the gnomons are simply the differences between $b^2$ and $x^2$, $b^2$ and $(2x)^2$, and so on

Instead therefore of Prop 2 we use the *Lemma to Prop* 2, *Cor* 1 given above [*On Spirals*, Prop 10] and instead of the ratio $(c+b)$ $\left(\frac{c}{2}+\frac{2b}{3}\right)$ we obtain the ratio 3 2 whence (segment $ABB$ ) (cone or segment of cone $ABB$ )$=2$ 1

PROPOSITIONS 31, 32

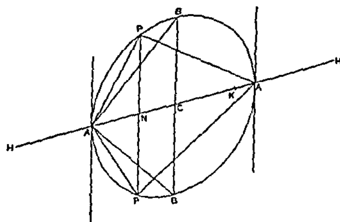
*If a plane divide a spheroid into two unequal segments and if $AN$ $A'N$ be the axes of the lesser and greater segments respectively while $C$ is the centre of the spheroid then*

*(greater segmt ) (cone or segmt of cone with same base and axis)*
$$=CA+AN \quad AN$$

Let the plane dividing the spheroid be that through $PP'$ perpendicular to the plane of the paper and let the latter plane be that through the axis of the spheroid which intersects the cutting plane in $PP$ and makes the elliptic section $PAP\ A$

Draw the tangents to the ellipse which are parallel to $PP$ let them touch the ellipse at $A$ $A'$, and through the tangents draw planes parallel to the base of the segments These planes will touch the spheroid at $A$, $A'$, the line $AA'$

will pass through the centre $C$ and bisect $PP'$ in $N$, while $AN$, $A'N$ will be the axes of the segments.




Then (1) if the cutting plane be perpendicular to the axis of the spheroid, $AA'$ will be that axis and $A$, $A'$ will be the vertices of the spheroid as well as of the segments. Also the sections of the spheroid by the cutting plane and all planes parallel to it will be circles.

(2) If the cutting plane be not perpendicular to the axis, the base of the segments will be an ellipse of which $PP'$ is an axis, and the sections of the spheroid by all planes parallel to the cutting plane will be similar ellipses.

Draw a plane through $C$ parallel to the base of the segments and meeting the plane of the paper in $BB'$.

Construct three cones or segments of cones, two having $A$ for their common vertex and the plane sections through $PP'$, $BB'$ for their respective bases, and a third having the plane section through $PP'$ for its base and $A'$ for its vertex.

Produce $CA$ to $H$ and $CA'$ to $H'$ so that

$$AH = A'H' = CA.$$

We have then to prove that

$$(\text{segment } APP') : (\text{cone or segment of cone } A'PP') = CA + A'N : A'N$$
$$= NH' : A'N.$$

Now half the spheroid is double of the cone or segment of a cone $ABB'$ [Props. 27, 28]. Therefore

$$(\text{the spheroid}) = 4(\text{cone or segment of cone } ABB').$$

But

$$(\text{cone or segmt. of cone } ABB') : (\text{cone or segmt. of cone } APP')$$
$$= (CA : AN) \cdot (BC^2 : PN^2)$$
$$= (CA : AN) \cdot (CA \cdot CA' : AN \cdot A'N). \qquad (\alpha)$$

If we measure $AK$ along $AA'$ so that

$$AK : AC = AC : AN,$$

we have $\qquad AK : A'N = AC : A'N = CA : AN,$

and the compound ratio in $(\alpha)$ becomes

$$(AK : A'N) \cdot (CA \cdot CA' : AN \cdot A'N),$$

i.e. $\qquad AK \cdot CA' : AN \cdot A'N.$

Thus

(cone or segmt of cone $ABB'$) (cone or segmt of cone $APP$)

$=AK\ CA'\ AN\ A'N$

But (cone or segment of cone $APP$) (segment $APP$)

$=A'N\ NH'$ [Props 29, 30]

$=AN\ A'N\ AN\ NH'$

Therefore *ex aequali*

(cone or segment of cone $ABB$) (segment $APP'$)

$=AK\ CA'\ AN\ NH'$,

so that (spheroid) (segment $APP$)

$=HH'\ AK\ AN\ NH'$,

since $HH'=4CA'$

Hence (segment $A'PP$) (segment $APP'$)

$=(HH\ AK-AN\ NH)\ AN\ NH'$

$=(AK\ NH+NH'\ NK)\ AN\ NH'$

Further,

(segment $APP$) (cone or segment of cone $APP'$)

$=NH\ A'N$

$=AN\ NH'\ AN\ A'N$,

and

(cone or segmt of cone $APP'$) (cone or segmt of cone $A'PP'$)

$=AN\ A'N$

$=AN\ A'N\ A'N^2$

From the last three proportions we obtain *ex aequali*

(segment $A\ PP'$) (cone or segment of cone $A'PP'$)

$=(AK\ NH+NH'\ NK)\ A'N^2$

$=(AK\ NH+NH\ NK)\ (CA^2+NH'\ CN)$

$=(AK\ NH+NH'\ NK)\ (AK\ AN+NH\ CN)$ $(\beta)$

But

$AK\ NH\ AK\ AN=NH\ AN$

$=CA+AN\ AN$

$=AK+CA\ CA$ (since $AK\ AC=AC\ AN$)

$=HK\ CA$

$=HK-NH\ CA-AN$

$=NK\ CN$

$=NH\ NK\ NH\ CN$

Hence the ratio in $(\beta)$ is equal to the ratio

$AK\ NH\ AK\ AN$, or $NH\ AN$

Therefore

(segment $A'PP$) (cone or segment of cone $A\ PP'$)

$=NH\ AN$

$=CA+AN\ AN$

# ON SPIRALS

"Archimedes to Dositheus greeting

Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs the demonstrations are already before you in the books brought to you by Heracleides and some more are also contained in that which I now send you Do not be surprised at my taking a considerable time before publishing these proofs This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them In fact, how many theorems in geometry which have seemed at first impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to otherwise he would have discovered and made manifest all these things and would have enriched geometry by many other discoveries besides For I know well that it was no common ability that he brought to bear on mathematics and that his industry was extraordinary But though many years have elapsed since Conon s death I do not find that any one of the problems has been stirred by a single person I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation [and which may serve as a warning] how those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible

What are the problems I mean, and what are those of which you have already received the proofs, and those of which the proofs are contained in this book respectively, I think it proper to specify The first of the problems was Given a sphere, to find a plane area equal to the surface of the sphere and this was first made manifest on the publication of the book concerning the sphere, for when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear that it is possible to find a plane area equal to the surface of the sphere The second was Given a cone or a cylinder to find a sphere equal to the cone or cylinder, the third To cut a given sphere by a plane so that the segments of it have to one another an assigned ratio the fourth To cut a given sphere by a plane so that the segments of the surface have to one another an assigned ratio the fifth To make a given segment of a sphere similar to a given segment of a sphere,[1] the sixth Given two segments of either the same or different spheres to find a segment of a sphere which shall be similar to one of the segments and have its surface equal to the surface of the other segment The seventh was From a given sphere to cut off a segment by a plane so that the segment bears to the cone which has the same base

[1]Cf *On the Sphere and Cylinder* II 5

as the segment and equal height an assigned ratio greater than that of three to two Of all the propositions just enumerated Heracleides brought you the proofs The proposition stated next after these was wrong viz that, if a sphere be cut by a plane into unequal parts the greater segment will have to the less the duplicate ratio of that which the greater surface has to the less That this is wrong is obvious by what I sent you before, for it included this proposition If a sphere be cut into unequal parts by a plane at right angles to any diameter in the sphere the greater segment of the surface will have to the less the same ratio as the greater segment of the diameter has to the less while the greater segment of the sphere has to the less a ratio less than the duplicate ratio of that which the greater surface has to the less but greater than the sesqui alterate[1] of that ratio The last of the problems was also wrong viz that, if the diameter of any sphere be cut so that the square on the greater segment is triple of the square on the lesser segment, and if through the point thus arrived at a plane be drawn at right angles to the diameter and cutting the sphere, the figure in such a form as is the greater segment of the sphere is the greatest of all the segments which have an equal surface That this is wrong is also clear from the theorems which I before sent you For it was there proved that the hemisphere is the greatest of all the segments of a sphere bounded by an equal surface

'After these theorems the following were propounded concerning the cone[2] If a section of a right-angled cone [a parabola] in which the diameter [axis] remains fixed be made to revolve so that the diameter [axis] is the axis [of revolution] let the figure described by the section of the right angled cone be called a *conoid* And if a plane touch the conoidal figure and another plane drawn parallel to the tangent plane cut off a segment of the conoid, let the *base* of the segment cut off be defined as the cutting plane and the *vertex* as the point in which the other plane touches the conoid Now, if the said figure be cut by a plane at right angles to the axis it is clear that the section will be a circle, but it needs to be proved that the segment cut off will be half as large again as the cone which has the same base as the segment and equal height And if two segments be cut off from the conoid by planes drawn in any manner, it is clear that the sections will be sections of acute-angled cones [ellipses] if the cutting planes be not at right angles to the axis but it needs to be proved that the segments will bear to one another the ratio of the squares on the lines drawn from their vertices parallel to the axis to meet the cutting planes The proofs of these propositions are not yet sent to you

After these came the following propositions about the *spiral* which are as it were another sort of problem having nothing in common with the foregoing and I have written out the proofs of them for you in this book They are as follows If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started and if at the same time as the straight line revolves a point move at a uniform rate along the straight line starting from the fixed extremity the point will describe a spiral in the plane I say then that the area bounded by the spiral and the straight line which has returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during

[1]See *On the Sphere and Cylinder* II 8

[2]This should be presumably the *conoid* not the cone

the one revolution And, if a straight line touch the spiral at the extreme end of the spiral and another straight line be drawn at right angles to the line which has revolved and resumed its position from the fixed extremity of it, so as to meet the tangent I say that the straight line so drawn to meet it is equal to the circumference of the circle Again, if the revolving line and the point moving along it make several revolutions and return to the position from which the straight line started I say that the area added by the spiral in the third revolution will be double of that added in the second that in the fourth three times, that in the fifth four times and generally the areas added in the later revolutions will be multiples of that added in the second revolution according to the successive numbers while the area bounded by the spiral in the first revolution is a sixth part of that added in the second revolution Also if on the spiral described in one revolution two points be taken and straight lines be drawn joining them to the fixed extremity of the revolving line and if two circles be drawn with the fixed point as centre and radii the lines drawn to the fixed extremity of the straight line and the shorter of the two lines be produced I say that (1) the area bounded by the circumference of the greater circle in the direction of (the part of) the spiral included between the straight lines the spiral (itself) and the produced straight line will bear to (2) the area bounded by the circumference of the lesser circle the same (part of the) spiral and the straight line joining their extremities the ratio which (3) the radius of the lesser circle together with two thirds of the excess of the radius of the greater circle over the radius of the lesser bears to (4) the radius of the lesser circle together with one third of the said excess

'The proofs then of these theorems and others relating to the spiral are given in the present book Prefixed to them, after the manner usual in other geometrical works are the propositions necessary to the proofs of them And here too as in the books previously published I assume the following lemma, that if there be (two) unequal lines or (two) unequal areas the excess by which the greater exceeds the less can by being [continually] added to itself be made to exceed any given magnitude among those which are comparable with [it and with] one another

## Proposition 1

*If a point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them*

Two unequal lengths are taken on a straight line and two lengths on another straight line representing the times and they are proved to be proportional by taking equimultiples of each length and the corresponding time after the manner of Eucl V Def 5

## Proposition 2

*If each of two points on different lines respectively move along them each at a uniform rate and if lengths be taken one on each line forming pairs, such that each pair are described in equal times the lengths will be proportionals*

This is proved at once by equating the ratio of the lengths taken on one line to that of the times of description which must also be equal to the ratio of the lengths taken on the other line

### Proposition 3

*Given any number of circles it is possible to find a straight line greater than the sum of all their circumferences*

For we have only to describe polygons about each and then take a straight line equal to the sum of the perimeters of the polygons

### Proposition 4

*Given two unequal lines, viz a straight line and the circumference of a circle, it is possible to find a straight line less than the greater of the two lines and greater than the less*

For by the Lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line

Thus e g , if $c > l$ (where $c$ is the circumference of the circle and $l$ the length of the straight line) we can find a number $n$ such that

$$n(c-l)>l$$

Therefore $$c-l>\frac{l}{n}$$

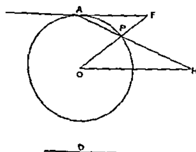
and $$c>l+\frac{l}{n}>l$$

Hence we have only to divide $l$ into $n$ equal parts and add one of them to $l$ The resulting line will satisfy the condition

### Proposition 5

*Given a circle with centre O and the tangent to it at a point A it is possible to draw from O a straight line OPF, meeting the circle in P and the tangent in F, such that, if c be the circumference of any given circle whatever*

$$FP\ \ OP < (\text{arc } AP)\ \ c$$




Take a straight line as $D$ greater than the circumference $c$ [Prop 3]

Through $O$ draw $OH$ parallel to the given tangent and draw through $A$ a line $APH$ meeting the circle in $P$ and $OH$ in $H$ such that the portion $PH$ intercepted between the circle and the line $OH$ may be equal to $D$ Join $OP$ and produce it to meet the tangent in $F$

Then $FP\ \ OP = AP\ \ PH$ by parallels
$= AP\ \ D$
$< (\text{arc } AP)\ \ c$

### Proposition 6

*Given a circle with centre O, a chord AB less than the diameter and OM the perpendicular on AB from O it is possible to draw a straight line OFP meeting the chord AB in F and the circle in P such that*

$$FP\ \ PB = D\ \ E$$

*where D E is any given ratio less than BM MO*

Draw $OH$ parallel to $AB$ and $BT$ perpendicular to $BO$ meeting $OH$ in $T$

Then the triangles $BMO$, $OBT$ are similar, and therefore

$$BM\;MO = OB\;BT,$$

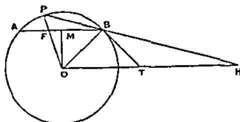
whence

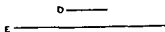
$$D\;E < OB\;BT$$

Suppose that a line $PH$ (greater than $BT$) is taken such that

$$D\;E = OB\;PH$$

and let $PH$ be so placed that it passes through $B$ and $P$ lies on the circumference of the circle, while $H$ is on the line $OH$ ($PH$ will fall outside $BT$, because $PH > BT$) Join $OP$ meeting $AB$ in $F$




We now have

$$FP\;PB = OP\;PH$$
$$= OB\;PH$$
$$= D\;E$$

PROPOSITION 7

*Given a circle with centre O a chord AB less than the diameter, and OM the perpendicular on it from O it is possible to draw from O a straight line OPF, meeting the circle in P and AB produced in F such that*

$$FP\;PB = D\;E$$

*where D E is any given ratio greater than BM MO*

Draw $OT$ parallel to $AB$ and $BT$ perpendicular to $BO$ meeting $OT$ in $T$

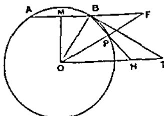
In this case

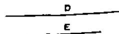
$$D\;E > BM\;MO$$
$$> OB\;BT, \text{ by similar triangles}$$

Take a line $PH$ (less than $BT$) such that

$$D\;E = OB\;PH,$$

and place $PH$ so that $P$ $H$ are on the circle and on $OT$ respectively while $HP$ produced passes through $B$




Then

$$FP\;PB = OP\;PH$$
$$= D\;E$$

PROPOSITION 8

*Given a circle with centre O, a chord AB less than the diameter the tangent at B, and the perpendicular OM from O on AB it is possible to draw from O a straight line OFP, meeting the chord AB in F the circle in P and the tangent in G, such that*

$$FP\;BG = D\;E$$

*where D E is any given ratio less than BM MO*

If $OT$ be drawn parallel to $AB$ meeting the tangent at $B$ in $T$,

$$BM\;MO = OB\;BT,$$

so that

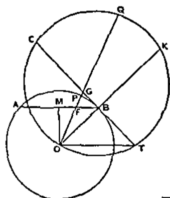
$$D\;E < OB\;BT$$

Take a point $C$ on $TB$ produced such that

$$D\;E = OB\;BC,$$

whence

$$BC > BT$$




D

E

Through the points $O, T, C$ describe a circle and let $OB$ be produced to meet this circle in $K$.

Then, since $BC > BT$, and $OB$ is perpendicular to $CT$ it is possible to draw from $O$ a straight line $OGQ$ meeting $CT$ in $G$ and the circle about $OTC$ in $Q$, such that $GQ = BK$.

Let $OGQ$ meet $AB$ in $F$ and the original circle in $P$.

Now $\quad CG\ GT = OG\ GQ$,

and $\quad OF\ OG = BT\ GT$,

so that $\quad OF\ GT = OG\ BT$.

It follows that

$$CG\ GT\ \ OF\ GT = OG\ GQ\ \ OG\ BT,$$

or

$$\begin{aligned} CG\ OF &= GQ\ BT \\ &= BK\ BT \text{ by construction}, \\ &= BC\ OB \\ &= BC\ OP \end{aligned}$$

Hence $\quad OP\ OF = BC\ CG$

and therefore $\quad PF\ OP = BG\ BC$

or $\quad PF\ BG = OP\ BC$

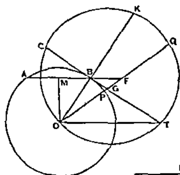
$$\begin{aligned} &= OB\ BC \\ &= D\ E \end{aligned}$$

### Proposition 9

*Given a circle with centre $O$ a chord $AB$ less than the diameter the tangent at $B$, and the perpendicular $OM$ from $O$ on $AB$ it is possible to draw from $O$ a straight line $OPGF$, meeting the circle in $P$, the tangent in $G$, and $AB$ produced in $F$, such that*

$$FP\ BG = D\ E,$$

*where $D\ E$ is any given ratio greater than $BM\ MO$.*




D

E

Let $OT$ be drawn parallel to $AB$ meeting the tangent at $B$ in $T$.

Then

$D\ E > BM\ MO$

$\quad > OB\ BT$ by similar triangles

Produce $TB$ to $C$ so that

$$D\ E = OB\ BC$$

whence $\quad BC < BT$

Describe a circle through the points $O\ T\ C$ and produce $OB$ to meet this circle in $K$.

Then since $TB > BC$, and $OB$ is perpendicular to $CT$ it is possible to draw from $O$ a line $OGQ$ meeting $CT$ in $G$ and the circle about $OTC$ in $Q$ such that $GQ = BK$. Let $OQ$ meet the original circle in $P$ and $AB$ produced in $F$.

We now prove, exactly as in the last proposition, that

$$CG\ \ OP = BK\ \ BT$$
$$= BC\ \ OP$$

Thus as before,

$$OP\ \ OF = BC\ \ CG,$$

and $$OP\ \ PF = BC\ \ BG,$$

whence $$PF\ \ BC = OP\ \ BC$$
$$= OB\ \ BC$$
$$= D\ \ F$$

### Proposition 10

*If $A_1, A_2, A_3 \quad A$ be n lines forming an ascending arithmetical progression in which the common difference is equal to $A_1$, the least term, then*

$$(n+1)A^{\ 2} + A_1(A_1+A_2+ \quad + A\ ) = 3(A_1^2+A_2^2+ \quad +A\ )$$

[Archimedes proof of this proposition is given above, pp 456–7, and it is there pointed out that the result is equivalent to

$$1^2+2\ +3^2+ \quad +n\ = \frac{n(n+1)(2n+1)}{6}\ ]$$

Cor 1 *It follows from this proposition that*

$$n\ \ A\ \ < 3(A_1^2+A_2^2+ \quad + A_n^2),$$

*and also that*

$$n\ A_n^2 > 3(A_1^2+A_2^2+ \quad + A_{\ -1}\ )$$

[For the proof of the latter inequality see p 457 above]

Cor 2 *All the results will equally hold if similar figures are substituted for squares*

### Proposition 11

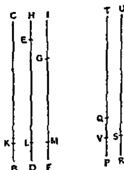
*If $A_1\ A_2 \quad A$ be n lines forming an ascending arithmetical progression [in which the common difference is equal to the least term $A_1$], then*

$$(n-1)A^{\ 2} \ \ (A\ \ + A_{n-1}^2 + \quad + A_2^2) < A_n^2 \ \ \{A \quad A_1 + \tfrac{1}{3}(A\ \ - A_1)^2\}$$

*but*

$$(n-1)A_n \ \ (A_{\ -1} + A_{n-2} + \quad + A_1^2) > A \quad \{A \ \ A_1 + \tfrac{1}{3}(A \ - A_1)^2\}$$

[Archimedes sets out the terms side by side in the manner shown in the figure where $BC = A \quad DE = A_{\ -1} \quad RS = A_1$ and produces $DE\ FG, \quad RS$ until they are respectively equal to $BC$ or $A$ so that $EH$, $GI, \quad SU$ in the figure are respectively equal to $A_1$ $A_2 \quad A_{n-1}$ He further measures lengths $BK \ \ DL$ $FM \quad PV$ along $BC$, $DE\ \ FG, \quad PQ$ respectively each equal to $RS$




The figure makes the relations between the terms easier to see with the eye but the use of so large a number of letters makes the proof somewhat difficult to follow, and it may be more clearly represented as follows ]

It is evident that $(A_n - A_1) = A_{n-1}$

The following proportion is therefore obviously true viz

$$(n-1)A_n^2 \ \ (n-1)(A \ \ A_1 + \tfrac{1}{3}A_{\ -1}\ ) = A_n^2 \ \ \{A \ \ A_1 + \tfrac{1}{3}(A_n - A_1)^2\}$$

In order therefore to prove the desired result, we have only to show that

$$(n-1)A_n \cdot A_1+\tfrac{1}{3}(n-1)A_{n-1}^2 <(A_n^2+A_{n-1}^2+\dots+A_2^2)$$

but

$$>(A_{n-1}^2+A_{n-2}^2+\dots+A_1^2).$$

I. To prove the first inequality, we have

$$(n-1)A_n \cdot A_1+\tfrac{1}{3}(n-1)A_{n-1}^2$$
$$=(n-1)A_1^2+(n-1)A_1 \cdot A_{n-1}+\tfrac{1}{3}(n-1)A_{n-1}^2 \quad \text{(1)}$$

And

$$\begin{aligned}A_n^2+A_{n-1}^2+\dots+A_2^2 &=(A_{n-1}+A_1)^2+(A_{n-2}+A_1)^2+\dots+(A_1+A_1)^2\\ &=(A_{n-1}^2+A_{n-2}^2+\dots+A_1^2)\\ &\quad+(n-1)A_1^2\\ &\quad+2A_1(A_{n-1}+A_{n-2}+\dots+A_1)\\ &=(A_{n-1}^2+A_{n-2}^2+\dots+A_1^2)\\ &\quad+(n-1)A_1^2\\ &\quad+A_1\{A_{n-1}+A_{n-2}+A_{n-3}+\dots+A_1\\ &\qquad+A_1+A_2+\dots+A_{n-2}+A_{n-1}\}\\ &=(A_{n-1}^2+A_{n-2}^2+\dots+A_1^2)\\ &\quad+(n-1)A_1^2\\ &\quad+nA_1 \cdot A_{n-1} \qquad \text{(2)}\end{aligned}$$

Comparing the right-hand sides of (1) and (2), we see that $(n-1)A_1^2$ is common to both sides, and

$$(n-1)A_1 \cdot A_{n-1}<nA_1 \cdot A_{n-1},$$

while by Prop. 10, Cor. 1,

$$\tfrac{1}{3}(n-1)A_{n-1}^2<A_{n-1}^2+A_{n-2}^2+\dots+A_1^2.$$

It follows therefore that

$$(n-1)A_n \cdot A_1+\tfrac{1}{3}(n-1)A_{n-1}^2<(A_n^2+A_{n-1}^2+\dots+A_2^2);$$

and hence the first part of the proposition is proved.

II. We have now, in order to prove the second result, to show that

$$(n-1)A_n \cdot A_1+\tfrac{1}{3}(n-1)A_{n-1}^2>(A_{n-1}^2+A_{n-2}^2+\dots+A_1^2).$$

The right-hand side is equal to

$$\begin{aligned}&(A_{n-2}+A_1)^2+(A_{n-3}+A_1)^2+\dots+(A_1+A_1)^2+A_1^2\\ &=A_{n-2}^2+A_{n-3}^2+\dots+A_1^2\\ &\quad+(n-1)A_1^2\\ &\quad+2A_1(A_{n-2}+A_{n-3}+\dots+A_1)\\ &=(A_{n-2}^2+A_{n-3}^2+\dots+A_1^2)\\ &\quad+(n-1)A_1^2\\ &\quad+A_1\left\{\begin{array}{l}A_{n-2}+A_{n-3}+\dots+A_1\\ +A_1+A_2+\dots+A_{n-2}\end{array}\right\}\\ &=(A_{n-2}^2+A_{n-3}^2+\dots+A_1^2)\\ &\quad+(n-1)A_1^2\\ &\quad+(n-2)A_1 \cdot A_{n-1} \qquad \text{(3)}\end{aligned}$$

Comparing this expression with the right-hand side of (1) above, we see that $(n-1)A_1^2$ is common to both sides, and

$$(n-1)A_1 \cdot A_{n-1}>(n-2)A_1 \cdot A_{n-1},$$

while by Prop. 10, Cor. 1,

$$\tfrac{1}{3}(n-1)A_{n-1}^2>(A_{n-2}^2+A_{n-3}^2+\dots+A_1^2).$$

Hence $(n-1)A_n \cdot A_1+\tfrac{1}{3}(n-1)A_{n-1}^2>(A_{n-1}^2+A_{n-2}^2+\dots+A_1^2)$,

and the second required result follows.

COR. *The results in the above proposition are equally true if similar figures be substituted for squares on the several lines.*

## DEFINITIONS

1 If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves a point move at a uniform rate along the straight line beginning from the extremity which remains fixed the point will describe a *spiral* (ἕλιξ) in the plane

2 Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin* of the spiral

3 And let the position of the line from which the straight line began to revolve be called the *initial line* in the revolution

4 Let the length which the point that moves along the straight line describes in one revolution be called the *first distance* that which the same point describes in the second revolution the *second distance*, and similarly let the distances described in further revolutions be called after the number of the particular revolution

5 Let the area bounded by the spiral described in the first revolution and the *first distance* be called the *first area* that bounded by the spiral described in the second revolution and the *second distance* the *second area*, and similarly for the rest in order

6 If from the origin of the spiral any straight line be drawn let that side of it which is in the same direction as that of the revolution be called *forward* (προαγούμενα), and that which is in the other direction *backward* (ἑπόμενα)

7 Let the circle drawn with the *origin* as centre and the *first distance* as radius be called the *first circle* that drawn with the same centre and twice the radius the *second circle*, and similarly for the succeeding circles

### Proposition 12

*If any number of straight lines drawn from the origin to meet the spiral make equal angles with one another the lines will be in arithmetical progression*

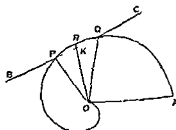
[The proof is obvious]

### Proposition 13

*If a straight line touch the spiral it will touch it in one point only*

Let $O$ be the origin of the spiral and $BC$ a tangent to it

If possible let $BC$ touch the spiral in two points $P$ $Q$ Join $OP$ $OQ$ and bisect the angle $POQ$ by the straight line $OR$ meeting the spiral in $R$




Then [Prop 12] $OR$ is an arithmetic mean between $OP$ and $OQ$ or

$$OP+OQ=2OR$$

But in any triangle $POQ$ if the bisector of the angle $POQ$ meets $PQ$ in $K$

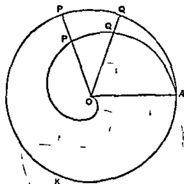
$$OP+OQ>2OK$$

Therefore $OK<OR$ and it follows that some point on $BC$ between $P$ and $Q$ lies within the spiral Hence $BC$ cuts the spiral, which is contrary to the hypothesis

### Proposition 14

*If $O$ be the origin, and $P$, $Q$ two points on the first turn of the spiral, and if $OP$, $OQ$ produced meet the 'first circle' $AKP'Q'$ in $P'$, $Q'$ respectively, $OA$ being the initial line, then*

$$OP : OQ = (\text{arc } AKP') : (\text{arc } AKQ')$$




For, while the revolving line $OA$ moves about $O$, the point $A$ on it moves uniformly along the circumference of the circle $AKP'Q'$, and at the same time the point describing the spiral moves uniformly along $OA$.

Thus, while $A$ describes the arc $AKP'$, the moving point on $OA$ describes the length $OP$, and while $A$ describes the arc $AKQ'$, the moving point on $OA$ describes the distance $OQ$.

Hence

$$OP : OQ = (\text{arc } AKP') : (\text{arc } AKQ')$$

[Prop 2]

### Proposition 15

*If $P$, $Q$ be points on the second turn of the spiral, and $OP$, $OQ$ meet the 'first circle' $AKP'Q'$ in $P'$, $Q'$, as in the last proposition, and if $c$ be the circumference of the "first circle," then*

$$OP : OQ = c + (\text{arc } AKP') : c + (\text{arc } AKQ')$$

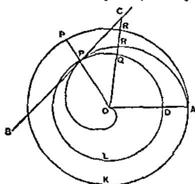
For, while the moving point on $OA$ describes the distance $OP$, the point $A$ describes the whole of the circumference of the "first circle" together with the arc $AKP'$; and, while the moving point on $OA$ describes the distance $OQ$, the point $A$ describes the whole circumference of the "first circle" together with the arc $AKQ'$.

Cor. Similarly, if $P$, $Q$ are on the $n$th turn of the spiral,

$$OP : OQ = (n-1)c + (\text{arc } AKP') : (n-1)c + (\text{arc } AKQ')$$

### Propositions 16, 17

*If $BC$ be the tangent at $P$, any point on the spiral, $PC$ being the "forward" part of $BC$, and if $OP$ be joined, the angle $OPC$ is obtuse while the angle $OPB$ is acute.*




I. Suppose $P$ to be on the first turn of the spiral.

Let $OA$ be the initial line, $AKP'$ the "first circle." Draw the circle $DLP$ with centre $O$ and radius $OP$, meeting $OA$ in $D$. This circle must then, in the "forward" direction from $P$, fall within the spiral, and in the "backward" direction outside it, since the radii vectores of the spiral are on the "forward" side greater, and on the "backward" side less, than $OP$. Hence the angle $OPC$ cannot be acute, since it cannot be less than the angle between $OP$ and

the tangent to the circle at $P$ which is a right angle

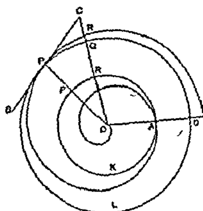
It only remains therefore to prove that $OPC$ is not a right angle

If possible let it be a right angle $BC$ will then touch the circle at $P$

Therefore [Prop 5] it is possible to draw a line $OQC$ meeting the circle through $P$ in $Q$ and $BC$ in $C$ such that

$$CQ\ \ OQ < (\text{arc } PQ)\ \ (\text{arc } DLP) \qquad (1)$$

Suppose that $OC$ meets the spiral in $R$ and the first circle in $R'$ and produce $OP$ to meet the "first circle" in $P'$




From (1) it follows *componendo* that

$$CO\ \ OQ < (\text{arc } DLQ)\ \ (\text{arc } DLP)$$
$$< (\text{arc } AKR')\ \ (\text{arc } AKP')$$
$$< OR\ \ OP \qquad [\text{Prop } 14]$$

But this is impossible because $OQ = OP$, and $OR < OC$

Hence the angle $OPC$ is not a right angle It was also proved not to be acute

Therefore the angle $OPC$ is obtuse and the angle $OPB$ consequently acute

II If $P$ is on the second or the $n$th turn the proof is the same except that in the proportion (1) above we have to substitute for the arc $DLP$ an arc equal to $(p + \text{arc } DLP)$ or $(n-1\ p + \text{arc } DLP)$ where $p$ is the perimeter of the circle $DLP$ through $P$ similarly in the later steps $p$ or $(n-1)p$ will be added to each of the arcs $DLQ$ and $DLP$ and $c$ or $(n-1)c$ to each of the arcs $AKR'$, $AKP'$ where $c$ is the circumference of the first circle $AKP'$

## Propositions 18 19

I *If $OA$ be the initial line $A$ the end of the first turn of the spiral, and if the tangent to the spiral at $A$ be drawn the straight line $OB$ drawn from $O$ perpendicular to $OA$ will meet the said tangent in some point $B$ and $OB$ will be equal to the circumference of the first circle*

II *If $A'$ be the end of the second turn the perpendicular $OB$ will meet the tangent at $A'$ in some point $B'$ and $OB'$ will be equal to 2 (circumference of second circle)*

III *Generally if $A_n$ be the end of the $n$th turn and $OB$ meet the tangent at $A_n$ in $B_n$ then*

$$OB_n = nc_n$$

*where $c_n$ is the circumference of the nth circle*

I Let $AKC$ be the first circle Then since the backward angle between $OA$ and the tangent at $A$ is acute [Prop 16] the tangent will meet the first circle in a second point $C$ And the angles $CAO$ $BOA$ are together less than two right angles therefore $OB$ will meet $AC$ produced in some point $B$

Then if $c$ be the circumference of the first circle we have to prove that

$$OB = c$$

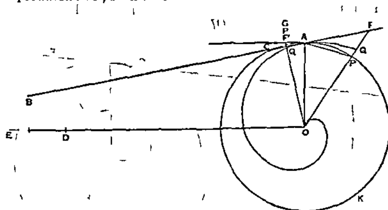
If not $OB$ must be either greater or less than $c$

(1) If possible suppose $OB > c$

Measure along $OB$ a length $OD$ less than $OB$ but greater than $c$

We have then a circle $AKC$ a chord $AC$ in it less than the diameter and a ratio $AO$ $OD$ which is greater than the ratio $AO$ $OB$ or (what is by similar

triangles equal to it) the ratio of $\frac{1}{2}AC$ to the perpendicular from $O$ on $AC$ Therefore [Prop 7] we can draw a straight line $OPF$, meeting the circle in $P$ and $CA$ produced in $F$, such that




$$FP \quad PA = AO \quad OD$$

Thus alternately, since $AO = PO$,

$$FP \quad PO = PA \quad OD$$
$$< (\text{arc } PA) \quad c$$

since (arc $PA$) $> PA$ and $OD > c$

*Componendo*

$$FO \quad PO < (c + \text{arc } PA) \quad c$$
$$< OQ \quad OA$$

where $OF$ meets the spiral in $Q$ [Prop 15]

Therefore since $OA = OP$, $FO < OQ$ which is impossible

Hence $OB \not> c$

(2) If possible suppose $OB < c$

Measure $OE$ along $OB$ so that $OE$ is greater than $OB$ but less than $c$

In this case since the ratio $AO \quad OE$ is less than the ratio $AO \quad OB$ (or the ratio of $\frac{1}{2}AC$ to the perpendicular from $O$ on $AC$), we can [Prop 8] draw a line $OF'P'G$ meeting $AC$ in $F'$ the circle in $P'$ and the tangent at $A$ to the circle in $G$, such that

$$F'P' \quad AG = AO \quad OE$$

Let $OP'G$ cut the spiral in $Q'$

Then we have alternately

$$F'P' \quad P'O = AG \quad OE$$
$$> (\text{arc } AP') \quad c$$

because $AG >$ (arc $AP'$) and $OE < c$

Therefore

$$F'O \quad P'O < (\text{arc } AKP') \quad c$$
$$< OQ' \quad OA$$ [Prop 14]

But this is impossible since $OA = OP'$ and $OQ' < OF'$

Hence $OB \not< c$

Since therefore $OB$ is neither greater nor less than $c$

$$OB = c$$

II Let $AKC$ be the second circle $AC$ being the tangent to the spiral

at $A'$ (which will cut the second circle, since the 'backward' angle $OA'C$ is acute). Thus, as before, the perpendicular $OB$ to $OA'$ will meet $A'C$ produced in some point $B$.

If then $c'$ is the circumference of the 'second circle,' we have to prove that

$$OB = 2c'.$$

For, if not, $OB$ must be either greater or less than $2c'$.

(1) If possible, suppose $OB > 2c'$.

Measure $OD$ along $OB$ so that $OD$ is less than $OB$ but greater than $2c'$.

Then, as in the case of the 'first circle' above, we can draw a straight line $OPF$ meeting the 'second circle' in $P$ and $CA'$ produced in $F$, such that

$$FP : PA' = A'O : OD.$$

Let $OF$ meet the spiral in $Q$.

We now have, since $A'O = PO$,

$$FP : PO = PA' : OD$$
$$< (\text{arc } A'P) : 2c',$$

because (arc $A'P$) $> A'P$ and $OD > 2c'$.

Therefore $\quad FO : PO < (2c' + \text{arc } A'P) : 2c'$
$$< OQ : OA'. \qquad \text{[Prop. 15, Cor.]}$$

Hence $FO < OQ$; which is impossible.

Thus $\quad OB \ngtr 2c'$.

Similarly, as in the case of the 'first circle,' we can prove that

$$OB \nless 2c'.$$

Therefore $\quad OB = 2c'$.

III. Proceeding in like manner to the 'third' and succeeding circles, we shall prove that

$$OB = nc.$$

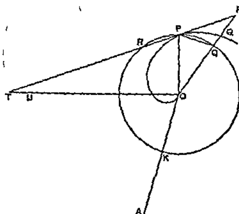
PROPOSITION 20.

I. *If $P$ be any point on the first turn of the spiral and $OT$ be drawn perpendicular to $OP$, $OT$ will meet the tangent at $P$ to the spiral in some point $T$; and, if the circle drawn with centre $O$ and radius $OP$ meet the initial line in $K$, then $OT$ is equal to the arc of this circle between $K$ and $P$ measured in the 'forward' direction of the spiral.*

II *Generally, if P be a point on the nth turn, and the notation be as before, while p represents the circumference of the circle with radius OP,*

$OT = (n-1)p + \text{arc } KP$ (*measured 'forward'*)

I Let $P$ be a point on the first turn of the spiral $OA$ the initial line, $PR$ the tangent at $P$ taken in the 'backward' direction




Then [Prop 16] the angle $OPR$ is acute Therefore $PR$ meets the circle through $P$ in some point $R$, and also $OT$ will meet $PR$ produced in some point $T$

If now $OT$ is not equal to the arc $KRP$, it must be either greater or less

(1) If possible let $OT$ be greater than the arc $KRP$

Measure $OU$ along $OT$ less than $OT$ but greater than the arc $KRP$

Then, since the ratio $PO : OU$ is greater than the ratio $PO : OT$, or (what is by similar triangles, equal to it) the ratio of $\frac{1}{2}PR$ to the perpendicular from $O$ on $PR$, we can draw a line $OQF$ meeting the circle in $Q$ and $RP$ produced in $F$ such that

$$FQ : PQ = PO : OU \quad \text{[Prop 7]}$$

Let $OF$ meet the spiral in $Q'$

We have then

$$FQ : QO = PQ : OU$$
$$< (\text{arc } PQ) : (\text{arc } KRP), \text{ by hypothesis}$$

*Componendo*

$$FO : QO < (\text{arc } KRQ) : (\text{arc } KRP)$$
$$< OQ' : OP \quad \text{[Prop 14]}$$

But $$QO = OP$$

Therefore $FO < OQ'$ which is impossible

Hence $$OT \not> (\text{arc } KRP)$$

(2) The proof that $OT \not< (\text{arc } KRP)$ follows the method of Prop 18 I (2), exactly as the above follows that of Prop 18 I (1)

Since then $OT$ is neither greater nor less than the arc $KRP$ it is equal to it

II If $P$ be on the second turn the same method shows that

$$OT = p + (\text{arc } KRP)$$

and, similarly we have for a point $P$ on the $n$th turn

$$OT = (n-1)p + (\text{arc } KRP)$$

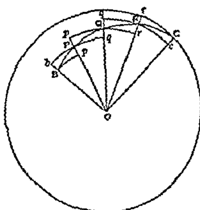
## Propositions 21, 22 23

*Given an area bounded by any arc of a spiral and the lines joining the extremities of the arc to the origin it is possible to circumscribe about the area one figure, and to inscribe in it another figure each consisting of similar sectors of circles and such that the circumscribed figure exceeds the inscribed by less than any assigned area*

For let $BC$ be any arc of the spiral $O$ the origin Draw the circle with centre

$O$ and radius $OC$, where $C$ is the "forward" end of the arc

Then, by bisecting the angle $BOC$ bisecting the resulting angles and so on continually we shall ultimately arrive at an angle $COr$ cutting off a sector of the circle less than any assigned area Let $COr$ be this sector




Let the other lines dividing the angle $BOC$ into equal parts meet the spiral in $P$ $Q$ and let $Or$ meet it in $R$ With $O$ as centre and radii $OB$ $OP$ $OQ$ $OR$ respectively describe arcs of circles $Bp$ $bBq$ $pQr$ $qRc'$ each meeting the adjacent radii as shown in the figure In each case the arc in the "forward" direction from each point will fall within and the arc in the "backward" direction outside the spiral

We have now a circumscribed figure and an inscribed figure each consisting of similar sectors of circles To compare their areas we take the successive sectors of each beginning from $OC$ and compare them

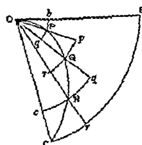
The sector $OCr$ in the circumscribed figure stands alone

And (sector $ORq$) = (sector $ORc$ ),
(sector $OQp$) = (sector $OQr$ )
(sector $OPb$) = (sector $OPq$ )

while the sector $OBp$ in the inscribed figure stands alone

Hence if the equal sectors be taken away the difference between the circumscribed and inscribed figures is equal to the difference between the sectors $OCr$ and $OBp$ and this difference is less than the sector $OCr$ which is itself less than any assigned area

The proof is exactly the same whatever be the number of angles into which the angle $BOC$ is divided the only difference being that when the arc begins from the origin the smallest sectors $OPb$ $OPq$ in each figure are equal and there is therefore no inscribed sector standing by itself so that the difference between the circumscribed and inscribed figures is equal to the sector $OCr$ itself




Thus the proposition is universally true

Cor Since the area bounded by the spiral is intermediate in magnitude between the circumscribed and inscribed figures it follows that

(1) *a figure can be circumscribed to the area such that it exceeds the area by less than any assigned space*

(2) *a figure can be inscribed such that the area exceeds it by less than any assigned space*

Proposition 24

*The area bounded by the first turn of the spiral and the initial line is equal to one third of the "first circle"* $[=\frac{1}{3}\pi(2\pi a)^2$ where the spiral is $r=a\theta]$

[The same proof shows equally that, *if OP be any radius vector in the first turn of the spiral, the area of the portion of the spiral bounded thereby is equal to one third of that sector of the circle drawn with radius OP which is bounded by the initial line and OP, measured in the "forward" direction from the initial line*.]

Let $O$ be the origin, $OA$ the initial line, $A$ the extremity of the first turn. Draw the "first circle," i.e. the circle with $O$ as centre and $OA$ as radius.

Then if $C_1$ be the area of the first circle, $R_1$ that of the first turn of the spiral bounded by $OA$, we have to prove that

$$R_1 = \tfrac{1}{3}C_1.$$

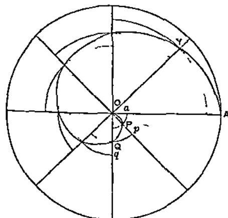
For, if not, $R_1$ must be either greater or less than $\tfrac{1}{3}C_1$.

I. If possible, suppose $R_1 < \tfrac{1}{3}C_1$.

We can then circumscribe a figure about $R_1$ made up of similar sectors of circles such that, if $F$ be the area of this figure,

$$F - R_1 < \tfrac{1}{3}C_1 - R_1,$$

whence $F < \tfrac{1}{3}C_1$.




Let $OP$, $OQ$, ... be the radii of the circular sectors beginning from the smallest. The radius of the largest is of course $OA$.

The radii then form an ascending arithmetical progression in which the common difference is equal to the least term $OP$. If $n$ be the number of the sectors, we have [by Prop. 10, Cor. 1]

$$n \cdot OA^2 < 3(OP^2 + OQ^2 + \ldots + OA^2);$$

and since the similar sectors are proportional to the squares on their radii, it follows that

$$C_1 < 3F,$$

or

$$F > \tfrac{1}{3}C_1.$$

But this is impossible, since $F$ was less than $\tfrac{1}{3}C_1$.

Therefore $R_1 \nless \tfrac{1}{3}C_1$.

II. If possible, suppose $R_1 > \tfrac{1}{3}C_1$.

We can then *inscribe* a figure made up of similar sectors of circles such that, if $f$ be its area,

$$R_1 - f < R_1 - \tfrac{1}{3}C_1,$$

whence $f > \tfrac{1}{3}C_1$.

If there are $(n-1)$ sectors, their radii, as $OP$, $OQ$, ..., form an ascending arithmetical progression in which the least term is equal to the common difference, and the greatest term, as $OY$, is equal to $(n-1)OP$.

Thus [Prop. 10, Cor. 1]

$$n \cdot OA^2 > 3(OP^2 + OQ^2 + \ldots + OY^2),$$

whence $C_1 > 3f$,

or $f < \tfrac{1}{3}C_1$;

which is impossible, since $f > \tfrac{1}{3}C_1$.

Therefore $R_1 \ngtr \tfrac{1}{3}C_1$.

Since then $R_1$ is neither greater nor less than $\tfrac{1}{3}C_1$,

$$R_1 = \tfrac{1}{3}C_1.$$

PROPOSITIONS 25, 26, 27

[Prop. 25.] *If $A_2$ be the end of the second turn of the spiral, the area bounded by the second turn and $OA_2$ is to the area of the "second circle" in the ratio of 7 to 12, being the ratio of $\{r_2r_1+\frac{1}{3}(r_2-r_1)^2\}$ to $r_2^2$, where $r_1, r_2$ are the radii of the "first" and "second" circles respectively.*

[Prop. 26.] *If BC be any arc measured in the "forward" direction on any turn of a spiral, not being greater than the complete turn, and if a circle be drawn with O as centre and OC as radius meeting OB in B', then*

$$(\textit{area of spiral between } OB, OC) : (\textit{sector } OB'C)$$
$$=\{OC \cdot OB+\tfrac{1}{3}(OC-OB)^2\} : OC^2.$$

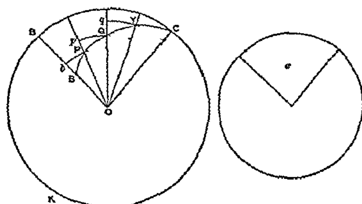
[Prop. 27.] *If $R_1$ be the area of the first turn of the spiral bounded by the initial line, $R_2$ the area of the ring added by the second complete turn, $R_3$ that of the ring added by the third turn, and so on, then*

$$R_3=2R_2,\quad R_4=3R_2,\quad R_5=4R_2, \quad \ldots \quad R_n=(n-1)R_2.$$

*Also* $$R_2=6R_1.$$

[Archimedes' proof of Prop. 25 is *mutatis mutandis* the same as his proof of the more general Prop. 26. The latter will accordingly be given here and applied to Prop. 25 as a particular case.]

Let $BC$ be an arc measured in the "forward" direction on any turn of the spiral, $CKB'$ the circle drawn with $O$ as centre and $OC$ as radius.




Take a circle such that the square of its radius is equal to

$$OC \cdot OB+\tfrac{1}{3}(OC-OB)^2,$$

and let $\sigma$ be a sector in it whose central angle is equal to the angle $BOC$.

Thus $\quad \sigma : (\text{sector } OB'C)=\{OC \cdot OB+\frac{1}{3}(OC-OB)^2\} : OC^2$,

and we have therefore to prove that

$$(\text{area of spiral } OBC)=\sigma.$$

For, if not, the area of the spiral $OBC$ (which we will call $S$) must be either greater or less than $\sigma$.

I. Suppose, if possible, $S<\sigma$.

Circumscribe to the area $S$ a figure made up of similar sectors of circles such that, if $F$ be the area of the figure,

$$F-S<\sigma-S,$$

whence $$F<\sigma.$$

Let the radii of the successive sectors starting from $OB$ be $OP, OQ, \dots OC$

Produce $OP, OQ \dots$ to meet the circle $CKB'$,

If then the lines $OB, OP, OQ \dots OC$ be $n$ in number, the number of sectors in the circumscribed figure will be $(n-1)$, and the sector $OB'C$ will also be divided into $(n-1)$ equal sectors Also $OB$ $OP, OQ, \dots OC$ will form an ascending arithmetical progression of $n$ terms

Therefore [see Prop 11 and Cor ]

$$(n-1)OC^2 : (OP^2+OQ^2+ \dots +OC^2) < OC^2 : \{OC \cdot OB+\tfrac{1}{3}(OC-OB)^2\}$$
$$<(\text{sector } OB'C) : \sigma \text{ by hypothesis}$$

Hence, since similar sectors are as the squares of their radii

$$(\text{sector } OB'C) : F < (\text{sector } OB'C) : \sigma$$

so that $F > \sigma$

But this is impossible because $F < \sigma$

Therefore $S \not< \sigma$

II Suppose if possible $S > \sigma$

Inscribe in the area $S$ a figure made up of similar sectors of circles such that if $f$ be its area,

$$S-f<S-\sigma,$$

whence $f > \sigma$

Suppose $OB$ $OP \dots OY$ to be the radii of the successive sectors making up the figure $f$ being $(n-1)$ in number

We shall have in this case [see Prop 11 and Cor ]

$$(n-1)OC^2 : (OB^2+OP^2+ \dots +OY^2) > OC^2 : \{OC \cdot OB+\tfrac{1}{3}(OC-OB)^2\},$$

whence $(\text{sector } OB'C) : f > (\text{sector } OB'C) : \sigma$

so that $f < \sigma$

But this is impossible because $f > \sigma$

Therefore $S \not> \sigma$

Since then $S$ is neither greater nor less than $\sigma$, it follows that

$$S=\sigma$$

In the particular case where $B$ coincides with $A_1$ the end of the first turn of the spiral, and $C$ with $A_2$ the end of the second turn the sector $OB'C$ becomes the complete 'second circle' that, namely with $OA_2$ (or $r_2$) as radius

Thus (area of spiral bounded by $OA_2$) : ('second circle')

$$=\{r_2r_1+\tfrac{1}{3}(r_2-r_1)^2\} : r_2^2$$
$$=(2+\tfrac{1}{3}) : 4 \quad (\text{since } r_2=2r_1)$$
$$=7 : 12$$

Again the area of the spiral bounded by $OA_2$ is equal to $R_1+R_2$ (i e the area bounded by the first turn and $OA_1$ together with the ring added by the second turn) Also the 'second circle' is four times the 'first circle' and therefore equal to $12R_1$

Hence $(R_1+R_2) : 12R_1=7 : 12$

or $R_1+R_2=7R_1$

Thus $R_2=6R_1$ (1)

Next, for the third turn we have

$$(R_1+R_2+R_3) : (\text{'third circle'})=\{r_3r_2+\tfrac{1}{3}(r_3-r_2)^2\} : r_3^2$$
$$=(3 \cdot 2+\tfrac{1}{3}) : 3^2$$
$$=19 : 27,$$

and ('third circle') $=9$('first circle')

$$=27R_1$$

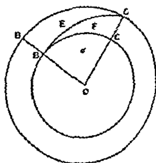
therefore $$R_1+R_2+R_3=19R_1,$$
and, by (1) above it follows that
$$R_3=12R_1$$
$$=2R_2 \qquad (2)$$
and so on

Generally, we have
$$(R_1+R_2+\quad +R\ )\quad (n\text{th circle})=\{r_n r_{-1}+\tfrac{1}{3}(r\ -r_{n-1})^2\}\quad r_n^2,$$
$$(R_1+R_2+\quad +R_{-1})\quad (\overline{n-1\text{th}}\text{ circle})=\{r_{n-1}\,r_{n-2}+\tfrac{1}{3}(r_{n-1}-r_{n-2})^2\}\quad r_{n-1}^2,$$
and $$(n\text{th circle})\quad (\overline{n-1\text{th}}\text{ circle})=r^2\quad r_{n-1}^2$$
Therefore
$$(R_1+R_2+\quad +R\ )\quad (R_1+R_2+\quad +R_{n-1})$$
$$=\{n(n-1)+\tfrac{1}{3}\}\quad \{(n-1)(n-2)+\tfrac{1}{3}\}$$
$$=\{3n(n-1)+1\}\quad \{3(n-1)(n-2)+1\}$$
*Dirimendo,*
$$R\quad (R_1+R_2+\quad +R_{n-1})=6(n-1)\quad \{3(n-1)(n-2)+1\} \qquad (\alpha)$$
Similarly
$$R_{-1}\quad (R_1+R_2+\quad +R_{n-2})=6(n-2)\quad \{3(n-2)(n-3)+1\},$$
from which we derive
$$R_{n-1}\quad (R_1+R_2+\quad +R_{-1})$$
$$=6(n-2)\quad \{6(n-2)+3(n-2)(n-3)+1\}$$
$$=6(n-2)\quad \{3(n-1)(n-2)+1\} \qquad (\beta)$$
Combining ($\alpha$) and ($\beta$) we obtain
$$R\quad R_{n-1}=(n-1)\quad (n-2)$$
Thus
$R_2\ R_3\ R_4\quad R$ are in the ratio of the successive numbers 1 2, 3 $(n-1)$

## Proposition 28

*If O be the origin and BC any arc measured in the 'forward direction on any turn of the spiral let two circles be drawn (1) with centre O and radius OB meeting OC in C and (2) with centre O and radius OC, meeting OB produced in B Then if E denote the area bounded by the larger circular arc B C, the line B B and the spiral BC while F denotes the area bounded by the smaller arc BC, the line CC and the spiral BC*
$$E\quad F=\{OB+\tfrac{2}{3}(OC-OB)\}\quad \{OB+\tfrac{1}{3}(OC-OB)\}$$
Let $\sigma$ denote the area of the lesser sector $OBC$ then the larger sector $OB\,C$ is equal to $\sigma+F+E$

Thus [Prop 26]
$$(\sigma+F)\quad (\sigma+F+E)=$$
$$\{OC\quad OB+\tfrac{1}{3}(OC-OB)^2\}\quad OC^2 \qquad (1)$$




whence
$$E\quad (\sigma+F)=\{OC(OC-OB)-\tfrac{1}{3}(OC-OB)^2\}$$
$$\{OC\quad OB+\tfrac{1}{3}(OC-OB)^2\}$$
$$=\{OB(OC-OB)+\tfrac{2}{3}(OC-OB)^2\}$$
$$\{OC\quad OB+\tfrac{1}{3}(OC-OB)^2\} \qquad (2)$$
Again
$$(\sigma+F+E)\quad \sigma=OC^2\quad OB^2$$
Therefore by the first proportion above *ex aequali*
$$(\sigma+F)\quad \sigma=\{OC\quad OB+\tfrac{1}{3}(OC-OB)^2\}\quad OB^2,$$

whence

$$(\sigma+F)\quad F=\{OC\quad OB+\tfrac{1}{3}(OC-OB)^2\}$$
$$\{OB(OC-OB)+\tfrac{1}{2}(OC-OB)^2\}$$

Combining this with (2) above we obtain

$$E\quad F=\{OB(OC-OB)+\tfrac{2}{3}(OC-OB)^2\}\quad\{OB(OC-OB)+\tfrac{1}{3}(OC-OB)^2\}$$
$$=\{OB+\tfrac{2}{3}(OC-OB)\}\quad\{OB+\tfrac{1}{3}(OC-OB)\}$$

# ON THE EQUILIBRIUM OF PLANES OR THE CENTRES OF GRAVITY OF PLANES

## BOOK ONE

"I POSTULATE the following

1 Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance '

2 If, when weights at certain distances are in equilibrium, something be added to one of the weights they are not in equilibrium but incline towards that weight to which the addition was made "

3 ' Similarly if anything be taken away from one of the weights they are not in equilibrium but incline towards the weight from which nothing was taken '

4 "When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide '

5 ' In figures which are unequal but similar, the centres of gravity will be similarly situated By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles they make equal angles with the corresponding sides

6 "If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances

7 "In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure

PROPOSITION 1

*Weights which balance at equal distances are equal*

For, if they are unequal take away from the greater the difference between the two The remainders will then not balance [*Post* 3], which is absurd

Therefore the weights cannot be unequal

PROPOSITION 2

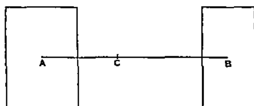
*Unequal weights at equal distances will not balance but will incline towards the greater weight*

For take away from the greater the difference between the two The equal remainders will therefore balance [*Post* 1] Hence if we add the difference again the weights will not balance but incline towards the greater [*Post* 2]

PROPOSITION 3

*Unequal weights will balance at unequal distances, the greater weight being at the lesser distance*

Let $A$ $B$ be two unequal weights (of which $A$ is the greater) balancing about $C$ at distances $AC$ $BC$ respectively




Then shall $AC$ be less than $BC$ For, if not, take away from $A$ the weight $(A-B)$ The remainders will then incline towards $B$ [*Post* 3] But this is impossible for (1) if $AC=CB$ the equal remainders will balance or (2) if $AC>CB$, they will incline towards $A$ at the greater distance [*Post* 1]

Hence $AC<CB$

*Conversely*, if the weights balance, and $AC<CB$, then $A>B$

PROPOSITION 4

*If two equal weights have not the same centre of gravity, the centre of gravity of both taken together is at the middle point of the line joining their centres of gravity*

[Proved from Prop 3 by *reductio ad absurdum*]

PROPOSITION 5

*If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude*

[This follows immediately from Prop 4]

COR 1 *The same is true of any odd number of magnitudes if those which are at equal distances from the middle one are equal while the distances between their centres of gravity are equal*

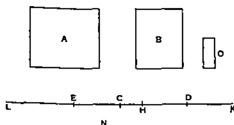
COR 2 *If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line and if the two middle ones be equal while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones*

PROPOSITIONS 6, 7

*Two magnitudes whether commensurable* [Prop 6] *or incommensurable* [Prop 7] *balance at distances reciprocally proportional to the magnitudes*

I Suppose the magnitudes $A$, $B$ to be commensurable, and the points $A$, $B$ to be their centres of gravity Let $DE$ be a straight line so divided at $C$ that

$$A \quad B = DC \quad CE$$




We have then to prove that if $A$ be placed at $E$ and $B$ at $D$ $C$ is the centre of gravity of the two taken together

Since $A$ $B$ are commensurable so are $DC$ $CE$ Let $N$ be a common measure of $DC$ $CE$ Make $DH$ $DK$ each equal to $CE$ and $EL$ (on $CE$ produced) equal to $CD$ Then $EH=CD$, since $DH=CE$ Therefore $LH$ is bisected at $E$ as $HK$ is bisected at $D$

Thus $LH$ $HK$ must each contain $N$ an even number of times

Take a magnitude $O$ such that $O$ is contained as many times in $A$ as $N$ is contained in $LH$, whence

$$A \quad O = LH \quad N$$

But $$B \quad A = CE \quad DC$$
$$= HK \quad LH$$

Hence *ex aequali* $B \quad O = HK \quad N$, or $O$ is contained in $B$ as many times as $N$ is contained in $HK$

Thus $O$ is a common measure of $A$ $B$

Divide $LH$, $HK$ into parts each equal to $N$ and $A$ $B$ into parts each equal to $O$ The parts of $A$ will therefore be equal in number to those of $LH$, and the parts of $B$ equal in number to those of $HK$ Place one of the parts of $A$ at the middle point of each of the parts $N$ of $LH$ and one of the parts of $B$ at the middle point of each of the parts $N$ of $HK$

Then the centre of gravity of the parts of $A$ placed at equal distances on $LH$ will be at $E$ the middle point of $LH$ [Prop 5 Cor 2] and the centre of gravity of the parts of $B$ placed at equal distances along $HK$ will be at $D$ the middle point of $HK$

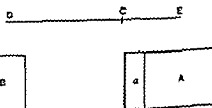
Thus we may suppose $A$ itself applied at $E$ and $B$ itself applied at $D$

But the system formed by the parts $O$ of $A$ and $B$ together is a system of equal magnitudes even in number and placed at equal distances along $LK$ And since $LE = CD$ and $EC = DK$ $LC = CK$ so that $C$ is the middle point of $LK$ Therefore $C$ is the centre of gravity of the system ranged along $LK$

Therefore $A$ acting at $E$ and $B$ acting at $D$ balance about the point $C$

II Suppose the magnitudes to be incommensurable, and let them be $(A+a)$ and $B$ respectively Let $DE$ be a line divided at $C$ so that

$$(A+a) \quad B = DC \quad CE$$




Then if $(A+a)$ placed at $E$ and $B$ placed at $D$ do not balance about $C$ $(A+a)$ is either too great to balance $B$ or not great enough

Suppose if possible that $(A+a)$ is too great to balance $B$ Take from $(A+a)$ a magnitude $a$ smaller than the deduction which would make the remainder balance $B$ but such that the remainder $A$ and the magnitude $B$ are commensurable

Then since $A$ $B$ are commensurable and

$$A \quad B < DC \quad CE$$

$A$ and $B$ will not balance [Prop 6] but $D$ will be depressed

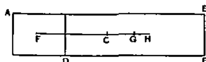
But this is impossible since the deduction $a$ was an insufficient deduction from $(A+a)$ to produce equilibrium so that $E$ was still depressed

Therefore $(A+a)$ is not too great to balance $B$ and similarly it may be proved that $B$ is not too great to balance $(A+a)$

Hence $(A+a)$ $B$ taken together have their centre of gravity at $C$

### Proposition 8

*If $AB$ be a magnitude whose centre of gravity is $C$ and $AD$ a part of it whose centre of gravity is $F$ then the centre of gravity of the remaining part will be a point $G$ on $FC$ produced such that*




GC CF=(AD) (DE)

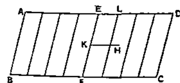
For, if the centre of gravity of the remainder (DE) be not G, let it be a point H Then an absurdity follows at once from Props 6, 7

PROPOSITION 9

*The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides*

Let ABCD be a parallelogram, and let EF join the middle points of the opposite sides AD, BC

If the centre of gravity does not lie on EF suppose it to be H, and draw HK parallel to AD or BC meeting EF in K




Then it is possible by bisecting ED then bisecting the halves and so on continually to arrive at a length EL less than KH Divide both AE and ED into parts each equal to EL and through the points of division draw parallels to AB or CD

We have then a number of equal and similar parallelograms, and if any one be applied to any other their centres of gravity coincide [*Post* 4] Thus we have an even number of equal magnitudes whose centres of gravity lie at equal distances along a straight line Hence the centre of gravity of the whole parallelogram will lie on the line joining the centres of gravity of the two middle parallelograms [Prop 5, Cor 2]

But this is impossible for H is outside the middle parallelograms

Therefore the centre of gravity cannot but lie on EF

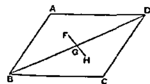
PROPOSITION 10

*The centre of gravity of a parallelogram is the point of intersection of its diagonals*

For by the last proposition the centre of gravity lies on each of the lines which bisect opposite sides Therefore it is at the point of their intersection, and this is also the point of intersection of the diagonals

*Alternative proof*

Let ABCD be the given parallelogram and BD a diagonal Then the triangles ABD CDB are equal and similar, so that [*Post* 4] if one be applied to the other their centres of gravity will fall one upon the other




Suppose F to be the centre of gravity of the triangle ABD Let G be the middle point of BD Join FG and produce it to H, so that FG=GH

If we then apply the triangle ABD to the triangle CDB so that AD falls on CB and AB on CD the point F will fall on H

But [by *Post* 4] F will fall on the centre of gravity of CDB Therefore H is the centre of gravity of CDB

Hence since F H are the centres of gravity of the two equal triangles the centre of gravity of the whole parallelogram is at the middle point of FH i e at the middle point of BD which is the intersection of the two diagonals

### Proposition 11

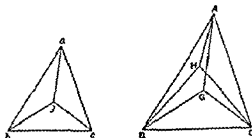
*If abc ABC be two similar triangles and g G two points in them similarly situated with respect to them respectively, then if g be the centre of gravity of the triangle abc G must be the centre of gravity of the triangle ABC*

Suppose

$ab \quad bc \quad ca = AB \quad BC \quad CA$

The proposition is proved by an obvious *reductio ad absurdum* For if $G$ be not the centre of gravity of the triangle $ABC$ suppose $H$ to be its centre of gravity

*Post* 5 requires that $g$ $H$ shall be similarly situated with respect to the triangles respectively and this leads at once to the absurdity that the angles $HAB$, $GAB$ are equal




### Proposition 12

*Given two similar triangles abc ABC and d D the middle points of bc BC respectively then, if the centre of gravity of abc lie on ad, that of ABC will lie on AD*

Let $g$ be the point on $ad$ which is the centre of gravity of $abc$

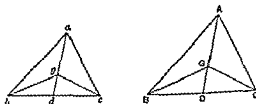
Take $G$ on $AD$ such that

$$ad \quad ag = AD \quad AG$$

and join $gb$, $gc$ $GB$ $GC$

Then since the triangles are similar and $bd$ $BD$ are the halves of $bc$ $BC$ respectively

$$ab \quad bd = AB \quad BD$$




and the angles $abd$ $ABD$ are equal

Therefore the triangles $abd$ $ABD$ are similar and

$$\angle bad = \angle BAD$$

Also $\quad ba \quad ad = BA \quad AD$

while from above $\quad ad \quad ag = AD \quad AG$

Therefore $ba \quad ag = BA \quad AG$ while the angles $bag$ $BAG$ are equal

Hence the triangles $bag$ $BAG$ are similar and

$$\angle abg = \angle ABG$$

And since the angles $abd$ $ABD$ are equal it follows that

$$\angle gbd = \angle GBD$$

In exactly the same manner we prove that

$$\angle gac = \angle GAC$$
$$\angle acg = \angle ACG$$
$$\angle gcd = \angle GCD$$

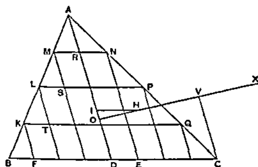
Therefore $g$ $G$ are similarly situated with respect to the triangles respectively whence [Prop 11] $G$ is the centre of gravity of $ABC$

PROPOSITION 13

*In any triangle the centre of gravity lies on the straight line joining any angle to the middle point of the opposite side*

Let $ABC$ be a triangle and $D$ the middle point of $BC$ Join $AD$ Then shall the centre of gravity lie on $AD$

For if possible let this not be the case and let $H$ be the centre of gravity Draw $HI$ parallel to $CB$ meeting $AD$ in $I$




Then if we bisect $DC$ then bisect the halves, and so on, we shall at length arrive at a length as $DE$ less than $HI$ Divide both $BD$ and $DC$ into lengths each equal to $DE$ and through the points of division draw lines each parallel to $DA$ meeting $BA$ and $AC$ in points as $K$, $L$, $M$ and $N$, $P$, $Q$ respectively

Join $MN$, $LP$ $KQ$ which lines will then be each parallel to $BC$

We have now a series of parallelograms as $FQ$ $TP$ $SN$ and $AD$ bisects opposite sides in each Thus the centre of gravity of each parallelogram lies on $AD$ [Prop 9] and therefore the centre of gravity of the figure made up of them all lies on $AD$

Let the centre of gravity of all the parallelograms taken together be $O$ Join $OH$ and produce it, also draw $CX$ parallel to $DA$ meeting $OH$ produced in $X$

Now if $n$ be the number of parts into which $AC$ is divided

$\triangle ADC$ (sum of triangles on $AN$ $NP$, ) $=AC^2$ ($AN^2+NP^2+$ )

$=n^2$ $n$

$=n$ $1$

$=AC$ $AN$

Similarly

$\triangle ABD$ (sum of triangles on $AM$ $ML$, ) $=AB$ $AM$

And $AC$ $AN=AB$ $AM$

It follows that

$\triangle ABC$ (sum of all the small $\triangle$s) $=CA$ $AN$

$>XO$ $OH$ by parallels

Suppose $OV$ produced to $X$ so that

$\triangle ABC$ (sum of small $\triangle$s) $=XO$ $OH$

whence *dividendo*

(sum of parallelograms) (sum of small $\triangle$s) $=XH$ $HO$

Since then the centre of gravity of the triangle $ABC$ is at $H$ and the centre of gravity of the part of it made up of the parallelograms is at $O$ it follows from Prop 8 that the centre of gravity of the remaining portion consisting of all the small triangles taken together is at $X$

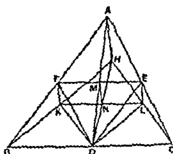
But this is impossible since all the triangles are on one side of the line through $X$ parallel to $AD$

Therefore the centre of gravity of the triangle cannot but lie on $AD$

*Alternative proof*

Suppose if possible that $H$ not lying on $AD$, is the centre of gravity of the triangle $ABC$ Join $AH$, $BH$ $CH$ Let $E$ $F$ be the middle points of $CA$, $AB$ respectively and join $DE$ $EF$ $FD$ Let $EF$ meet $AD$ in $M$

Draw $FK$ $EL$ parallel to $AH$ meeting $BH$ $CH$ in $K$ $L$ respectively Join $KD$ $HD$ $LD$, $KL$ Let $KL$ meet $DH$ in $N$ and join $MN$




Since $DE$ is parallel to $AB$ the triangles $ABC$ $EDC$ are similar

And, since $CE=EA$ and $EL$ is parallel to $AH$ it follows that $CL=LH$ And $CD=DB$ Therefore $BH$ is parallel to $DL$

Thus in the similar and similarly situated triangles $ABC$ $EDC$ the straight lines $AH$ $BH$ are respectively parallel to $EL$ $DL$ and it follows that $H$ $L$ are similarly situated with respect to the triangles respectively

But $H$ is by hypothesis the centre of gravity of $ABC$ Therefore $L$ is the centre of gravity of $EDC$ [Prop 11]

Similarly the point $K$ is the centre of gravity of the triangle $FBD$

And the triangles $FBD$ $EDC$ are equal so that the centre of gravity of both together is at the middle point of $KL$ i e at the point $N$

The remainder of the triangle $ABC$ after the triangles $FBD$ $EDC$ are deducted is the parallelogram $AFDE$ and the centre of gravity of this parallelogram is at $M$ the intersection of its diagonals

It follows that the centre of gravity of the whole triangle $ABC$ must lie on $MN$ that is $MN$ must pass through $H$ which is impossible (since $MN$ is parallel to $AH$)

Therefore the centre of gravity of the triangle $ABC$ cannot but lie on $AD$

PROPOSITION 14

It follows at once from the last proposition that *the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively*

PROPOSITION 15

*If $AD$ $BC$ be the two parallel sides of a trapezium $ABCD$ $AD$ being the smaller, and if $AD$ $BC$ be bisected at $E$ $F$ respectively then the centre of gravity of the trapezium is at a point $G$ on $EF$ such that*

$$GE : GF=(2BC+AD) : (2AD+BC)$$

Produce $BA$ $CD$ to meet at $O$ Then $FE$ produced will also pass through $O$, since $AE=ED$ and $BF=FC$

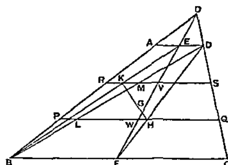
Now the centre of gravity of the triangle $OAD$ will lie on $OE$ and that of the triangle $OBC$ will lie on $OF$ [Prop 13]

It follows that the centre of gravity of the remainder the trapezium $ABCD$ will also lie on $OF$ [Prop 8]

Join $BD$ and divide it at $L$ $M$ into three equal parts Through $L$ $M$ draw $PQ$ $RS$ parallel to $BC$ meeting $BA$ in $P$ $R$ $FE$ in $W$, $V$ and $CD$ in $Q$ $S$ respectively

Join $DF$, $BE$ meeting $PQ$ in $H$ and $RS$ in $K$ respectively.

Now, since $$BL = \tfrac{1}{3}BD,$$ $$FH = \tfrac{1}{3}FD.$$




Therefore $H$ is the centre of gravity of the triangle $DBC$.

Similarly, since $EK = \tfrac{1}{3}BE$, it follows that $K$ is the centre of gravity of the triangle $ADB$.

Therefore the centre of gravity of the triangles $DBC$, $ADB$ together, i.e. of the trapezium, lies on the line $HK$.

But it also lies on $OF$.

Therefore, if $OF$, $HK$ meet in $G$, $G$ is the centre of gravity of the trapezium.

Hence [Props. 6, 7]

$$\triangle DBC \quad \triangle ABD = KG \quad GH$$
$$= VG \quad GW.$$

But $$\triangle DBC \quad \triangle ABD = BC \quad AD.$$

Therefore $$BC \quad AD = VG \quad GW.$$

It follows that

$$(2BC + AD) \quad (2AD + BC) = (2VG + GW) \quad (2GW + VG)$$
$$= EG \quad GF.$$

Q.E.D.

# ON THE EQUILIBRIUM OF PLANES

## BOOK TWO

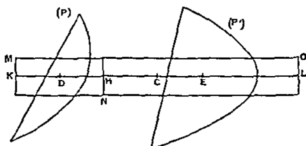
PROPOSITION 1

*If $P$, $P'$ be two parabolic segments and $D$, $E$ their centres of gravity respectively the centre of gravity of the two segments taken together will be at a point $C$ on $DE$ determined by the relation*

$$P \quad P = CE \quad CD$$

In the same straight line with $DE$ measure $EH$ $EL$ each equal to $DC$, and $DK$ equal to $DH$ whence it follows at once that $DK = CE$, and also that

$$KC = CL$$




Apply a rectangle $MN$ equal in area to the parabolic segment $P$ to a base equal to $KH$, and place the rectangle so that $KH$ bisects it, and is parallel to its base

Then $D$ is the centre of gravity of $MN$ since $KD = DH$

Produce the sides of the rectangle which are parallel to $KH$, and complete the rectangle $NO$ whose base is equal to $HL$ Then $E$ is the centre of gravity of the rectangle $NO$

Now $\qquad (MN) \quad (NO) = KH \quad HL$

$\qquad\qquad = DH \quad EH$

$\qquad\qquad = CE \quad CD$

$\qquad\qquad = P \quad P$

But $\qquad (MN) = P$

Therefore $\qquad (NO) = P$

Also since $C$ is the middle point of $KL$ $C$ is the centre of gravity of the whole parallelogram made up of the two parallelograms $(MN)$ $(NO)$, which are equal to, and have the same centres of gravity as $P$, $P'$ respectively

Hence $C$ is the centre of gravity of $P$ $P'$ taken together

## DEFINITION AND LEMMAS PRELIMINARY TO PROPOSITION 2

If in a segment bounded by a straight line and a section of a right angled cone [a parabola] a triangle be inscribed having the same base as the segment and equal height, if again triangles be inscribed in the remaining segments having the same bases as the segments and equal height and if in the remaining segments triangles be inscribed in the same manner let the resulting figure be said to be *inscribed in the recognised manner* in the segment

"And it is plain

(1) that *the lines joining the two angles of the figure so inscribed which are nearest to the vertex of the segment and the next pairs of angles in order will be parallel to the base of the segment* "

(2) "that *the said lines will be bisected by the diameter of the segment*, and

(3) ' that *they will cut the diameter in the proportions of the successive odd numbers, the number one having reference to [the length adjacent to] the vertex of the segment*

"And these properties will have to be proved in their proper places '

### Proposition 2

*If a figure be "inscribed in the recognised manner' in a parabolic segment, the centre of gravity of the figure so inscribed will lie on the diameter of the segment*

For, in the figure of the foregoing lemmas, the centre of gravity of the trapezium $BRrb$ must lie on $XO$, that of the trapezium $RQqr$ on $WX$, and so on while the centre of gravity of the triangle $PAp$ lies on $AV$

Hence the centre of gravity of the whole figure lies on $AO$

### Proposition 3

*If $BAB'$, $bab'$ be two similar parabolic segments whose diameters are $AO$ $ao$ respectively and if a figure be inscribed in each segment in the recognised manner ' the number of sides in each figure being equal, the centres of gravity of the inscribed figures will divide $AO$ $ao$ in the same ratio* [1]

Suppose $BRQPAP'Q'R'B'$ $brqpap'q'r'b'$ to be the two figures inscribed "in the recognised manner Join $PP'$, $QQ'$ $RR'$ meeting $AO$ in $L$ $M$, $N$, and $pp'$ $qq'$ $rr'$ meeting $ao$ in $l$ $m$ $n$

Then [Lemma (3)]

$$AL \quad LM \quad MN \quad NO = 1 \quad 3 \quad 5 \quad 7$$
$$= al \quad lm \quad mn \quad no$$

so that $AO$ $ao$ are divided in the same proportion

Also, by reversing the proof of Lemma (3) we see that

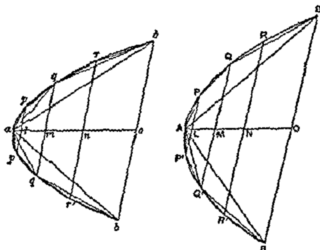
$$PP' \quad pp' = QQ' \quad qq' = RR' \quad rr' = BB' \quad bb'$$

Since then $RR'$ $BB' = rr'$ $bb'$, and these ratios respectively determine the proportion in which $NO$, $no$ are divided by the centres of gravity of the trapezia $BRR'B'$ $brr'b'$ [I 15] it follows that the centres of gravity of the trapezia divide $NO$ $no$ in the same ratio

Similarly the centres of gravity of the trapezia $RQQ'R'$ $rqq'r'$ divide $MN$ $mn$ in the same ratio respectively, and so on

[1] Archimedes enunciates this proposition as true of *similar* segments but it is equally true of segments which are not similar as the course of the proof will show

Lastly, the centres of gravity of the triangles $PAP'$, $pap'$ divide $AL$, $al$ respectively in the same ratio




Moreover the corresponding trapezia and triangles are each to each in the same proportion (since their sides and heights are respectively proportional), while $AO$, $ao$ are divided in the same proportion

Therefore the centres of gravity of the complete inscribed figures divide $AO$, $ao$ in the same proportion

## Proposition 4

*The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment*

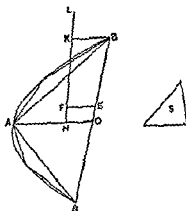
Let $BAB'$ be a parabolic segment, $A$ its vertex and $AO$ its diameter

Then if the centre of gravity of the segment does not lie on $AO$ suppose it to be, if possible, the point $F$ Draw $FE$ parallel to $AO$ meeting $BB'$ in $E$

Inscribe in the segment the triangle $ABB'$ having the same vertex and height as the segment, and take an area $S$ such that

$$\triangle ABB' : S = BE : EO$$

We can then inscribe in the segment "in the recognised manner" a figure such that the segments of the parabola left over are together less than $S$ [1]




[1] For Prop 20 of the *Quadrature of the Parabola* proves that, if in any segment the triangle with the same base and height be inscribed, the triangle is greater than half the segment, whence it appears that, each time that we increase the number of the sides of the figure inscribed "in the recognised manner," we take away more than half of the remaining segments

Let the inscribed figure be drawn accordingly, its centre of gravity then lies on $AO$ [Prop 2] Let it be the point $H$

Join $HF$ and produce it to meet in $K$ the line through $B$ parallel to $AO$

Then we have

(inscribed figure) (remainder of segmt ) $> \triangle ABB \quad S$

$> BE \quad EO$

$> KF \quad FH$

Suppose $L$ taken on $HK$ produced so that the former ratio is equal to the ratio $LF \quad FH$

Then since $H$ is the centre of gravity of the inscribed figure, and $F$ that of the segment, $L$ must be the centre of gravity of all the segments taken together which form the remainder of the original segment [I 8]

But this is impossible since all these segments lie on one side of the line drawn through $L$ parallel to $AO$ (Cf *Post* 7]

Hence the centre of gravity of the segment cannot but lie on $AO$

## Proposition 5

*If in a parabolic segment a figure be inscribed ' in the recognised manner, the centre of gravity of the segment is nearer to the vertex of the segment than the centre of gravity of the inscribed figure is*

Let $BAB$ be the given segment and $AO$ its diameter *First* let $ABB$ be the *triangle* inscribed ' in the recognised manner

Divide $AO$ in $F$ so that $AF = 2FO$ $F$ is then the centre of gravity of the triangle $ABB$

Bisect $AB$ $AB$ in $D$ $D$ respectively and join $DD$ meeting $AO$ in $E$ Draw $DQ$ $DQ$ parallel to $OA$ to meet the curve $QD$ $QD$ will then be the diameters of the segments whose bases are $AB$ $AB'$, and the centres of gravity of those segments will lie respectively on $QD$ $QD$ [Prop 4] Let them be $H$, $H$ and join $HH$ meeting $AO$ in $K$

Now $QD$ $QD$ are equal ¹ and therefore the segments of which they are the diameters are equal [*On Conoids and Spheroids* Prop 3]

Also since $QD$ $QD$ are parallel and $DE = ED$ $K$ is the middle point of $HH$

Hence the centre of gravity of the equal segments $AQB$ $AQB'$ taken together is $K$ where $K$ lies between $E$ and $A$ And the centre of gravity of the triangle $ABB$ is $F$

It follows that the centre of gravity of the whole segment $BAB$ lies between $K$ and $F$ and is therefore nearer to the vertex $A$ than $F$ is

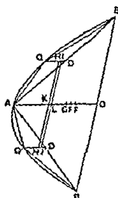
*Secondly* take the *five sided* figure $BQAQB$ inscribed in the recognised manner $QD$ $QD$ being as before the diameters of the segments $AQB$ $AQB$

Then by the first part of this proposition the centre of gravity of the segment $AQB$ (lying of course on $QD$) is nearer to $Q$ than the centre of gravity of

¹This may either be inferred from Lemma (1) above (since $QQ$ $DD$ are both parallel to $BB'$) or from Prop 19 of the *Quadrature of the Parabola* which applies equally to $Q$ or $Q$

the triangle $AQB$ is Let the centre of gravity of the segment be $H$ and that of the triangle $I$

Similarly let $H'$ be the centre of gravity of the segment $AQ B$ ,and $I'$ that of the triangle $AQ B'$




It follows that the centre of gravity of the two segments $AQB$, $AQ B'$ taken together is $K$, the middle point of $HH'$ and that of the two triangles $AQB$ $AQ B'$ is $L$, the middle point of $II$

If now the centre of gravity of the triangle $ABB$ be $F$, the centre of gravity of the whole segment $BAB$ (i e that of the triangle $ABB$ and the two segments $AQB$ $AQ'B'$ taken together) is a point $G$ on $KF$ determined by the proportion

(sum of segments $AQB$ $AQ B'$) $\triangle ABB = FG\ GK$ [I 6 7]

And the centre of gravity of the inscribed figure $BQAQ B$ is a point $F$ on $LF$ determined by the proportion

$$(\triangle AQB+\triangle AQ B)\ \ \triangle ABB = FF\ \ F L \qquad [I\ 6\ 7]$$

[Hence $FG\ \ GK > FF\ \ F L$

or $GK\ \ FG < F L\ \ FF$

and *componendo* $FK\ \ FG < FL\ \ FF$ while $FK > FL$ ]

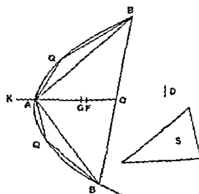
Therefore $FG > FF$ or $G$ lies nearer than $F$ to the vertex $A$

Using this last result and proceeding in the same way we can prove the proposition for *any* figure inscribed in the recognised manner

PROPOSITION 6

*Given a segment of a parabola cut off by a straight line it is possible to inscribe in it in the recognised manner a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length*

Let $BAB$ be the segment $AO$ its diameter $G$ its centre of gravity and $ABB$ the triangle inscribed in the recognised manner




Let $D$ be the assigned length and $S$ an area such that

$$AG\ \ D = \triangle ABB\ \ S$$

In the segment inscribe in the recognised manner a figure such that the sum of the segments left over is less than $S$ Let $F$ be the centre of gravity of the inscribed figure

We shall prove that $FG < D$

For if not $FG$ must be either equal to or greater than $D$

And clearly

(inscribed fig ) (sum of remaining segmts )

$> \triangle ABB\ \ S$

$> AG\ \ D$

$> AG\ \ FG$ by hypothesis (since $FG \not< D$)

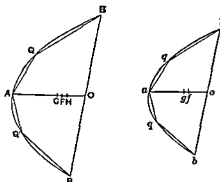
Let the first ratio be equal to the ratio $KG$  $FG$ (where $K$ lies on $GA$ produced), and it follows that $K$ is the centre of gravity of the small segments taken together [I 8]

But this is impossible, since the segments are all on the same side of a line drawn through $K$ parallel to $BB'$

Hence $FG$ cannot but be less than $D$

### Proposition 7

*If there be two similar parabolic segments their centres of gravity divide their diameters in the same ratio*




Let $BAB'$ $bab'$ be the two similar segments $AO$, $ao$ their diameters and $G$, $g$ their centres of gravity respectively

Then if $G$, $g$ do not divide $AO$ $ao$ respectively in the same ratio, suppose $H$ to be such a point on $AO$ that

$$AH \quad HO = ag \quad go$$

and inscribe in the segment $BAB'$ "in the recognised manner a figure such that if $F$ be its centre of gravity,

$$GF < GH \qquad \text{[Prop 6]}$$

Inscribe in the segment $bab$ ' in the recognised manner a similar figure then if $f$ be the centre of gravity of this figure

$$ag < af \qquad \text{[Prop 5]}$$

And by Prop 3 $\quad af \quad fo = AF \quad FO$

But $\quad AF \quad FO < AH \quad HO$

$\quad < ag \quad go$ by hypothesis

Therefore $\quad af \quad fo < ag \quad go$ which is impossible

It follows that $G$, $g$ cannot but divide $AO$ $ao$ in the same ratio

### Proposition 8

*If $AO$ be the diameter of a parabolic segment, and $G$ its centre of gravity then*

$$AG = \tfrac{3}{2}GO$$

Let the segment be $BAB'$ Inscribe the triangle $ABB$ in the recognised manner and let $F$ be its centre of gravity

Bisect $AB$ $AB$ in $D$ $D$, and draw $DQ$ $D Q$ parallel to $OA$ to meet the curve so that $QD$ $Q D$ are the diameters of the segments $AQB$ $AQ B$ respectively

Let $H$ $H'$ be the centres of gravity of the segments $AQB$ $AQ B$ respectively Join $QQ$ $HH'$ meeting $AO$ in $V$ $K$ respectively

$K$ is then the centre of gravity of the two segments $AQB$, $AQ B$ taken together

Now $\quad AG \quad GO = QH \quad HD$ [Prop 7]

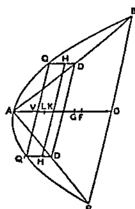
whence $\quad AO \quad OG = QD \quad HD$

But $AO = 4QD$ [as is easily proved by means of Lemma (3) p 511]

Therefore $\quad OG = 4HD$

and by subtraction $\quad AG = 4QH$

Also by Lemma (2) $QQ$ is parallel to $BB'$ and therefore to $DD'$ It follows from Prop 7 that $HH'$ is also parallel to $QQ$ or $DD$,




and hence $QH = VK$

Therefore $AG = 4VK$

and $AV + KG = 3VK$

Measuring $VL$ along $VK$ so that $VL = \frac{1}{3}AV$, we have

$$KG = 3LK \qquad (1)$$

Again $AO = 4AV$ [Lemma (3)]

$$= 3AL, \text{ since } AV = 3VL$$

whence $AL = \frac{1}{3}AO = OF$ (2)

Now, by I 6, 7

$\triangle ABB'$ (sum of segmts $AQB$, $AQ'B$) $= KG\ \ GF$,

and $\triangle ABB = 3$(sum of segments $AQB$, $AQ B'$)

[since the segment $ABB'$ is equal to $\frac{4}{3}\triangle ABB'$ (*Quadrature of the Parabola* Props 17, 24)]

Hence $KG = 3GF$

But $KG = 3LK$ from (1) above

Therefore $LF = LK + KG + GF$

$$= 5GF$$

And, from (2),

$$LF = (AO - AL - OF) = \frac{1}{3}AO = OF$$

Therefore $OF = 5GF$,

and $OG = 6GF$

But $AO = 3OF = 15GF$

Therefore, by subtraction

$$AG = 9GF$$
$$= \frac{3}{2}GO$$

## Proposition 9 (Lemma)

*If a b c d be four lines in continued proportion and in descending order of magnitude and if*

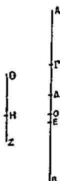
$$d\ \ (a-d) = x\ \ \tfrac{3}{5}(a-c),$$

*and* $(2a+4b+6c+3d)\ \ (5a+10b+10c+5d) = y\ \ (a-c)$,

*it is required to prove that*

$$x + y = \tfrac{2}{5}a$$

[The following is the proof given by Archimedes with the only difference that it is set out in algebraical instead of geometrical notation This is done in the particular case simply in order to make the proof easier to follow Archimedes exhibits his lines in the figure reproduced in the margin but now that it is possible to use algebraical notation there is no advantage in using the figure and the more cumbrous notation which only obscures the course of the proof The relation between Archimedes figure and the letters used below is as follows




AB $= a$ ΓB $= b$ ΔB $= c$ EB $= d$, ZH $= x$, HΘ $= y$ ΔO $= z$]

We have $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d}$ (1)

whence $$\frac{a-b}{b}=\frac{b-c}{c}=\frac{c-d}{d}$$

and therefore $$\frac{a-b}{b-c}=\frac{b-c}{c-d}=\frac{a}{b}=\frac{b}{c}=\frac{c}{d} \tag{2}$$

Now $$\frac{2(a+b)}{2c}=\frac{a+b}{c}=\frac{a+b}{b}\cdot\frac{b}{c}=\frac{a-c}{b-c}\cdot\frac{b-c}{c-d}=\frac{a-c}{c-d}$$

And in like manner,

$$\frac{b+c}{d}=\frac{b+c}{c}\cdot\frac{c}{d}=\frac{a-c}{c-d}$$

It follows from the last two relations that

$$\frac{a-c}{c-d}=\frac{2a+3b+c}{2c+d} \tag{3}$$

Suppose $z$ to be so taken that

$$\frac{2a+4b+4c+2d}{2c+d}=\frac{a-c}{z} \tag{4}$$

so that $z<(c-d)$

Therefore $$\frac{a-c+z}{a-c}=\frac{2a+4b+6c+3d}{2(a+d)+4(b+c)}$$

And by hypothesis

$$\frac{a-c}{y}=\frac{5(a+d)+10(b+c)}{2a+4b+6c+3d},$$

so that $$\frac{a-c+z}{y}=\frac{5(a+d)+10(b+c)}{2(a+d)+4(b+c)}=\frac{5}{2} \tag{5}$$

Again dividing (3) by (4) crosswise we obtain

$$\frac{z}{c-d}=\frac{2a+3b+c}{2(a+d)+4(b+c)},$$

whence $$\frac{c-d-z}{c-d}=\frac{b+3c+2d}{2(a+d)+4(b+c)} \tag{6}$$

But by (2)

$$\frac{c-d}{d}=\frac{a-b}{b}=\frac{3(b-c)}{3c}=\frac{2(c-d)}{2d},$$

so that $$\frac{c-d}{d}=\frac{(a-b)+3(b-c)+2(c-d)}{b+3c+2d} \tag{7}$$

Combining (6) and (7) we have

$$\frac{c-d-z}{d}=\frac{(a-b)+3(b-c)+2(c-d)}{2(a+d)+4(b+c)}$$

whence $$\frac{c-z}{d}=\frac{3a+6b+3c}{2(a+d)+4(b+c)} \tag{8}$$

And since [by (1)]

$$\frac{c-d}{c+d}=\frac{b-c}{b+c}=\frac{a-b}{a+b}$$

we have $$\frac{c-d}{a-c}=\frac{c+d}{b+c+a+b}$$

whence $$\frac{a-d}{a-c}=\frac{a+2b+2c+d}{a+2b+c}=\frac{2(a+d)+4(b+c)}{2(a+c)+4b} \tag{9}$$

Thus $$\frac{a-d}{\frac{3}{5}(a-c)}=\frac{2(a+d)+4(b+c)}{\frac{3}{5}\{2(a+c)+4b\}},$$

and therefore, by hypothesis

$$\frac{d}{x}=\frac{2(a+d)+4(b+c)}{\frac{3}{5}\{2(a+c)+4b\}}$$

But by (8), $$\frac{c-z}{d}=\frac{3a+6b+3c}{2(a+d)+4(b+c)}$$

and it follows *ex aequali*, that

$$\frac{c-z}{x}=\frac{3(a+c)+6b}{\frac{3}{5}\{2(a+c)+4b\}}=\frac{5}{3}\ \frac{3}{2}=\frac{5}{2}$$

And, by (5) $$\frac{a-c+z}{y}=\frac{5}{2}$$

Therefore $$\frac{5}{2}=\frac{a}{x+y}$$

or $$x+y=\tfrac{2}{5}a$$

## Proposition 10

*If PP B B be the portion of a parabola intercepted between two parallel chords PP BB′ bisected respectively in N O by the diameter ANO (N being nearer than O to A, the vertex of the segments), and if NO be divided into five equal parts of which LM is the middle one (L being nearer than M to N), then, if G be a point on LM such that*

$$LG\ \ GM=BO^2\ (2PN+BO)\ \ PN^2\ (2BO+PN),$$

*G will be the centre of gravity of the area PP B B*

Take a line $ao$ equal to $AO$ and $an$ on it equal to $AN$ Let $p$ $q$ be points on the line $ao$ such that

$$ao\ \ aq=aq\ \ an \qquad (1)$$
$$ao\ \ an=aq\ \ ap \qquad (2)$$

[whence $ao\ \ aq=aq\ \ an=an\ \ ap$ or $ao$ $aq$ $an$ $ap$ are lines in continued proportion and in descending order of magnitude]

Measure along $GA$ a length $GF$ such that

$$op\ \ ap=OL\ \ GF \qquad (3)$$

Then since $PN$ $BO$ are ordinates to $ANO$

$$BO\ \ PN^2=AO\ \ AN$$
$$=ao\ \ an$$
$$=ao^2\ \ aq^2 \text{ by (1)},$$

so that $$BO\ \ PN=ao\ \ aq \qquad (4)$$

and $$BO^3\ \ PN^3=ao^3\ \ aq^3$$
$$=(ao\ \ aq)\ (aq\ \ an)\ (an\ \ ap)$$
$$=ao\ \ ap \qquad (5)$$

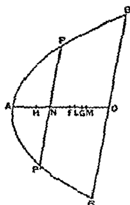
Thus (segment $BAB$ ) (segment $PAP'$)

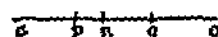
$$=\triangle BAB\ \ \triangle PAP$$
$$=BO^3\ \ PN^3$$
$$=ao\ \ ap$$

whence

(area $PP\,B\,B$) (segment $PAP$ ) $=op\ \ ap$

$= OL : GF$, by (3)
$= \frac{2}{5}ON : GF$ (6)




Now
$BO : (2PN + BO) = BO^3 : (2PN + BO) BO^2$
$= (2aq + ao) : ao$, by (4)
$BO^3 : PN^3$
$= ao : ap$, by (5),
and
$PN^3 : PN^2 (2BO + PN)$
$= PN : (2BO + PN)$
$= aq : (2ao + aq)$, by (4)
$= ap : (2an + ap)$, by (2)
Hence, *ex aequali*,
$BO^2 (2PN + BO) : PN^2 (2BO + PN) = (2aq + ao) : (2an + ap)$,

so that, by hypothesis,

$$LG : GM = (2aq + ao) : (2an + ap)$$

*Componendo* and multiplying the antecedents by 5,

$$OM : GM = \{5(ao + ap) + 10(aq + an)\} : (2an + ap)$$

But

$$OM : OM = 5 : 2 = \{5(ao + ap) + 10(aq + an)\} : \{2(ao + ap) + 4(aq + an)\}$$

It follows that

$$OM : OG = \{5(ao + ap) + 10(aq + an)\} : (2ao + 4aq + 6an + 3ap)$$

Therefore

$$(2ao + 4aq + 6an + 3ap) : \{5(ao + ap) + 10(aq + an)\} = OG : ON$$
$$= OG : on$$

And $ap : (ao - ap) = ap : op$
$= GF : OL$, by hypothesis,
$= GF : \frac{2}{5}on$

while $ao$, $aq$, $an$, $ap$ are in continued proportion.

Therefore, by Prop. 9,

$$GF + OG = OF = \frac{3}{5}ao = \frac{3}{5}OA$$

Thus $F$ is the centre of gravity of the segment $BAB'$. [Prop. 8]

Let $H$ be the centre of gravity of the segment $PAP'$, so that $AH = \frac{3}{5}AN$.

And, since $AF = \frac{3}{5}AO$,

we have, by subtraction, $HF = \frac{3}{5}ON$.

But, by (6) above,

$$(\text{area } PP'B'B) : (\text{segment } PAP') = \frac{3}{5}ON : GF$$
$$= HF : FG$$

Thus, since $F$, $H$ are the centres of gravity of the segments $BAB'$, $PAP'$ respectively, it follows [by I. 6, 7] that $G$ is the centre of gravity of the area $PP'B'B$.

# THE SAND-RECKONER

'There are some King Gelon who think that the number of the sand is infinite in multitude and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited Again there are some who without regarding it as infinite yet think that no number has been named which is great enough to exceed its multitude And it is clear that they who hold this view if they imagined a mass made up of sand in other respects as large as the mass of the earth including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains would be many times further still from recognising that any number could be expressed which exceeded the multitude of the sand so taken But I will try to show you by means of geometrical proofs which you will be able to follow that of the numbers named by me and given in the work which I sent to Zeuxippus some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described but also that of a mass equal in magnitude to the universe Now you are aware that universe is the name given by most astronomers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth This is the common account (τὰ γραφόμενα) as you have heard from astronomers But Aristarchus of Samos brought out a book consisting of some hypotheses in which the premisses lead to the result that the universe is many times greater than that now so called His hypotheses are that the fixed stars and the sun remain unmoved that the earth revolves about the sun in the circumference of a circle the sun lying in the middle of the orbit, and that the sphere of the fixed stars situated about the same centre as the sun is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface Now it is easy to see that this is impossible for since the centre of the sphere has no magnitude we cannot conceive it to bear any ratio whatever to the surface of the sphere We must however take Aristarchus to mean this since we conceive the earth to be as it were the centre of the universe the ratio which the earth bears to what we describe as the universe is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars For he adapts the proofs of his results to a hypothesis of this kind and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the universe

'I say then that even if a sphere were made up of the sand as great as Aristarchus supposes the sphere of the fixed stars to be I shall still prove that,

of the numbers named in the *Principles*,[1] some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made "

1 ' *The perimeter of the earth is about* 3 000,000 *stadia and not greater*

' It is true that some have tried as you are of course aware to prove that the said perimeter is about 300 000 stadia But I go further and, putting the magnitude of the earth at ten times the size that my predecessors thought it, I suppose its perimeter to be about 3,000 000 stadia and not greater '

2 ' *The diameter of the earth is greater than the diameter of the moon, and the diameter of the sun is greater than the diameter of the earth*

"In this assumption I follow most of the earlier astronomers "

3 "*The diameter of the sun is about* 30 *times the diameter of the moon and not greater*

' It is true that, of the earlier astronomers, Eudoxus declared it to be about nine times as great and Pheidias my father twelve times while Aristarchus tried to prove that the diameter of the sun is greater than 18 times but less than 20 times the diameter of the moon But I go even further than Aristarchus, in order that the truth of my proposition may be established beyond dispute, and I suppose the diameter of the sun to be about 30 times that of the moon and not greater '

4 ' *The diameter of the sun is greater than the side of the chiliagon inscribed in the greatest circle in the (sphere of the) universe*

"I make this assumption because Aristarchus discovered that the sun appeared to be about $\frac{1}{720}$th part of the circle of the zodiac and I myself tried by a method which I will now describe to find experimentally (ὀργανικῶς) the angle subtended by the sun and having its vertex at the eye

[Up to this point the treatise has been literally translated because of the historical interest attaching to the *ipsissima verba* of Archimedes on such a subject The rest of the work can now be more freely reproduced and, before proceeding to the mathematical contents of it it is only necessary to remark that Archimedes next describes how he arrived at a higher and a lower limit for the angle subtended by the sun This he did by taking a long rod or ruler fastening on the end of it a small cylinder or disc pointing the rod in the direction of the sun just after its rising (so that it was possible to look directly at it), then putting the cylinder at such a distance that it just concealed and just failed to conceal, the sun and lastly measuring the angles subtended by the cylinder He explains also the correction which he thought it necessary to make because the eye does not see from one point but from a certain area ']

The result of the experiment was to show that the angle subtended by the diameter of the sun was less than $\frac{1}{164}$th part and greater than $\frac{1}{200}$th part of a right angle

*To prove that (on this assumption) the diameter of the sun is greater than the side of a chiliagon, or figure with* 1000 *equal sides inscribed in a great circle of the universe*

Suppose the plane of the paper to be the plane passing through the centre of the sun the centre of the earth and the eye at the time when the sun has

[1] A lost work of Archimedes

just risen above the horizon. Let the plane cut the earth in the circle $EHL$ and the sun in the circle $FKG$, the centres of the earth and sun being $C$, $O$ respectively, and $E$ being the position of the eye.

Further, let the plane cut the sphere of the 'universe' (i.e. the sphere whose centre is $C$ and radius $CO$) in the great circle $AOB$.

Draw from $E$ two tangents to the circle $FKG$ touching it at $P$, $Q$, and from $C$ draw two other tangents to the same circle touching it in $F$, $G$ respectively.

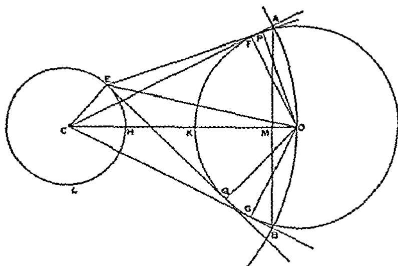
Let $CO$ meet the sections of the earth and sun in $H$, $K$ respectively; and let $CF$, $CG$ produced meet the great circle $AOB$ in $A$, $B$.

Join $EO$, $OF$, $OG$, $OP$, $OQ$, $AB$, and let $AB$ meet $CO$ in $M$.

Now $CO > EO$, since the sun is just above the horizon.

Therefore $\angle PEQ > \angle FCG$.

And $\angle PEQ > \frac{1}{200}R$
but $< \frac{1}{164}R$, where $R$ represents a right angle.




Thus $\angle FCG < \frac{1}{164}R$, *a fortiori*,
and the chord $AB$ subtends an arc of the great circle which is less than $\frac{1}{656}$th of the circumference of that circle, i.e.

$AB <$ (side of 656-sided polygon inscribed in the circle).

Now the perimeter of any polygon inscribed in the great circle is less than $\frac{44}{7}CO$. [Cf. *Measurement of a circle*, Prop. 3.]

Therefore $AB : CO < 11 : 1148$,

and, *a fortiori*, $AB < \frac{1}{100}CO$. (α)

Again, since $CA = CO$, and $AM$ is perpendicular to $CO$, while $OF$ is perpendicular to $CA$,

$AM = OF$.

Therefore $AB = 2AM =$ (diameter of sun).

Thus (diameter of sun) $< \frac{1}{100}CO$, by (α),

and, *a fortiori*, (diameter of earth) $< \frac{1}{100}CO$. [Assumption 2.]

Hence $CH + OK < \frac{1}{100}CO$,

so that $HK > \frac{99}{100}CO$.

or $CO \quad HK < 100 \quad 99$

And $CO > CF$

while $HK < EQ$

Therefore $CF \quad EQ < 100 \quad 99$ $(\beta)$

Now in the right-angled triangles $CFO$ $EQO$ of the sides about the right angles $OF = OQ$, but $EQ < CF$ (since $EO < CO$)

Therefore $\angle OEQ \quad \angle OCF > CO \quad EO$

but $< CF \quad EQ$ [1]

Doubling the angles

$$\angle PEQ \quad \angle ACB < CF \quad EQ$$

$$< 100 \quad 99, \text{ by } (\beta) \text{ above}$$

But $\angle PEQ > \frac{1}{200}R$, by hypothesis

Therefore $\angle ACB > \frac{99}{20000}R$

$$> \frac{1}{203}R$$

It follows that the arc $AB$ is greater than $\frac{1}{812}$th of the circumference of the great circle $AOB$

Hence, *a fortiori*

$AB >$ (side of chiliagon inscribed in great circle)

and $AB$ is equal to the diameter of the sun as proved above

*The following results can now be proved*

(*diameter of ' universe '*) < 10 000 (*diameter of earth*)

*and* (*diameter of "universe* ) < 10 000 000 000 *stadia*

(1) Suppose, for brevity that $d$ represents the diameter of the ' universe " $d$ that of the sun $d$ that of the earth and $d_m$ that of the moon

By hypothesis $d \not> 30 d_m$ [Assumption 3]

and $d > d_m$, [Assumption 2]

therefore $d < 30d$

Now by the last proposition,

$d >$ (side of chiliagon inscribed in great circle),

so that (perimeter of chiliagon) $< 1000d$

$< 30\,000d$

But the perimeter of any regular polygon with more sides than 6 inscribed in a circle is greater than that of the inscribed regular hexagon and therefore greater than three times the diameter Hence

(perimeter of chiliagon) $> 3d$

It follows that $d < 10{,}000d$

(2) (Perimeter of earth) $\not>$ 3 000 000 stadia [Assumption 1]

and (perimeter of earth) $> 3d$

Therefore $d <$ 1 000 000 stadia

whence $d <$ 10 000 000 000 stadia

*Assumption 5*

Suppose a quantity of sand taken not greater than a poppy seed and suppose that it contains not more than 10 000 grains

[1] The proposition here assumed is of course equivalent to the trigonometrical formula which states that, if $\alpha$ $\beta$ are the circular measures of two angles each less than a right angle of which $\alpha$ is the greater then

$$\frac{\tan \alpha}{\tan \beta} > \frac{\alpha}{\beta} > \frac{\sin \alpha}{\sin \beta}$$

Next suppose the diameter of the poppy seed to be not less than $\frac{1}{40}$th of a finger breadth

ORDERS AND PERIODS OF NUMBERS

I We have traditional names for numbers up to a myriad (10 000), we can therefore express numbers up to a myriad myriads (100 000,000) Let these numbers be called numbers of the *first order*

Suppose the 100 000 000 to be the unit of the *second order* and let the *second order* consist of the numbers from that unit up to $(100\,000\,000)^2$

Let this again be the unit of the *third order* of numbers ending with $(100\,000\,000)^3$ and so on until we reach the 100 000 000*th order* of numbers ending with $(100\,000\,000)^{100\,000\,000}$ which we will call $P$

II Suppose the numbers from 1 to $P$ just described to form the *first period*

Let $P$ be the unit of the *first order of the second period* and let this consist of the numbers from $P$ up to $100\,000\,000P$

Let the last number be the unit of the *second order of the second period* and let this end with $(100\,000\,000)^2P$

We can go on in this way till we reach the 100 000 000*th order of the second period* ending with $(100\,000\,000)^{100\,000\,000}P$ or $P^2$

III Taking $P^2$ as the unit of the *first order of the third period* we proceed in the same way till we reach the 100 000 000*th order of the third period* ending with $P^3$

IV Taking $P^3$ as the unit of the *first order of the fourth period*, we continue the same process until we arrive at the 100 000 000*th order of the 100 000 000th period* ending with $P^{100\,000\,000}$ This last number is expressed by Archimedes as a myriad myriad units of the myriad myriad th order of the myriad myriad th period (αἱ μυριακισμυριοστᾶς περιόδου μυριακισμυριοστῶν ἀριθμῶν μυρίαι μυριάδες) which is easily seen to be 100 000 000 times the product of $(100\,000\,000)^{99\,999\,999}$ and $P^{99\,999\,999}$ i e $P^{100\,000\,000}$

OCTADS

Consider the series of terms in continued proportion of which the first is 1 and the second 10 [i e the geometrical progression 1 $10^1$ $10^2$, $10^3$, ] The *first octad* of these terms [*i e* 1 $10^1$ $10^2$ $10^7$] fall accordingly under the *first order of the first period* above described the *second octad* [i e $10^8$ $10^9$ $10^{15}$] under the *second order of the first period* the first term of the octad being the unit of the corresponding order in each case Similarly for the *third octad*, and so on We can in the same way place any number of octads

THEOREM

*If there be any number of terms of a series in continued proportion say* $A_1$ $A_2$ $A_3$ $A_m$ $A_n$ $A_{m+n-1}$ *of which* $A_1=1$ $A_2=10$ [so that the series forms the geometrical progression 1 $10^1$ $10^2$ $10^{m-1}$ $10^{n-1}$ $10^{m+n-2}$, ] *and if any two terms as* $A_m$ $A_n$ *be taken and multiplied the product* $A_m$ $A_n$ *will be a term in the same series and will be as many terms distant from* $A_n$ *as* $A_m$ *is distant from* $A_1$ *also it will be distant from* $A_1$ *by a number of terms less by one than the sum of the numbers of terms by which* $A_m$ *and* $A_n$ *respectively are distant from* $A_1$

Take the term which is distant from $A_n$ by the same number of terms as $A_m$

is distant from $A_1$. This number of terms is $m$ (the first and last being both counted). Thus the term to be taken is $m$ terms distant from $A_n$, and is therefore the term $A_{m+n-1}$.

We have therefore to prove that

$$A_m \cdot A_n = A_{m+n-1}.$$

Now terms equally distant from other terms in the continued proportion are proportional.

Thus $\qquad \dfrac{A_m}{A_1} = \dfrac{A_{m+n-1}}{A_n}$.

But $\qquad A_m = A_m \cdot A_1$, since $A_1 = 1$.

Therefore $\qquad A_{m+n-1} = A_m \cdot A_n. \qquad (1)$

The second result is now obvious, since $A_m$ is $m$ terms distant from $A_1$, $A_n$ is $n$ terms distant from $A_1$, and $A_{m+n-1}$ is $(m+n-1)$ terms distant from $A_1$.

APPLICATION TO THE NUMBER OF THE SAND.

By Assumption 5 [p. 523],

(diam. of poppy seed) $\not< \frac{1}{40}$(finger breadth),

and since spheres are to one another in the triplicate ratio of their diameters it follows that

(sphere of diam. 1 finger breadth) $\not>$ 64 000 poppy seeds

$\not>$ 64 000 × 10 000 grains of sand
$\not>$ 640 000 000 grains of sand
$\not>$ 6 units of *second order* + 40,000,000 units of *first order* grains of sand
(*a fortiori*) < 10 units of *second order* of numbers grains of sand.

We now gradually increase the diameter of the supposed sphere, multiplying it by 100 each time. Thus, remembering that the sphere is thereby multiplied by $100^3$ or 1 000 000, the number of grains of sand which would be contained in a sphere with each successive diameter may be arrived at as follows.

| *Diameter of sphere* | *Corresponding number of grains of sand* |
|---|---|
| (1) 100 finger breadths | < 1 000 000 × 10 units of *second order* |
| | < (7th term of series) × (10th term of series) |
| | < 16th term of series [i.e. $10^{15}$] |
| | < [$10^7$ or] 10 000 000 units of the *second order*. |
| (2) 10 000 finger breadths | < 1 000 000 × (last number) |
| | < (7th term of series) × (16th term) |
| | < 22nd term of series [i.e. $10^{21}$] |
| | < [$10^5$ or] 100 000 units of *third order*. |
| (3) 1 stadium (< 10 000 finger breadths) | < 100 000 units of *third order*. |
| (4) 100 stadia | < 1 000 000 × (last number) |
| | < (7th term of series) × (22nd term) |
| | < 28th term of series [$10^{27}$] |
| | < [$10^3$ or] 1 000 units of *fourth order*. |
| (5) 10 000 stadia | < 1 000 000 × (last number) |
| | < (7th term of series) × (28th term) |
| | < 34th term of series [$10^{33}$] |
| | < 10 units of *fifth order*. |

| | | |
|---|---|---|
| (6) 1 000 000 stadia | <(7th term of series)×(34th term) | |
| | <40th term | [$10^{39}$] |
| | <[$10^7$ or] 10 000 000 units of *fifth order* | |
| (7) 100 000 000 stadia | <(7th term of series)×(40th term) | |
| | <46th term | [$10^{45}$] |
| | <[$10^5$ or] 100 000 units of *sixth order* | |
| (8) 10 000 000 000 stadia | <(7th term of series)×(46th term) | |
| | <52nd term of series | [$10^{51}$] |
| | <[$10^3$ or] 1 000 units of *seventh order* | |

But, by the proposition above [p 523]

(diameter of "universe")<10,000 000 000 stadia

Hence *the number of grains of sand which could be contained in a sphere of the size of our "universe" is less than* 1,000 *units of the seventh order of numbers* [or $10^{51}$]

From this we can prove further that *a sphere of the size attributed by Aristarchus to the sphere of the fixed stars would contain a number of grains of sand less than* 10 000 000 *units of the eighth order of numbers* [or $10^{56+7}=10^{63}$]

For by hypothesis

(earth) : ("universe") = ("universe") : (sphere of fixed stars)

And [p 523]

(diameter of "universe")<10 000 (diam of earth),

whence

(diam of sphere of fixed stars)<10 000 (diam of "universe")

Therefore

(sphere of fixed stars)<$(10\,000)^3$ ("universe")

It follows that the number of grains of sand which would be contained in a sphere equal to the sphere of the fixed stars

<$(10,000)^3$×1 000 units of *seventh order*

<(13th term of series)×(52nd term of series)

<64th term of series [i e $10^{63}$]

<[$10^7$ or] 10,000,000 units of *eighth order* of numbers

Conclusion

I conceive that these things, King Gelon will appear incredible to the great majority of people who have not studied mathematics but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth, the sun and moon and the whole universe the proof will carry conviction And it was for this reason that I thought the subject would be not inappropriate for your consideration

# QUADRATURE OF THE PARABOLA

ARCHIMEDES to DOSITHEUS greeting

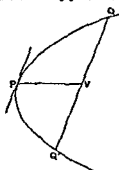
"When I heard that Conon, who was my friend in his lifetime was dead but that you were acquainted with Conon and withal versed in geometry while I grieved for the loss not only of a friend but of an admirable mathematician I set myself the task of communicating to you, as I had intended to send to Conon a certain geometrical theorem which had not been investigated before but has now been investigated by me and which I first discovered by means of mechanics and then exhibited by means of geometry Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle, and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line assuming lemmas not easily conceded so that it was recognised by most people that the problem was not solved But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution For it is here shown that every segment bounded by a straight line and a section of a right angled cone [a parabola] is *four thirds* of the triangle which has the same base and equal height with the segment and for the demonstration of this property the following lemma is assumed that the excess by which the greater of (two) unequal areas exceeds the less can by being added to itself be made to exceed any given finite area The earlier geometers have also used this lemma for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters and that spheres are to one another in the triplicate ratio of their diameters and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid And in the result each of the aforesaid theorems has been accepted no less than those proved without the lemma As therefore my work now published has satisfied the same test as the propositions referred to I have written out the proof and send it to you, first as investigated by means of mechanics and afterwards too as demonstrated by geometry Prefixed are also the elementary propositions in conics which are of service in the proof Farewell

### PROPOSITION 1

*If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as $PV$, and if $QQ'$ be a chord parallel to the tangent to the parabola at $P$ and meeting $PV$ in $V$, then*

$$QV = VQ'.$$

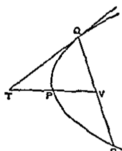
Conversely, *if $QV = VQ'$, the chord $QQ'$ will be parallel to the tangent at $P$.*




### PROPOSITION 2

*If in a parabola $QQ'$ be a chord parallel to the tangent at $P$, and if a straight line be drawn through $P$ which is either itself the axis or parallel to the axis, and which meets $QQ'$ in $V$ and the tangent at $Q$ to the parabola in $T$, then*

$$PV = PT.$$




### PROPOSITION 3

*If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as $PV$, and if from two other points $Q$, $Q'$ on the parabola straight lines be drawn parallel to the tangent at $P$ and meeting $PV$ in $V$, $V'$ respectively, then* $PV : PV' = QV^2 : Q'V'^2$.

*"And these propositions are proved in the elements of conics.*[1]"

### PROPOSITION 4

*If $Qq$ be the base of any segment of a parabola, and $P$ the vertex of the segment, and if the diameter through any other point $R$ meet $Qq$ in $O$ and $QP$ (produced if necessary) in $F$, then* $QV : VO = OF : FR$.

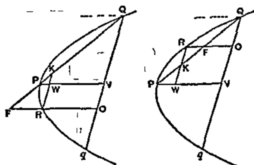
Draw the ordinate $RW$ to $PV$, meeting $QP$ in $K$.

Then $PV : PW = QV^2 : RW^2$,

whence, by parallels, $PQ : PK = PQ^2 : PF^2$.

[1] i.e. in the treatises on conics by Euclid and Aristaeus.

In other words, $PQ$, $PF$, $PK$ are in continued proportion therefore



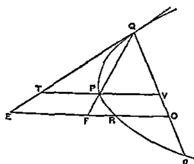

$$PQ \quad PF = PF \quad PK$$
$$= PQ \pm PF \quad PF \pm PK$$
$$= QF \quad KF$$

Hence by parallels, $\quad QV \quad VO = OF \quad FR$

### Proposition 5

*If $Qq$ be the base of any segment of a parabola $P$ the vertex of the segment, and $PV$ its diameter and if the diameter of the parabola through any other point $R$ meet $Qq$ in $O$ and the tangent at $Q$ in $E$, then*

$$QO \quad Oq = ER \quad RO$$




Let the diameter through $R$ meet $QP$ in $F$

Then, by Prop 4,

$$QV \quad VO = OF \quad FR$$

Since $QV = Vq$ it follows that

$$QV \quad qO = OF \quad OR \qquad (1)$$

Also if $VP$ meet the tangent in $T$,

$$PT = PV, \text{ and therefore } EF = OF$$

Accordingly doubling the antecedents in (1), we have

$$Qq \quad qO = OE \quad OR$$

whence $\quad QO \quad Oq = ER \quad RO$

### Propositions 6 7[1]

*Suppose a lever $AOB$ placed horizontally and supported at its middle point $O$ Let a triangle $BCD$ in which the angle $C$ is right or obtuse be suspended from $B$ and $O$, so that $C$ is attached to $O$ and $CD$ is in the same vertical line with $O$ Then if $P$ be such an area as when suspended from $A$ will keep the system in equilibrium*

$$P = \tfrac{1}{3}\triangle BCD$$

[1] In Prop 6 Archimedes takes the separate case in which the angle $BCD$ of the triangle is a right angle so that $C$ coincides with $O$ in the figure and $F$ with $E$ He then proves in Prop 7 the same property for the triangle in which $BCD$ is an obtuse angle by treating the triangle as the difference between two right-angled triangles $BOD$ $BOC$ and using the result of Prop 6 I have combined the two propositions in one proof for the sake of brevity The same remark applies to the propositions following Props 6 7

Take a point $E$ on $OB$ such that $BE=2OE$, and draw $EFH$ parallel to $OCD$ meeting $BC$, $BD$ in $F$, $H$ respectively

Let $G$ be the middle point of $FH$

Then $G$ is the centre of gravity of the triangle $BCD$

Hence, if the angular points $B$, $C$ be set free and the triangle be suspended by attaching $F$ to $E$, the triangle will hang in the same position as before, because $EFG$ is a vertical straight line "For this is proved"[1]

Therefore as before, there will be equilibrium

Thus $\qquad P \ \triangle BCD = OE \ AO$

$\qquad\qquad = 1 \ 3$

or $\qquad P = \frac{1}{3}\triangle BCD$

PROPOSITIONS 8 9

*Suppose a lever $AOB$ placed horizontally and supported at its middle point $O$ Let a triangle $BCD$ right angled or obtuse-angled at $C$, be suspended from the points $B$, $E$ on $OB$ the angular point $C$ being so attached to $E$ that the side $CD$ is in the same vertical line with $E$ Let $Q$ be an area such that*

$$AO \ OE = \triangle BCD \ Q$$

*Then if an area $P$ suspended from $A$ keep the system in equilibrium,*

$$P < \triangle BCD \text{ but} > Q$$

Take $G$ the centre of gravity of the triangle $BCD$ and draw $GH$ parallel to $DC$ i e vertically meeting $BO$ in $H$

We may now suppose the triangle $BCD$ suspended from $H$ and since there is equilibrium

$$\triangle BCD \ P = AO \ OH, \qquad (1)$$

whence $\qquad P < \triangle BCD$

Also $\qquad \triangle BCD \ Q = AO \ OE$

Therefore by (1), $\qquad \triangle BCD \ Q > \triangle BCD \ P,$

and $\qquad P > Q$

PROPOSITIONS 10 11

*Suppose a lever $AOB$ placed horizontally and supported at $O$ its middle point Let $CDEF$ be a trapezium which can be so placed that its parallel sides $CD$, $FE$ are vertical while $C$ is vertically below $O$, and the other sides $CF$, $DE$ meet in $B$ Let $EF$ meet $BO$ in $H$ and let the trapezium be suspended by attaching $F$ to $H$ and $C$ to $O$ Further suppose $Q$ to be an area such that*

$$AO \ OH = (\text{trapezium } CDEF) \ Q$$

*Then if $P$ be the area which when suspended from $A$, keeps the system in equilibrium,*

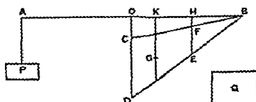
$$P < Q$$

[1]Doubtless in the lost book περὶ ζυγῶν

*The same is true in the particular case where the angles at C, F are right and consequently C F coincide with O, H respectively*

Divide $OH$ in $K$ so that

$$(2CD+FE) \quad (2FE+CD)=HK \quad KO$$




Draw $KG$ parallel to $OD$ and let $G$ be the middle point of the portion of $KG$ intercepted within the trapezium Then $G$ is the centre of gravity of the trapezium [*On the equilibrium of planes* I 15]

Thus we may suppose the trapezium suspended from $K$ and the equilibrium will remain undisturbed

Therefore $AO \quad OK=(\text{trapezium } CDEF) \quad P$

and, by hypothesis $AO \quad OH=(\text{trapezium } CDEF) \quad Q$

Since $OK<OH$, it follows that

$$P<Q$$

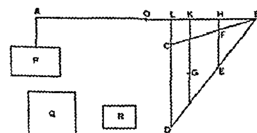
PROPOSITIONS 12 13

*If the trapezium CDEF be placed as in the last propositions except that CD is vertically below a point L on OB instead of being below O and the trapezium is suspended from L, H suppose that Q R are areas such that*

$$AO \quad OH=(\textit{trapezium } CDEF) \quad Q$$

*and*

$$AO \quad OL=(\textit{trapezium } CDEF) \quad R$$




*If then an area P suspended from A keep the system in equilibrium*

$$P>R \text{ but} <Q$$

Take the centre of gravity $G$ of the trapezium as in the last propositions and let the line through $G$ parallel to $DC$ meet $OB$ in $K$

Then we may suppose the trapezium suspended from $K$ and there will still be equilibrium

Therefore $(\text{trapezium } CDEF) \quad P=AO \quad OK$

Hence $(\text{trapezium } CDEF) \quad P>(\text{trapezium } CDEF) \quad Q$

but $<(\text{trapezium } CDEF) \quad R$

It follows that $P<Q$ but $>R$

PROPOSITIONS 14 15

Let $Qq$ be the base of any segment of a parabola Then if two lines be drawn from $Q$ $q$ each parallel to the axis of the parabola and on the same side of $Qq$ as the segment is either (1) the angles so formed at $Q$ $q$ are both right angles or (2) one is acute and the other obtuse In the latter case let the angle at $q$ be the obtuse angle

Divide $Qq$ into any number of equal parts at the points $O_1$ $O_2$ $O$ Draw

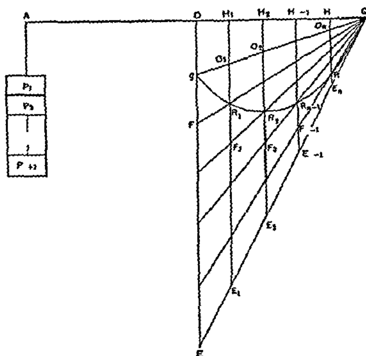
through $q$, $O_1$ $O_2$, $O$ diameters of the parabola meeting the tangent at $Q$ in $E$, $E_1$, $E_2$ $E$ and the parabola itself in $q$ $R_1$ $R_2$ $R$ Join $QR_1$, $QR_2$ $QR$ meeting $qE$, $O_1E_1$ $O_2E_2$ $O_{n-1}E_{n-1}$ in $F$ $F_1$ $F_2$ $F_{n-1}$

Let the diameters $Eq$ $E_1O_1$, $E$ $O$ meet a straight line $QOA$ drawn through $Q$ perpendicular to the diameters in the points $O$ $H_1$ $H_2$ $H_n$ respectively (In the particular case where $Qq$ is itself perpendicular to the diameters $q$ will coincide with $O$, $O_1$ with $H_1$ and so on )

*It is required to prove that*

(1) $\triangle EqQ < 3$(*sum of trapezia* $FO_1$ $F_1O_2$, $F_{n-1}O$ *and* $\triangle E$ $O$ $Q$),

(2) $\triangle EqQ > 3$(*sum of trapezia* $R_1O_2$ $R_2O_3$ $R_{n-1}O$ *and* $\triangle R$ $O$ $Q$)




Suppose $AO$ made equal to $OQ$ and conceive $QOA$ as a lever placed horizontally and supported at $O$ Suppose the triangle $EqQ$ suspended from $OQ$ in the position drawn and suppose that the trapezium $EO_1$ in the position drawn is balanced by an area $P_1$ suspended from $A$ the trapezium $E_1O_2$ in the position drawn is balanced by the area $P_2$ suspended from $A$ and so on the triangle $E$ $O$ $Q$ being in like manner balanced by $P_{n+1}$

Then $P_1+P_2+$ $+P$ $_{+1}$ will balance the whole triangle $EqQ$ as drawn and therefore $P_1+P_2+$ $+P_{n+1}=\frac{1}{3}\triangle EqQ$ [Props 6 7]

Again $AO$ $OH_1=QO$ $OH_1$

$=Qq$ $qO_1$

$=E_1O_1$ $O_1R_1$ [by means of Prop 5]

$=$(trapezium $EO_1$) (trapezium $FO_1$),

whence [Props 10 11] $(FO_1)>P_1$

Next $$AO \quad OH_1 = E_1O_1 \quad O_1R_1$$
$$= (E_1O_2) \quad (R_1O_2), \qquad (\alpha)$$
while $$AO \quad OH_2 = E_2O_2 \quad O_2R_2$$
$$= (E_1O_2) \quad (F_1O_2) \qquad (\beta)$$
and, since $(\alpha)$ and $(\beta)$ are simultaneously true we have by Props 12, 13,
$$(F_1O_2) > P_2 > (R_1O_2)$$
Similarly it may be proved that
$$(F_2O_3) > P_3 > (R_2O_3),$$
and so on

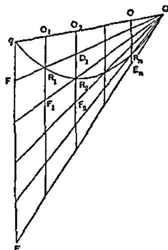
Lastly [Props 8, 9] $\quad \triangle E\ O_nQ > P_{n+1} > \triangle R_nO_nQ$

By addition, we obtain

(1) $\quad (FO_1)+(F_1O_2)+ \quad +(F_{n-1}O\ )+\triangle E\ O\ Q > P_1+P_2+ \quad +P_{n+1}$
$$> \tfrac{1}{3}\triangle EqQ,$$
or $$\triangle EqQ < 3(FO_1+F_1O_2+ \quad +F_{-1}O\ +\triangle E\ O\ Q)$$
(2) $\quad (R_1O_2)+(R_2O_3)+ \quad +(R_{n-1}O\ )+\triangle R_nO\ Q < P_2+P_3+ \quad +P_{n+1}$
$$< P_1+P_2+ \quad +P_{n+1} \text{ a fortiori},$$
$$< \tfrac{1}{3}\triangle EqQ,$$
or $$\triangle EqQ > 3(R_1O_2+R\ O_3+ \quad +R_{n-1}O\ +\triangle R\ O\ Q)$$

## Proposition 16

*Suppose $Qq$ to be the base of a parabolic segment, $q$ being not more distant than $Q$ from the vertex of the parabola Draw through $q$ the straight line $qE$ parallel to the axis of the parabola to meet the tangent at $Q$ in $E$ It is required to prove that*
$$(\textit{area of segment}) = \tfrac{1}{3}\triangle EqQ$$




For, if not, the area of the segment must be either greater or less than $\frac{1}{3}\triangle EqQ$

I Suppose the area of the segment greater than $\frac{1}{3}\triangle EqQ$ Then the excess can if continually added to itself be made to exceed $\triangle EqQ$ And it is possible to find a submultiple of the triangle $EqQ$ less than the said excess of the segment over $\frac{1}{3}\triangle EqQ$

Let the triangle $FqQ$ be such a submultiple of the triangle $EqQ$ Divide $Eq$ into equal parts each equal to $qF$, and let all the points of division including $F$ be joined to $Q$ meeting the parabola in $R_1$ $R_2$ $\quad R$ respectively Through $R_1$ $R_2$ $\quad R$ draw diameters of the parabola meeting $qQ$ in $O_1$ $O_2$ $\quad O$ respectively

Let $O_1R_1$ meet $QR_2$ in $F_1$

Let $O_2R_2$ meet $QR_1$ in $D_1$ and $QR_3$ in $F_2$

Let $O_3R_3$ meet $QR_2$ in $D_2$ and $QR_4$ in $F_3$, and so on

We have, by hypothesis,
$$\triangle FqQ < (\text{area of segment}) - \tfrac{1}{3}\triangle EqQ$$
or $$(\text{area of segment}) - \triangle FqQ > \tfrac{1}{3}\triangle EqQ \qquad (\alpha)$$
Now, since all the parts of $qE$ as $qF$ and the rest are equal $O_1R_1 = R_1F_1$, $O_2D_1 = D_1R_2 = R_2F_2$ and so on therefore

$$\triangle FqQ = (FO_1 + R_1O_2 + D_1O_3 + \quad )$$
$$= (FO_1 + F_1D_1 + F_2D_2 + \quad + F_{n-1}D_{-1} + \triangle E\ R\ Q) \qquad (\beta)$$

But (area of segment) $< (FO_1 + F_1O_2 + \quad + F_{-1}O + \triangle E\ O\ Q)$

Subtracting we have

(area of segment) $- \triangle FqQ < (R_1O_2 + R_2O_3 + \quad + R_{-1}O + \triangle R\ O\ Q)$,

whence *a fortiori*, by $(\alpha)$

$$\tfrac{1}{3}\triangle EqQ < (R_1O_2 + R_2O_3 + \quad + R_{-1}O + \triangle R\ O\ Q)$$

But this is impossible since [Props 14, 15]

$$\tfrac{1}{3}\triangle EqQ > (R_1O + R\ O_3 + \quad + R_{-1}O + \triangle R\ O\ Q)$$

Therefore (area of segment) $\ngtr \tfrac{1}{3}\triangle EqQ$

II If possible suppose the area of the segment less than $\tfrac{1}{3}\triangle EqQ$

Take a submultiple of the triangle $EqQ$ as the triangle $FqQ$ less than the excess of $\tfrac{1}{3}\triangle EqQ$ over the area of the segment, and make the same construction as before

Since $\triangle FqQ < \tfrac{1}{3}\triangle EqQ -$ (area of segment),

it follows that

$\triangle FqQ +$ (area of segment) $< \tfrac{1}{3}\triangle EqQ$

$< (FO_1 + F_1O_2 + \quad + F_{n-1}O + \triangle E\ O\ Q)$

[Props 14, 15]

Subtracting from each side the area of the segment, we have

$\triangle FqQ <$ (sum of spaces $qFR_1$ $R_1F_1R_2$ $\quad E\ R\ Q$)

$< (FO_1 + F_1D_1 + \quad + F_{n-1}D_{-1} + \triangle E\ R\ Q)$, a *fortiori*

which is impossible because, by $(\beta)$ above

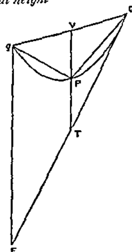
$$\triangle FqQ = FO_1 + F_1D_1 + \quad + F_{-1}D_{-1} + \triangle E\ R\ Q$$

Hence (area of segment) $\nless \tfrac{1}{3}\triangle EqQ$

Since then the area of the segment is neither less nor greater than $\tfrac{1}{3}\triangle EqQ$ it is equal to it

## Proposition 17

It is now manifest that *the area of any segment of a parabola is four thirds of the triangle which has the same base as the segment and equal height*




Let $Qq$ be the base of the segment $P$ its vertex Then $PQq$ is the inscribed triangle with the same base as the segment and equal height

Since $P$ is the vertex of the segment the diameter through $P$ bisects $Qq$ Let $V$ be the point of bisection

Let $VP$ and $qE$ drawn parallel to it, meet the tangent at $Q$ in $T$ $E$ respectively

Then by parallels

$$qE = 2VT,$$

and $PV = PT$ [Prop 2]

so that $VT = 2PV$

Hence $\triangle EqQ = 4\triangle PQq$

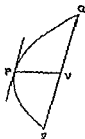
But, by Prop 16 the area of the segment is equal to $\tfrac{1}{3}\triangle EqQ$

Therefore (area of segment) $= \tfrac{4}{3}\triangle PQq$

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DEF In segments bounded by a straight line and any curve I call the

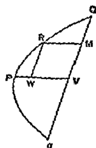
straight line the *base* and the *height* the greatest perpendicular drawn from the curve to the base of the segment, and the *vertex* the point from which the greatest perpendicular is drawn '




PROPOSITION 18

*If $Qq$ be the base of a segment of a parabola and $V$ the middle point of $Qq$, and if the diameter through $V$ meet the curve in $P$, then $P$ is the vertex of the segment*

For $Qq$ is parallel to the tangent at $P$ [Prop 1] Therefore, of all the perpendiculars which can be drawn from points on the segment to the base $Qq$, that from $P$ is the greatest Hence, by the definition $P$ is the vertex of the segment




PROPOSITION 19

*If $Qq$ be a chord of a parabola bisected in $V$ by the diameter $PV$ and if $RM$ be a diameter bisecting $QV$ in $M$, and $RW$ be the ordinate from $R$ to $PV$ then*

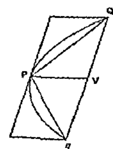
$$PV = \tfrac{4}{3}RM$$

For, by the property of the parabola,

$$PV : PW = QV^2 : RW^2$$
$$= 4RW^2 : RW^2,$$

so that $PV = 4PW$

whence $PV = \frac{4}{3}RM$



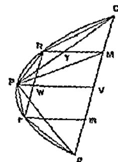

PROPOSITION 20

*If $Qq$ be the base, and $P$ the vertex, of a parabolic segment, then the triangle $PQq$ is greater than half the segment $PQq$*

For the chord $Qq$ is parallel to the tangent at $P$, and the triangle $PQq$ is half the parallelogram formed by $Qq$ the tangent at $P$ and the diameters through $Q$ $q$

Therefore the triangle $PQq$ is greater than half the segment

COR It follows that *it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area*




PROPOSITION 21

*If $Qq$ be the base and $P$ the vertex of any parabolic segment and if $R$ be the vertex of the segment cut off by $PQ$, then*

$$\triangle PQq = 8\triangle PRQ$$

The diameter through $R$ will bisect the chord $PQ$, and therefore also $QV$ where $PV$ is the diameter bisecting $Qq$ Let the diameter through $R$ bisect $PQ$ in $Y$ and $QV$ in $M$ Join $PM$

By Prop 19 $PV = \frac{4}{3}RM$

Also $PV = 2YM$

Therefore $YM = 2RY$

and $\triangle PQM = 2\triangle PRQ$

Hence $\triangle PQV = 4\triangle PRQ$,

and $\triangle PQq = 8\triangle PRQ$

Also, if $RW$, the ordinate from $R$ to $PV$, be produced to meet the curve again in $r$, $RW = rW$,
and the same proof shows that

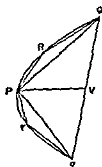
$$\triangle PQq = 8\triangle Prq$$

## Proposition 22

*If there be a series of areas $A$, $B$, $C$, $D$, each of which is four times the next in order, and if the largest, $A$ be equal to the triangle $PQq$ inscribed in a parabolic segment $PQq$ and having the same base with it and equal height, then*

$(A+B+C+D+ \quad ) <$ (*area of segment* $PQq$)

For, since $\triangle PQq = 8\triangle PRQ = 8\triangle Pqr$ where $R$ $r$ are the vertices of the segments cut off by $PQ$ $Pq$ as in the last proposition,




$$\triangle PQq = 4(\triangle PQR + \triangle Pqr)$$

Therefore since $\triangle PQq = A$,

$$\triangle PQR + \triangle Pqr = B$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area $C$, and so on

Therefore $A+B+C+D+ \quad$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment

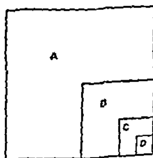
## Proposition 23

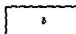
*Given a series of areas $A$ $B$, $C$, $D$ $Z$ of which $A$ is the greatest and each is equal to four times the next in order, then*

$$A+B+C+ \quad +Z+\tfrac{1}{3}Z = \tfrac{4}{3}A$$

Take areas $b$, $c$, $d$ such that

$$b = \tfrac{1}{3}B,$$
$$c = \tfrac{1}{3}C$$
$$d = \tfrac{1}{3}D \text{ and so on}$$










Then since $b = \tfrac{1}{3}B$

and $B = \tfrac{1}{4}A$

$$B+b = \tfrac{1}{3}A$$

Similarly $C+c = \tfrac{1}{3}B$

Therefore

$$B+C+D+ \quad +Z+b+c+d+ \quad +z = \tfrac{1}{3}(A+B+C+ \quad +Y)$$

But

$$b+c+d+ \quad +y = \tfrac{1}{3}(B+C+D+ \quad +Y)$$

Therefore, by subtraction

$$B+C+D+ \quad +Z+z = \tfrac{1}{3}A$$

or

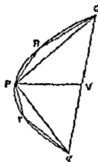
$$A+B+C+ \quad +Z+\tfrac{1}{3}Z = \tfrac{4}{3}A$$

PROPOSITION 24

*Every segment bounded by a parabola and a chord Qq is equal to four thirds of the triangle which has the same base as the segment and equal height*

Suppose $K = \frac{4}{3}\triangle PQq$

where $P$ is the vertex of the segment, and we have then to prove that the area of the segment is equal to $K$




For if the segment be not equal to $K$, it must either be greater or less

I Suppose the area of the segment greater than $K$

If then we inscribe in the segments cut off by $PQ$ $Pq$ triangles which have the same base and equal height, i e triangles with the same vertices $R$ $r$ as those of the segments and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment $PQq$ exceeds $K$

Therefore the polygon so formed must be greater than the area $K$, which is impossible since [Prop 23]

$$A+B+C+ \quad +Z<\tfrac{4}{3}A,$$

where

$$A=\triangle PQq$$

Thus the area of the segment cannot be greater than $K$

II Suppose if possible that the area of the segment is less than $K$

If then $\triangle PQq=A$, $B=\frac{1}{4}A$ $C=\frac{1}{4}B$, and so on until we arrive at an area $X$ such that $X$ is less than the difference between $K$ and the segment we have

$$A+B+C+ \quad +X+\tfrac{1}{3}X=\tfrac{4}{3}A \qquad \text{[Prop 23]}$$
$$=K$$

Now since $K$ exceeds $A+B+C+ \quad +X$ by an area less than $X$ and the area of the segment by an area greater than $X$ it follows that

$$A+B+C+ \quad +X>\text{(the segment)}$$

which is impossible by Prop 22 above

Hence the segment is not less than $K$

Thus, since the segment is neither greater nor less than $K$,

$$\text{(area of segment } PQq)=K=\tfrac{4}{3}\triangle PQq$$

# ON FLOATING BODIES

## BOOK ONE

### POSTULATE 1

"Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more, and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in any thing and compressed by anything else"

PROPOSITION 1

*If a surface be cut by a plane always passing through a certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere*

For if not there will be some two lines drawn from the point to the surface which are not equal

Suppose $O$ to be the fixed point, and $A$ $B$ to be two points on the surface such that $OA$ $OB$ are unequal Let the surface be cut by a plane passing through $OA$ $OB$ Then the section is by hypothesis a circle whose centre is $O$

Thus $OA = OB$, which is contrary to the assumption Therefore the surface cannot but be a sphere

PROPOSITION 2

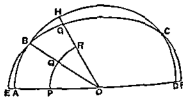
*The surface of any fluid at rest is the surface of a sphere whose centre is the same as that of the earth*

Suppose the surface of the fluid cut by a plane through $O$, the centre of the earth in the curve $ABCD$

$ABCD$ shall be the circumference of a circle

For if not, some of the lines drawn from $O$ to the curve will be unequal Take one of them $OB$ such that $OB$ is greater than some of the lines from $O$ to the curve and less than others Draw a circle with $OB$ as radius Let it be $EBF$ which will therefore fall partly within and partly without the surface of the fluid

Draw $OGH$ making with $OB$ an angle equal to the angle $EOB$ and meeting the surface in $H$ and the circle in $G$ Draw also in the plane an arc of a circle $PQR$ with centre $O$ and within the fluid




Then the parts of the fluid along $PQR$ are uniform and continuous and the part $PQ$ is compressed by the part between it and $AB$, while the part $QR$ is compressed by the part between $QR$ and $BH$

Therefore the parts along $PQ$, $QR$ will be unequally compressed, and the part which is compressed the less will be set in motion by that which is compressed the more

Therefore there will not be rest, which is contrary to the hypothesis

Hence the section of the surface will be the circumference of a circle whose centre is $O$ and so will all other sections by planes through $O$

Therefore the surface is that of a sphere with centre $O$

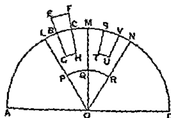
PROPOSITION 3

*Of solids those which size for size, are of equal weight with a fluid will if let down into the fluid be immersed so that they do not project above the surface but do not sink lower*

If possible, let a certain solid $EFHG$ of equal weight volume for volume with the fluid remain immersed in it so that part of it $EBCF$, projects above the surface

Draw through $O$ the centre of the earth, and through the solid a plane cutting the surface of the fluid in the circle $ABCD$

Conceive a pyramid with vertex $O$ and base a parallelogram at the surface of the fluid such that it includes the immersed portion of the solid Let this pyramid be cut by the plane of $ABCD$ in $OL$, $OM$ Also let a sphere within the fluid and below $GH$ be described with centre $O$ and let the plane of $ABCD$ cut this sphere in $PQR$




Conceive also another pyramid in the fluid with vertex $O$, continuous with the former pyramid and equal and similar to it Let the pyramid so described be cut in $OM$ $ON$ by the plane of $ABCD$

Lastly let $STUV$ be a part of the fluid within the second pyramid equal and similar to the part $BGHC$ of the solid, and let $SV$ be at the surface of the fluid

Then the pressures on $PQ$ $QR$ are unequal that on $PQ$ being the greater Hence the part at $QR$ will be set in motion by that at $PQ$, and the fluid will not be at rest, which is contrary to the hypothesis

Therefore the solid will not stand out above the surface

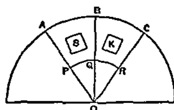
Nor will it sink further, because all the parts of the fluid will be under the same pressure

PROPOSITION 4

*A solid lighter than a fluid will if immersed in it not be completely submerged but part of it will project above the surface*

In this case after the manner of the previous proposition we assume the solid if possible to be completely submerged and the fluid to be at rest in that position and we conceive (1) a pyramid with its vertex at $O$ the centre of the earth, including the solid (2) another pyramid continuous with the former and equal and similar to it with the same vertex $O$ (3) a portion of the fluid within this latter pyramid equal to the immersed solid in the other pyramid (4) a sphere with centre $O$ whose surface is below the immersed solid and the part of the fluid in the second pyramid corresponding thereto We suppose a plane to be drawn through the centre $O$ cutting the surface of the fluid in the circle

$ABC$, the solid in $S$, the first pyramid in $OA$, $OB$, the second pyramid in $OB$, $OC$, the portion of the fluid in the second pyramid in $K$, and the inner sphere in $PQR$



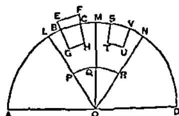

Then the pressures on the parts of the fluid at $PQ$, $QR$ are unequal, since $S$ is lighter than $K$ Hence there will not be rest, which is contrary to the hypothesis

Therefore the solid $S$ cannot, in a condition of rest, be completely submerged

## Proposition 5

*Any solid lighter than a fluid will, if placed in the fluid be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced*

For let the solid be $EGHF$, and let $BGHC$ be the portion of it immersed when the fluid is at rest As in Prop 3, conceive a pyramid with vertex $O$ including the solid, and another pyramid with the same vertex continuous with the former and equal and similar to it Suppose a portion of the fluid $STUV$ at the base of the second pyramid to be equal and similar to the immersed portion of the solid and let the construction be the same as in Prop 3

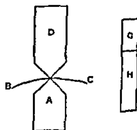



Then since the pressure on the parts of the fluid at $PQ$, $QR$ must be equal in order that the fluid may be at rest, it follows that the weight of the portion $STUV$ of the fluid must be equal to the weight of the solid $EGHF$ And the former is equal to the weight of the fluid displaced by the immersed portion of the solid $BGHC$

## Proposition 6

*If a solid lighter than a fluid be forcibly immersed in it the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced*

For let $A$ be completely immersed in the fluid, and let $G$ represent the weight of $A$ and $(G+H)$ the weight of an equal volume of the fluid Take a solid $D$, whose weight is $H$ and add it to $A$ Then the weight of $(A+D)$ is less than that of an equal volume of the fluid, and if $(A+D)$ is immersed in the fluid, it will project so that its weight will be equal to the weight of the fluid displaced But its weight is $(G+H)$




Therefore the weight of the fluid displaced is $(G+H)$ and hence the volume of the fluid displaced is the volume of the solid $A$ There will accordingly be rest with $A$ immersed and $D$ projecting

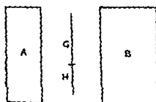
Thus the weight of $D$ balances the upward force exerted by the fluid on $A$ and therefore the latter force is equal to $H$ which is the difference between the weight of $A$ and the weight of the fluid which $A$ displaces

### Proposition 7

*A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced*

(1) The first part of the proposition is obvious, since the part of the fluid under the solid will be under greater pressure, and therefore the other parts will give way until the solid reaches the bottom

(2) Let $A$ be a solid heavier than the same volume of the fluid and let $(G+H)$ represent its weight, while $G$ represents the weight of the same volume of the fluid




Take a solid $B$ lighter than the same volume of the fluid and such that the weight of $B$ is $G$, while the weight of the same volume of the fluid is $(G+H)$

Let $A$ and $B$ be now combined into one solid and immersed Then since $(A+B)$ will be of the same weight as the same volume of fluid both weights being equal to $(G+H)+G$, it follows that $(A+B)$ will remain stationary in the fluid

Therefore the force which causes $A$ by itself to sink must be equal to the upward force exerted by the fluid on $B$ by itself This latter is equal to the difference between $(G+H)$ and $G$ [Prop 6] Hence $A$ is depressed by a force equal to $H$, i e its weight in the fluid is $H$ or the difference between $(G+H)$ and $G$

### POSTULATE 2

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity

### Proposition 8

*If a solid in the form of a segment of a sphere and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface the solid will rest in such a posit on that its axis is perpendicular to the surface and if the solid be forced into such a position that its base touches the fluid on one side and be then set free it will not remain in that position but will return to the symmetrical position*

### Proposition 9

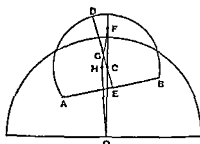
*If a solid in the form of a segment of a sphere and of a substance lighter than a fluid be immersed in it so that its base is completely below the surface, the solid will rest in such a position that its axis is perpendicular to the surface*

[The proof of this proposition has only survived in a mutilated form It deals moreover with only one case out of three which are distinguished at the beginning viz that in which the segment is greater than a hemisphere ]

Suppose first that the segment is greater than a hemisphere Let it be cut by a plane through its axis and the centre of the earth and if possible let it be at rest in the position shown in the figure where $AB$ is the intersection of

the plane with the base of the segment, $DE$ its axis, $C$ the centre of the sphere of which the segment is a part, $O$ the centre of the earth

*The centre of gravity of the portion of the segment outside the fluid as F, lies on OC produced its axis passing through C*




Let $G$ be the centre of gravity of the segment Join $FG$, and produce it to $H$ so that

$FG$ $GH$ = (volume of immersed portion) (rest of solid)

Join $OH$

Then the weight of the portion of the solid outside the fluid acts along $FO$ and the pressure of the fluid on the immersed portion along $OH$, while the weight of the immersed portion acts along $HO$ and is by hypothesis less than the pressure of the fluid acting along $OH$

Hence there will not be equilibrium but the part of the segment towards $A$ will ascend and the part towards $B$ descend, until $DE$ assumes a position perpendicular to the surface of the fluid

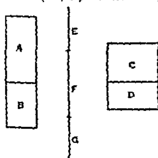
# ON FLOATING BODIES

## BOOK TWO

PROPOSITION 1

*If a solid lighter than a fluid be at rest in it the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole*

Let $(A+B)$ be the solid, $B$ the portion immersed in the fluid




Let $(C+D)$ be an equal volume of the fluid, $C$ being equal in volume to $A$ and $B$ to $D$

Further suppose the line $E$ to represent the weight of the solid $(A+B)$ $(F+G)$ to represent the weight of $(C+D)$, and $G$ that of $D$

Then

$$\text{weight of } (A+B) \quad \text{weight of } (C+D) = E \quad (F+G) \qquad (1)$$

And the weight of $(A+B)$ is equal to the weight of a volume $B$ of the fluid [I 5], i e to the weight of $D$

That is to say $E=G$

Hence by (1)

$$\begin{aligned}\text{weight of } (A+B) \quad \text{weight of } (C+D) &= G \quad F+G \\ &= D \quad C+D \\ &= B \quad A+B\end{aligned}$$

PROPOSITION 2

*If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ (where $p$ is the principal parameter of the generating parabola) and whose specific gravity is less than that of a fluid be placed in the fluid with its axis inclined to the vertical at any angle but so that the base of the segment does not touch the surface of the fluid the segment of the paraboloid will not remain in that position but will return to the position in which its axis is vertical*

Let the axis of the segment of the paraboloid be $AN$ and through $AN$ draw a plane perpendicular to the surface of the fluid Let the plane intersect the paraboloid in the parabola $BAB$ the base of the segment of the paraboloid in $BB$ and the plane of the surface of the fluid in the chord $QQ$ of the parabola

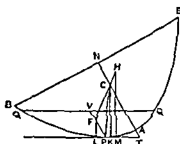
Then since the axis $AN$ is placed in a position not perpendicular to $QQ$, $BB$ will not be parallel to $QQ$

Draw the tangent $PT$ to the parabola which is parallel to $QQ$ and let $P$ be the point of contact [1]

[1] The rest of the proof is given in brackets as supplied by Commandinus

[From $P$ draw $PV$ parallel to $AN$ meeting $QQ'$ in $V$. Then $PV$ will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid.

Let $C$ be the centre of gravity of the paraboloid $BAB'$, and $F$ that of the portion immersed in the fluid. Join $FC$ and produce it to $H$ so that $H$ is the centre of gravity of the remaining portion of the paraboloid above the surface.




Then, since $AN = \frac{3}{2}AC$,

and $AN \not> \frac{3}{4}p$,

it follows that $AC \not> \frac{p}{2}$.

Therefore if $CP$ be joined, the angle $CPT$ is acute. Hence if $CK$ be drawn perpendicular to $PT$, $K$ will fall between $P$ and $T$. And if $FL$, $HM$ be drawn parallel to $CK$ to meet $PT$, they will each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the paraboloid will act upwards along $LF$, while the weight of the portion outside the fluid will act downwards along $HM$.

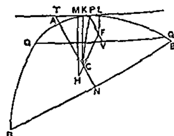
Therefore there will not be equilibrium, but the segment will turn so that $B$ will rise and $B'$ will fall, until $AN$ takes the vertical position.]

## Proposition 3

*If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ (where $p$ is the parameter), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.*

Let the axis of the paraboloid be $AN$, and through $AN$ draw a plane perpendicular to the surface of the fluid intersecting the paraboloid in the parabola $BAB'$, the base of the segment in $BNB'$, and the plane of the surface of the fluid in the chord $QQ'$ of the parabola.

Then, since $AN$, as placed, is not perpendicular to the surface of the fluid, $QQ'$ and $BB'$ will not be parallel.




Draw $PT$ parallel to $QQ'$ and touching the parabola at $P$. Let $PT$ meet $NA$ produced in $T$. Draw the diameter $PV$ bisecting $QQ'$ in $V$. $PV$ is then the axis of the portion of the paraboloid above the surface of the fluid.

Let $C$ be the centre of gravity of the whole segment of the paraboloid, $F$ that of the portion above the surface. Join $FC$ and produce it to $H$ so that $H$ is the centre of gravity of the immersed portion.

Then, since $AC \not> \frac{p}{2}$, the angle $CPT$ is an acute angle, as in the last proposition.

Hence, if $CK$ be drawn perpendicular to $PT$, $K$ will fall between $P$ and $T$. Also, if $HM$, $FL$ be drawn parallel to $CK$, they will be perpendicular to the surface of the fluid.

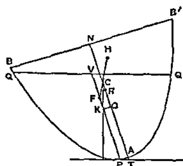
And the force acting on the submerged portion will act upwards along $HM$, while the weight of the rest will act downwards along $LF$ produced.

Thus the paraboloid will turn until it takes the position in which $AN$ is vertical.

### Proposition 4

*Given a right segment of a paraboloid of revolution whose axis $AN$ is greater than $\frac{3}{4}p$ (where p is the parameter), and whose specific gravity is less than that of a fluid but bears to it a ratio not less than $(AN-\frac{3}{4}p)^2 : AN^2$, if the segment of the paraboloid be placed in the fluid with its axis at any inclination to the vertical, but so that its base does not touch the surface of the fluid, it will not remain in that position but will return to the position in which its axis is vertical.*

Let the axis of the segment of the paraboloid be $AN$, and let a plane be drawn through $AN$ perpendicular to the surface of the fluid and intersecting the segment in the parabola $BAB'$, the base of the segment in $BB'$, and the surface of the fluid in the chord $QQ'$ of the parabola.




Then $AN$ as placed will not be perpendicular to $QQ'$.

Draw $PT$ parallel to $QQ'$ and touching the parabola at $P$. Draw the diameter $PV$ bisecting $QQ'$ in $V$. Thus $PV$ will be the axis of the submerged portion of the solid.

Let $C$ be the centre of gravity of the whole solid, $F$ that of the immersed portion. Join $FC$ and produce it to $H$ so that $H$ is the centre of gravity of the remaining portion.

Now, since $AN=\frac{3}{2}AC$

and $AN>\frac{3}{4}p$

it follows that $AC>\frac{p}{2}$.

Measure $CO$ along $CA$ equal to $\frac{p}{2}$, and $OR$ along $OC$ equal to $\frac{1}{2}AO$.

Then since $AN=\frac{3}{2}AC$,

and $AR=\frac{3}{2}AO$,

we have, by subtraction, $NR=\frac{3}{2}OC$.

That is, $AN-AR=\frac{3}{2}OC$

$=\frac{3}{4}p$,

or $AR=(AN-\frac{3}{4}p)$.

Thus $(AN-\frac{3}{4}p)^2 : AN^2=AR^2 : AN^2$,

and therefore the ratio of the specific gravity of the solid to that of the fluid is, by the enunciation, not less than the ratio $AR^2 : AN^2$.

But, by Prop. 1, the former ratio is equal to the ratio of the immersed portion to the whole solid, i.e. to the ratio $PV^2 : AN^2$ [*On Conoids and Spheroids*, Prop. 24].

Hence $PV^2 : AN^2 \nless AR^2 : AN^2$.

or $$PV \nless AR$$

It follows that $$PF(=\tfrac{2}{3}PV) \nless \tfrac{2}{3}AR$$
$$\nless AO$$

If, therefore $OK$ be drawn from $O$ perpendicular to $OA$, it will meet $PF$ between $P$ and $F$

Also if $CK$ be joined the triangle $KCO$ is equal and similar to the triangle formed by the normal the subnormal and the ordinate at $P$ (since $CO=\frac{1}{2}p$ or the subnormal and $KO$ is equal to the ordinate)

Therefore $CK$ is parallel to the normal at $P$, and therefore perpendicular to the tangent at $P$ and to the surface of the fluid

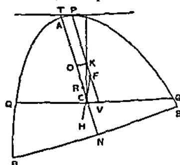
Hence if parallels to $CK$ be drawn through $F$ $H$ they will be perpendicular to the surface of the fluid and the force acting on the submerged portion of the solid will act upwards along the former while the weight of the other portion will act downwards along the latter

Therefore the solid will not remain in its position but will turn until $AN$ assumes a vertical position

## Proposition 5

*Given a right segment of a paraboloid of revolution such that its axis $AN$ is greater than $\frac{3}{4}p$ (where p is the parameter), and its specific gravity is less than that of a fluid but in a ratio to it not greater than the ratio $\{AN^2-(AN-\frac{3}{4}p)^2\}$ $AN^2$, if the segment be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is completely submerged, it will not remain in that position but will return to the position in which $AN$ is vertical*

Let a plane be drawn through $AN$ as placed perpendicular to the surface of the fluid and cutting the segment of the paraboloid in the parabola $BAB'$, the base of the segment in $BB'$ and the plane of the surface of the fluid in the chord $QQ'$ of the parabola




Draw the tangent $PT$ parallel to $QQ'$ and the diameter $PV$ bisecting $QQ'$ will accordingly be the axis of the portion of the paraboloid above the surface of the fluid

Let $F$ be the centre of gravity of the portion above the surface $C$ that of the whole solid and produce $FC$ to $H$ the centre of gravity of the immersed portion

As in the last proposition $AC>\dfrac{p}{2}$ and we measure $CO$ along $CA$ equal to $\dfrac{p}{2}$ and $OR$ along $OC$ equal to $\frac{1}{2}AO$

Then $$AN=\tfrac{3}{2}AC \text{ and } AR=\tfrac{3}{2}AO$$

and we derive as before $$AR=(AN-\tfrac{3}{4}p)$$

Now by hypothesis

$$(\text{spec gravity of solid}) \quad (\text{spec gravity of fluid})$$
$$\ngtr \{AN^2-(AN-\tfrac{3}{4}p)^2\} \quad AN^2$$
$$\ngtr (AN^2-AR^2) \quad AN^2$$

Therefore

$$(\text{portion submerged}) \quad (\text{whole solid})$$
$$\ngtr (AN^2-AR^2) \quad AN^2$$

and (whole solid) (portion above surface)

$$\not> AN^2 \quad AR^2$$

Thus $$AN^2 \quad PV \not> AN^2 \quad AR^2,$$

whence $$PV \not< AR$$

and $$PF \not< \tfrac{2}{3}AR$$

$$\not< AO$$

Therefore if a perpendicular to $AC$ be drawn from $O$ it will meet $PF$ in some point $K$ between $P$ and $F$

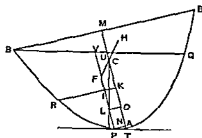
And, since $CO = \frac{1}{2}p$ $CK$ will be perpendicular to $PT$ as in the last proposition

Now the force acting on the submerged portion of the solid will act upwards through $H$ and the weight of the other portion downwards through $F$ in directions parallel in both cases to $CK$ whence the proposition follows

## Proposition 6

*If a right segment of a paraboloid lighter than a fluid be such that its axis $AM$ is greater than $\frac{3}{4}p$ but $AM$ $\frac{1}{2}p < 15$ $4$ and if the segment be placed in the fluid with its axis so inclined to the vertical that its base touches the fluid it will never remain in such a position that the base touches the surface in one point only*

Suppose the segment of the paraboloid to be placed in the position described and let the plane through the axis $AM$ perpendicular to the surface of the fluid intersect the segment of the paraboloid in the parabolic segment $BAB$ and the plane of the surface of the fluid in $BQ$




Take $C$ on $AM$ such that $AC = 2CM$ (or so that $C$ is the centre of gravity of the segment of the paraboloid) and measure $CK$ along $CA$ such that

$$AM \quad CK = 15 \quad 4$$

Thus $AM$ $CK > AM$ $\frac{1}{2}p$ by hypothesis therefore $CK < \frac{1}{2}p$

Measure $CO$ along $CA$ equal to $\frac{1}{2}p$ Also draw $KR$ perpendicular to $AC$ meeting the parabola in $R$

Draw the tangent $PT$ parallel to $BQ$ and through $P$ draw the diameter $PV$ bisecting $BQ$ in $V$ and meeting $KR$ in $I$

Then $$PV \quad PI \underset{\text{or} >}{=} KM \quad AK,$$

*for this is proved*

And $$CK = \tfrac{4}{15}AM = \tfrac{2}{5}AC$$

whence $$AK = AC - CK = \tfrac{3}{5}AC = \tfrac{2}{5}AM$$

Thus $$KM = \tfrac{3}{5}AM$$

Therefore $$KM = \tfrac{3}{2}AK$$

It follows that $$PV \underset{\text{or} >}{=} \tfrac{3}{2}PI$$

so that $$PI \underset{\text{or} <}{=} 2IV$$

Let $F$ be the centre of gravity of the immersed portion of the paraboloid so that $PF = 2FV$ Produce $FC$ to $H$ the centre of gravity of the portion above the surface

Draw $OL$ perpendicular to $PV$

Then since $CO = \frac{1}{2}p$ $CL$ must be perpendicular to $PT$ and therefore to the surface of the fluid

And the forces acting on the immersed portion of the paraboloid and the portion above the surface act respectively upwards and downwards along lines through $F$ and $H$ parallel to $CL$

Hence the paraboloid cannot remain in the position in which $B$ just touches the surface but must turn in the direction of increasing the angle $PTM$

The proof is the same in the case where the point $I$ is not on $VP$ but on $VP$ produced as in the second figure

## Proposition 7

*Given a right segment of a paraboloid of revolution lighter than a fluid and such that its axis $AM$ is greater than $\frac{3}{4}p$, but $AM : \frac{1}{2}p < 15 : 4$, if the segment be placed in the fluid so that its base is entirely submerged, it will never rest in such a position that the base touches the surface of the fluid at one point only*

Suppose the solid so placed that one point of the base only ($B$) touches the surface of the fluid Let the plane through $B$ and the axis $AM$ cut the solid in the parabolic segment $BAB'$ and the plane of the surface of the fluid in the chord $BQ$ of the parabola

Let $C$ be the centre of gravity of the segment so that $AC = 2CM$ and measure $CK$ along $CA$ such that

$$AM : CK = 15 : 4$$

It follows that $CK < \frac{1}{2}p$

Measure $CO$ along $CA$ equal to $\frac{1}{2}p$ Draw $KR$ perpendicular to $AM$ meeting the parabola in $R$

Let $PT$ touching at $P$ be the tangent to the parabola which is parallel to $BQ$, and $PV$ the diameter bisecting $BQ$ i e the axis of the portion of the paraboloid above the surface

Then as in the last proposition we prove that

$$PV \underset{\text{or} >}{=} \tfrac{3}{2}PI$$

and

$$PI \underset{\text{or} <}{=} 2IV$$

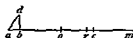
Let $F$ be the centre of gravity of the portion of the solid above the surface, join $FC$ and produce it to $H$ the centre of gravity of the portion submerged

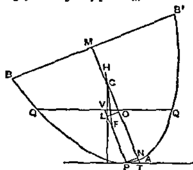
Draw $OL$ perpendicular to $PV$ and as before since $CO = \frac{1}{2}p$ $CL$ is perpendicular to the tangent $PT$ And the lines through $H$ $F$ parallel to $CL$ are perpendicular to the surface of the fluid thus the proposition is established as before

The proof is the same if the point $I$ is not on $VP$ but on $VP$ produced

PROPOSITION 8

*Given a solid in the form of a right segment of a paraboloid of revolution whose axis $AM$ is greater than $\frac{3}{4}p$ but such that $AM$ $\frac{1}{2}p < 15$ $4$, and whose specific gravity bears to that of a fluid a ratio less than $(AM - \frac{3}{4}p)^2$ $AM^2$ then, if the solid be placed in the fluid so that its base does not touch the fluid and its axis is inclined at an angle to the vertical the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid a certain angle to be described*




Let $am$ be taken equal to the axis $AM$, and let $c$ be a point on $am$ such that $ac = 2cm$ Measure $co$ along $ca$ equal to $\frac{1}{2}p$ and $or$ along $oc$ equal to $\frac{1}{2}ao$

Let $X + Y$ be a straight line such that

(spec gr of solid) (spec gr of fluid) $= (X + Y)^2$ $am^2$, $(\alpha)$

and suppose $X = 2Y$

Now $ar = \frac{3}{2}ao = \frac{3}{2}(\frac{2}{3}am - \frac{1}{2}p)$
$= am - \frac{3}{4}p$
$= AM - \frac{3}{4}p$

Therefore by hypothesis
$(X + Y)^2$ $am^2 < ar^2$ $am^2$,
whence $(X + Y) < ar$, and therefore $X < ao$

Measure $ob$ along $oa$ equal to $X$, and draw $bd$ perpendicular to $ab$ and of such length that

$$bd^2 = \tfrac{1}{2}co \quad ab \qquad (\beta)$$

Join $ad$

Now let the solid be placed in the fluid with its axis $AM$ inclined at an angle to the vertical Through $AM$ draw a plane perpendicular to the surface of the fluid and let this plane cut the paraboloid in the parabola $BAB'$ and the plane of the surface of the fluid in the chord $QQ'$ of the parabola

Draw the tangent $PT$ parallel to $QQ'$ touching at $P$, and let $PV$ be the diameter bisecting $QQ'$ in $V$ (or the axis of the immersed portion of the solid), and $PN$ the ordinate from $P$

Measure $AO$ along $AM$ equal to $ao$ and $OC$ along $OM$ equal to $oc$ and draw $OL$ perpendicular to $PV$

I Suppose the angle $OTP$ greater than the angle $dab$

| | |
|---|---|
| Thus | $PN^2$ $NT^2 > db^2$ $ba^2$ |
| But | $PN^2$ $NT^2 = p$ $4AN$ |
| | $= co$ $NT$ |
| and | $db^2$ $ba^2 = \frac{1}{2}co$ $ab$ by $(\beta)$ |
| Therefore | $NT < 2ab$ |
| or | $AN < ab$ |
| whence | $NO > bo$ (since $ao = AO$) |
| | $> X$ |

Now $(X + Y)^2$ $am^2 =$ (spec gr of solid) (spec gr of fluid)
$=$ (portion immersed) (rest of solid)

$$= PV^2 \quad AM^2,$$

so that $\quad X + Y = PV$

But $\quad PL(= NO) > X$

$$> \tfrac{2}{3}(X + Y), \text{ since } X = 2Y,$$
$$> \tfrac{2}{3}PV,$$

or $\quad PV < \tfrac{3}{2}PL$

and therefore $\quad PL > 2LV$

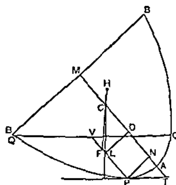
Take a point $F$ on $PV$ so that $PF = 2FV$, i e so that $F$ is the centre of gravity of the immersed portion of the solid

Also $AC = ac = \frac{2}{3}am = \frac{2}{3}AM$, and therefore $C$ is the centre of gravity of the whole solid

Join $FC$ and produce it to $H$ the centre of gravity of the portion of the solid above the surface

Now since $CO = \frac{1}{2}p$ $CL$ is perpendicular to the surface of the fluid therefore so are the parallels to $CL$ through $F$ and $H$ But the force on the immersed portion acts upwards through $F$ and that on the rest of the solid downwards through $H$

Therefore the solid will not rest but turn in the direction of diminishing the angle $MTP$




II Suppose the angle $OTP$ less than the angle $dab$ In this case we shall have instead of the above results, the following,

$$AN > ab$$
$$NO < X$$

Also $\quad PV > \tfrac{3}{2}PL$

and therefore $\quad PL < 2LV$

Make $PF$ equal to $2FV$ so that $F$ is the centre of gravity of the immersed portion

And, proceeding as before we prove in this case that the solid will turn in the direction of *increasing* the angle $MTP$

III When the angle $MTP$ is equal to the angle $dab$, equalities replace inequalities in the results obtained, and $L$ is itself the centre of gravity of the immersed portion Thus all the forces act in one straight line the perpendicular $CL$ therefore there is equilibrium and the solid will rest in the position described

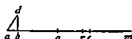
## Proposition 9

*Given a solid in the form of a right segment of a paraboloid of revolution whose axis $AM$ is greater than $\frac{3}{4}p$, but such that $AM \;\, \frac{1}{2}p < 15 \;\, 4$ and whose specific gravity bears to that of a fluid a ratio greater than $\{AM^2 - (AM - \frac{3}{4}p)^2\} \;\, AM^2$, then, if the solid be placed in the fluid with its axis inclined at an angle to the vertical but so that its base is entirely below the surface the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid an angle equal to that described in the last proposition*

Take $am$ equal to $AM$ and take $c$ on $am$ such that $ac = 2cm$ Measure $co$ along $ca$ equal to $\frac{1}{2}p$ and $ar$ along $ac$ such that $ar = \frac{3}{4}ao$

Let $X + Y$ be such a line that

(spec gr of solid) (spec gr of fluid) $=\{am^2-(X+Y)^2\}\ \ am^2$, and suppose $X=2Y$




Now $ar=\frac{2}{3}ao$

$=\frac{2}{3}(\frac{2}{3}am-\frac{1}{2}p)$

$=AM-\frac{3}{4}p$

Therefore, by hypothesis

$am^2-ar^2\ \ am^2<\{am^2-(X+Y)^2\}\ \ am^2$,

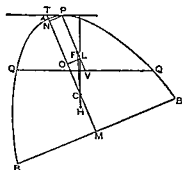
whence $X+Y<ar$,

and therefore $X<ao$

Make $ob$ (measured along $oa$) equal to $Y$ and draw $bd$ perpendicular to $ba$ and of such length that

$$bd^2=\tfrac{1}{2}co\ \ ab$$

Join $ad$




Now suppose the solid placed as in the figure with its axis $AM$ inclined to the vertical Let the plane through $AM$ perpendicular to the surface of the fluid cut the solid in the parabola $BAB$ and the surface of the fluid in $QQ$

Let $PT$ be the tangent parallel to $QQ$ $PV$ the diameter bisecting $QQ$ or) the axis of the portion of the paraboloid above the surface), $PN$ the ordinate from $P$

I Suppose the angle $MTP$ greater than the angle $dab$ Let $AM$ be cut as before in $C$ and $O$ so that $AC=2CM$, $OC=\frac{1}{2}p$ and accordingly $AM$, $am$ are equally divided Draw $OL$ perpendicular to $PV$

Then we have, as in the last proposition

$$PN^2\ \ NT^2>db^2\ \ ba^2$$

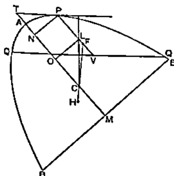
whence $co\ \ NT>\frac{1}{2}co\ \ ab$,

and therefore $AN<ab$

It follows that $NO>bo$

$>X$

Again since the specific gravity of the solid is to that of the fluid as the immersed portion of the solid to the whole




$AM^2-(X+Y)^2\ \ AM^2=AM^2-PV^2\ \ AM^2$,

or $(X+Y)^2\ \ AM^2=PV^2\ \ AM^2$

That is $X+Y=PV$

And $PL$ (or $NO$) $>X$

$>\frac{2}{3}PV$

so that $PL>2LV$

Take $F$ on $PV$ so that $PF=2FV$ Then $F$ is the centre of gravity of the portion of the solid above the surface

Also $C$ is the centre of gravity of the whole solid Join $FC$ and produce it to $H$, the centre of gravity of the immersed portion

Then since $CO=\frac{1}{2}p$ $CL$ is perpendicular to $PT$ and to the surface of the fluid, and the force acting on the immersed portion of the solid acts upwards along the

parallel to $CL$ through $H$, while the weight of the rest of the solid acts downwards along the parallel to $CL$ through $F$.

Hence the solid will not rest but turn in the direction of diminishing the angle $MTP$.

II. Exactly as in the last proposition we prove that, if the angle $MTP$ be less than the angle $dab$ the solid will not remain in its position but will turn in the direction of increasing the angle $MTP$.

III. If the angle $MTP$ is equal to the angle $dab$, the solid will rest in that position because $L$ and $F$ will coincide and all the forces will act along the one line $CL$.

## Proposition 10

*Given a solid in the form of a right segment of a paraboloid of revolution in which the axis $AM$ is of a length such that $AM : \frac{1}{2}p > 15 : 4$ and supposing the solid placed in a fluid of greater specific gravity so that its base is entirely above the surface of the fluid, to investigate the positions of rest.*

(Preliminary.)

Suppose the segment of the paraboloid to be cut by a plane through its axis $AM$ in the parabolic segment $BAB_1$ of which $BB_1$ is the base.

Divide $AM$ at $C$ so that $AC = 2CM$, and measure $CK$ along $CA$ so that

$$AM : CK = 15 : 4, \qquad (\alpha)$$

whence by the hypothesis $CK > \frac{1}{2}p$.

Suppose $CO$ measured along $CA$ equal to $\frac{1}{2}p$ and take a point $R$ on $AM$ such that

$$MR = \tfrac{3}{2}CO.$$

Thus $AR = AM - MR$

$$= \tfrac{3}{2}(AC - CO)$$

$$= \tfrac{3}{2}AO.$$

Join $BA$, draw $KA_2$ perpendicular to $AM$ meeting $BA$ in $A_2$, bisect $BA$ in $A_3$, and draw $A_2M_2$, $A_3M_3$ parallel to $AM$ meeting $BM$ in $M_2$, $M_3$ respectively.

On $A_2M_2$, $A_3M_3$ as axes describe parabolic segments similar to the segment $BAB_1$. (It follows by similar triangles that $BM$ will be the base of the segment whose axis is $A_3M_3$, and $BB_2$ the base of that whose axis is $A_2M_2$, where $BB_2 = 2BM_2$.)

The parabola $BA_2B_2$ will then pass through $C$.

[For

$$BM_2 : M_2M = BM_2 : A_2K$$

$$= KM : AK$$

$$=CM+CK \quad AC-CK$$
$$=(\tfrac{1}{3}+\tfrac{4}{15})AM \quad (\tfrac{2}{3}-\tfrac{4}{15})AM$$
$$=9 \quad 6 \qquad (\beta)$$
$$=MA \quad AC$$

Thus $C$ is seen to be on the parabola $BA_2B_2$ by the converse of Prop 4 of the *Quadrature of the Parabola* ]

Also if a perpendicular to $AM$ be drawn from $O$ it will meet the parabola $BA_2B_2$ in two points, as $Q_2$ $P$ Let $Q_1Q$ $Q_3D$ be drawn through $Q_2$ parallel to $AM$ meeting the parabolas $BAB_1$ $BA_2M$ respectively in $Q_1$ $Q_3$ and $BM$ in $D$, and let $P_1P_2P_3$ be the corresponding parallel to $AM$ through $P_2$ Let the tangents to the outer parabola at $P_1$ $Q_1$ meet $MA$ produced in $T_1$ $U$ respectively

Then, since the three parabolic segments are similar and similarly situated with their bases in the same straight line and having one common extremity, and since $Q_1Q_2Q_3D$ is a diameter common to all three segments, it follows that

$$Q_1Q_2 \quad Q_2Q_3=(B_2B_1 \quad B_1B) \quad (BM \quad MB_2)$$

Now $\quad B_2B_1 \quad B_1B=MM_2 \quad BM$ (dividing by 2)
$\quad =2 \quad 5$ by means of $(\beta)$ above

And $\quad BM \quad MB_2=BM \quad (2BM_2-BM)$
$\quad =5 \quad (6-5)$ by means of $(\beta)$,
$\quad =5 \quad 1$

It follows that $\quad Q_1Q \quad Q_2Q_3=2 \quad 1$

or $\quad Q_1Q_2=2Q_2Q_3$

Similarly $\quad P_1P_2=2P_2P_3$

Also, since $\quad MR=\tfrac{3}{2}CO=\tfrac{3}{4}p$
$$AR=AM-MR$$
$$=AM-\tfrac{3}{4}p$$

(Enunciation)

*If the segment of the paraboloid be placed in the fluid with its base entirely above the surface then*

(I) *if*

$$(spec\ gr\ of\ solid) \quad (spec\ gr\ of\ fluid) \not< AR^2 \quad AM^2$$
$$[\not<(AM-\tfrac{3}{4}p)^2 \quad AM^2]$$

*the solid will rest in the position in which its axis $AM$ is vertical,*

(II) *if*

$$(spec\ gr\ of\ solid) \quad (spec\ gr\ of\ fluid) < AR^2 \quad AM^2$$
$$but > Q_1Q_3^2 \quad AM^2$$

*the solid will not rest with its base touching the surface of the fluid in one point only but in such a position that its base does not touch the surface at any point and its axis makes with the surface an angle greater than $U$*

(III *a*) *if*

$$(spec\ gr\ of\ solid) \quad (spec\ gr\ of\ fluid) = Q_1Q_3^2 \quad AM^2,$$

*the solid will rest and remain in the position in which the base touches the surface of the fluid at one point only and the axis makes with the surface an angle equal to $U$*

(III *b*) *if*

$$(spec\ gr\ of\ solid) \quad (spec\ gr\ of\ fluid) = P_1P_3^2 \quad AM^2,$$

*the solid will rest with its base touching the surface of the fluid at one point only and with its axis inclined to the surface at an angle equal to $T_1$,*

(IV.) if

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) > P_1P_3^2 : AM^2$$
$$\text{but} < Q_1Q_3^2 : AM^2,$$

*the solid will rest and remain in a position with its base more submerged,*

(V.) if

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) < P_1P_3^2 : AM^2,$$

*the solid will rest in a position in which its axis is inclined to the surface of the fluid at an angle less than $T_1$, but so that the base does not even touch the surface at one point.*

(PROOF.)

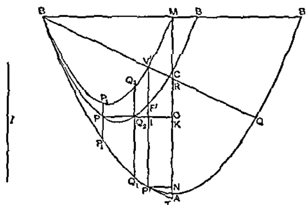
(I.) Since $AM > \frac{3}{4}p$, and

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) \nless (AM - \tfrac{3}{4}p)^2 : AM^2,$$

it follows by Prop. 4 that the solid will be in stable equilibrium with its axis vertical.

(II.) In this case

$$(\text{spec. gr. of solid}) : (\text{spec. gr. of fluid}) < AR^2 : AM^2$$
$$\text{but} > Q_1Q_3^2 : AM^2.$$




Suppose the ratio of the specific gravities to be equal to $l^2 : AM^2$, so that $l < AR$ but $> Q_1Q_3$.

Place $PV'$ between the two parabolas $BAB_1$, $BP_3Q_3M$ equal to $l$ and parallel to $AM$; and let $PV'$ meet the intermediate parabola in $F$.

Then by the same proof as before we obtain

$$P'F' = 2F'V'.$$

Let $P'T$, the tangent at $P'$ to the outer parabola, meet $MA$ in $T$, and let $P'N'$ be the ordinate at $P'$.

Join $BV'$ and produce it to meet the outer parabola in $Q$. Let $OQ_2P_2$ meet $PV'$ in $I$.

Now, since in two similar and similarly situated parabolic segments with bases $BM$, $BB_1$ in the same straight line $BV'$, $BQ$ are drawn making the same angle with the bases,

$$BV' : BQ = BM : BB_1$$
$$= 1 : 2,$$

so that $$BV' = V'Q.$$

Suppose the segment of the paraboloid placed in the fluid as described with its axis inclined at an angle to the vertical, and with its base touching the surface at one point $B$ only. Let the solid be cut by a plane through the axis and perpendicular to the surface of the fluid, and let the plane intersect the solid in the parabolic segment $BAB'$ and the plane of the surface of the fluid in $BQ$.

Take the points C, O on $AM$ as before described. Draw the tangent parallel to $BQ$ touching the parabola in $P$ and meeting $AM$ in $T$, and let $PV$ be the diameter bisecting $BQ$ (i.e. the axis of the immersed portion of the solid).

Then $\quad l^2 : AM^2 =$ (spec. gr. of solid) : (spec. gr. of fluid)

$=$ (portion immersed) : (whole solid)

$= PV'^2 : AM^2$,

whence $\quad P'V' = l = PV$.

Thus the segments in the two figures, namely $B'P'Q'$, $BPQ$, are equal and similar.

Therefore $\quad \angle PTN = \angle P'T'N'$.

Also $\quad AT = AT'$, $AN = AN'$, $PN = P'N'$.

Now, in the first figure, $P'L' < 2L'V'$.

Therefore, if $OL$ be perpendicular to $PV$ in the second figure,

$$PL < 2LV.$$

Take $F$ on $LV$ so that $PF = 2FV$, i.e. so that $F$ is the centre of gravity of the immersed portion of the solid. And $C$ is the centre of gravity of the whole solid. Join $FC$ and produce it to $H$, the centre of gravity of the portion above the surface.

Now, since $CO = \frac{1}{2}p$, $CL$ is perpendicular to the tangent at $P$ and to the surface of the fluid. Thus, as before, we prove that the solid will not rest with $B$ touching the surface, but will turn in the direction of increasing the angle $PTN$.

Hence in the position of rest the axis $AM$ must make with the surface of the fluid an angle greater than the angle $U$ which the tangent at $Q_1$ makes with $AM$.

(III. *a*) *In this case*

(spec. gr. of solid) : (spec. gr. of fluid) $= Q_1Q_1'^2 : AM^2$.

Let the segment of the paraboloid be placed in the fluid so that its base nowhere touches the surface of the fluid and its axis is inclined at an angle to the vertical.

Let the plane through $AM$ perpendicular to the surface of the fluid cut the paraboloid in the parabola $BAB'$ and the plane of the surface of the fluid in $QQ'$. Let $PT$ be the tangent parallel to $QQ'$, $PV$ the diameter bisecting $QQ'$, $PN$ the ordinate at $P$.

Divide $AM$ as before at $C$, $O$

In the other figure let $Q_1N'$ be the ordinate at $Q_1$ Join $BQ_3$ and produce it to meet the outer parabola in $q$ Then $BQ_3 = Q_3q$, and the tangent $Q_1U$ is parallel to $Bq$ Now

$$Q_1Q_3^2 \quad AM^2 = (\text{spec gr of solid}) \quad (\text{spec gr of fluid})$$
$$= (\text{portion immersed}) \quad (\text{whole solid})$$
$$= PV^2 \quad AM^2$$

Therefore $Q_1Q_3 = PV$, and the segments $QPQ'$, $BQ_1q$ of the paraboloid are equal in volume And the base of one passes through $B$, while the base of the other passes through $Q$ a point nearer to $A$ than $B$ is

It follows that the angle between $QQ'$ and $BB$ is less than the angle $B_1Bq$

Therefore

$$\angle U < \angle PTN,$$

whence $\quad AN' > AN,$

and therefore

$$N\,O (\text{or } Q_1Q_2) < PL,$$

where $OL$ is perpendicular to $PV$

It follows since $Q_1Q_2 = 2Q_2Q_3$, that

$$PL > 2LV$$

Therefore $F$, the centre of gravity of the immersed portion of the solid, is between $P$ and $L$ while as before, $CL$ is perpendicular to the surface of the fluid

Producing $FC$ to $H$ the centre of gravity of the portion of the solid above the surface we see that the solid must turn in the direction of diminishing the angle $PTN$ until one point $B$ of the base just touches the surface of the fluid

When this is the case we shall have a segment $BPQ$ equal and similar to the segment $BQ_1q$, the angle $PTN$ will be equal to the angle $U$ and $AN$ will be equal to $AN'$

Hence in this case $PL = 2LV$ and $F$, $L$ coincide, so that $F$, $C$, $H$ are all in one vertical straight line

Thus the paraboloid will remain in the position in which one point $B$ of the base touches the surface of the fluid, and the axis makes with the surface an angle equal to $U$

(III *b*) In the case where

(spec gr of solid) (spec gr of fluid) $= P_1P_2^2 \quad AM^2$

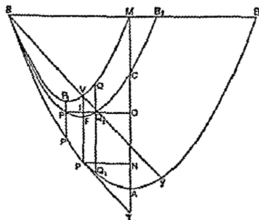
we can prove in the same way that if the solid be placed in the fluid so that its axis is inclined to the vertical and its base does not anywhere touch the surface of the fluid the solid will take up and rest in the position in which one point only of the base touches the surface and the axis is inclined to it at an angle equal to $T_1$ (in the figure on p 552)

(IV) In this case

(spec gr of solid) (spec gr of fluid) $> P_1P_2^2 \quad AM^2$

but $< Q_1Q_2^2 \quad AM^2$

Suppose the ratio to be equal to $l \quad AM^2$, so that $l$ is greater than $P_1P_2$ but less than $Q_1Q_2$



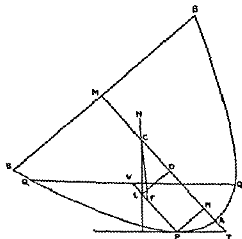

Place $PV'$ between the parabolas $BP_1Q_1$ $BP_2Q_2$ so that $PV'$ is equal to $l$ and parallel to $AM$, and let $PV'$ meet the intermediate parabola in $F$ and $OQ_2P_2$ in $I$

Join $BV$ and produce it to meet the outer parabola in $q$

Then as before $BV = Vq$ and accordingly the tangent $P'T'$ at $P'$ is parallel to $Bq$ Let $P'N'$ be the ordinate of $P'$

1 Now let the segment be placed in the fluid *first*, with *its axis so inclined to the vertical* that its base does not anywhere touch the surface of the fluid

Let the plane through $AM$ perpendicular to the surface of the fluid cut the paraboloid in the parabola $BAB$ and the plane of the surface of the fluid in $QQ$ Let $PT$ be the tangent parallel to $QQ$ $PV$ the diameter bisecting $QQ$ Divide $AM$ at $C$ $O$ as before and draw $OL$ perpendicular to $PV$




Then as before we have $PV' = l = PV$

Thus the segments $BPq$ $QPQ$ of the paraboloid are equal in volume and it follows that the angle between $QQ$ and $BB$ is less than the angle $B_1Bq$

Therefore

$\angle PTN < \angle PTN$

and hence $AN > AN$

so that $NO > NO$

i e $PL > PI$

$> PF$ *a fortiori*

Thus $PL > 2LV$, so that $F$, the centre of gravity of the immersed portion of the solid is between $L$ and $P$, while $CL$ is perpendicular to the surface of the fluid

If then we produce $FC$ to $H$ the centre of gravity of the portion of the solid above the surface we prove that the solid will not rest but turn in the direction of diminishing the angle $PTN$

2 Next let the paraboloid be so placed in the fluid that its base touches the surface of the fluid at one point $B$ only, and let the construction proceed as before

Then $PV = P'V'$, and the segments $BPQ$ $BP'q$ are equal and similar, so that

$$\angle PTN = \angle P'T'N$$

It follows that $AN = AN$ $NO = N'O$

and therefore $P'I = PL$

whence $IL > 2LV$

Thus $F$ again lies between $P$ and $L$ and as before the paraboloid will turn in the direction of diminishing the angle $PTN$, i e so that the base will be more submerged

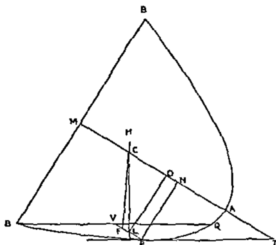
(V) In this case

$$(\text{spec gr of solid}) \quad (\text{spec gr of fluid}) < P_1P_3{}^2 \quad AM^2$$

If then the ratio is equal to $l^2 \quad AM^2 \quad l < P_1P_3$ Place $PV'$ between the parabolas $BP_1Q_1$ and $BP_3Q_3$ equal in length to $l$ and parallel to $AM$ Let $PV'$ meet the intermediate parabola in $F'$ and $OP_2$ in $I$

Join $BV'$ and produce it to meet the outer parabola in $q$ Then, as before, $BV' = Vq$ and the tangent $PT'$ is parallel to $Bq$

1 Let the paraboloid be so placed in the fluid that its base touches the surface at one point only




Let the plane through $AM$ perpendicular to the surface of the fluid cut the paraboloid in the parabolic section $BAB$ and the plane of the surface of the fluid in $BQ$

Making the usual construction, we find

$$PV = l = PV'$$

and the segments $BPQ$ $BP_1q$ are equal and similar

Therefore $\qquad \angle PTN = \angle PTN'$

and $\qquad AN = AN \quad NO = NO$

Therefore $\qquad PL = P'I$

whence it follows that $\qquad PL < 2LV$

Thus $F$ the centre of gravity of the immersed portion of the solid, lies between $L$ and $V$ while $CL$ is perpendicular to the surface of the fluid

Producing $FC$ to $H$ the centre of gravity of the portion above the surface we prove as usual that there will not be rest but the solid will turn in the direction of increasing the angle $PTN$ so that the base will not anywhere touch the surface

2 The solid will however rest in a position where its axis makes with the surface of the fluid an angle less than $T_1$

For let it be placed so that the angle $PTN$ is not less than $T_1$

Then with the same construction as before $PV = l = P'V'$

And since $\angle T \nless \angle T_1$

$AN \ngtr AN_1$

and therefore $NO \nless N_1O$, where $P_1N_1$ is the ordinate of $P_1$

Hence $PF \nless P_1P_2$

But $P_1P_2 > P'F'$

Therefore $PF > \frac{2}{3}PV$,

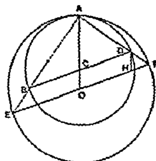
so that $F$ the centre of gravity of the immersed portion of the solid lies between $P$ and $L$

Thus the solid will turn in the direction of diminishing the angle $PTN$ until that angle becomes less than $T_1$

# BOOK OF LEMMAS

PROPOSITION 1

*If two circles touch at A, and if BD EF be parallel diameters in them ADF is a straight line*




Let $O$ $C$ be the centres of the circles and let $OC$ be joined and produced to $A$ Draw $DH$ parallel to $AO$ meeting $OF$ in $H$

Then since $OH = CD = CA$,
and $OF = OA$
we have by subtraction $HF = CO = DH$
Therefore $\angle HDF = \angle HFD$

Thus both the triangles $CAD$ $HDF$ are isosceles and the third angles $ACD$ $DHF$ in each are equal Therefore the equal angles in each are equal to one another and

$$\angle ADC = \angle DFH$$

Add to each the angle $CDF$ and it follows that

$$\angle ADC + \angle CDF = \angle CDF + \angle DFH$$
$$= \text{(two right angles)}$$

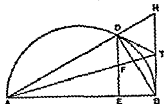
Hence $ADF$ is a straight line

The same proof applies if the circles touch externally

PROPOSITION 2

*Let AB be the diameter of a semicircle and let the tangents to it at B and at any other point D on it meet in T If now DE be drawn perpendicular to AB and if AT DE meet in F,*

$$DF = FE$$




Produce $AD$ to meet $BT$ produced in $H$ Then the angle $ADB$ in the semicircle is right therefore the angle $BDH$ is also right And $TB$, $TD$ are equal

Therefore $T$ is the centre of the semicircle on $BH$ as diameter, which passes through $D$

Hence $HT = TB$

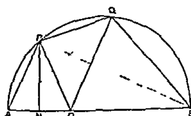
And since $DE$ $HB$ are parallel it follows that $DF = FE$

PROPOSITION 3

*Let P be any point on a segment of a circle whose base is AB, and let PN be perpendicular to AB Take D on AB so that AN=ND If now PQ be an arc equal to the arc PA, and BQ be joined,*

*BQ BD shall be equal*

Join $PA$, $PQ$ $PD$ $DQ$




Then since the arcs $PA$ $PQ$ are equal,

$$PA = PQ$$

But, since $AN = ND$, and the angles at $N$ are right,

$$PA = PD$$

Therefore $PQ = PD$

and $\angle PQD = \angle PDQ$

Now, since $A$ $P$, $Q$, $B$ are concyclic,

$$\angle PAD + \angle PQB = \text{(two right angles)},$$

whence $\angle PDA + \angle PQB = \text{(two right angles)}$

$$= \angle PDA + \angle PDB$$

Therefore $\angle PQB = \angle PDB$,

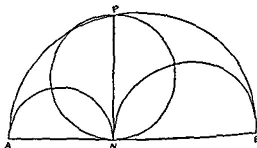
and since the parts, the angles $PQD$, $PDQ$ are equal,

$$\angle BQD = \angle BDQ$$

and $BQ = BD$

PROPOSITION 4

*If AB be the diameter of a semicircle and N any point on AB, and if semicircles be described within the first semicircle and having AN, BN as diameters respectively, the figure included between the circumference of the three semicircles is what Archimedes called an* αρβηλος [1] *and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P*




For

$$AB^2 = AN^2 + NB + 2AN\ NB$$

$$= AN + NB + 2PN^2$$

But circles (or semicircles) are to one another as the squares of their radii (or diameters)

Hence

(semicircle on $AB$) = (sum of semicircles on $AN$ $NB$) + 2(semicircle on $PN$)

That is the circle on $PN$ as diameter is equal to the difference between the semicircle on $AB$ and the sum of the semicircles on $AN$ $NB$, i e is equal to the area of the αρβηλος

PROPOSITION 5

*Let AB be the diameter of a semicircle C any point on AB and CD perpendicular to it and let semicircles be described within the first semicircle and having AC, CB as diameters Then if two circles be drawn touching CD on different sides and each touching two of the semicircles the circles so drawn will be equal*

Let one of the circles touch $CD$ at $E$ the semicircle on $AB$ in $F$, and the semicircle on $AC$ in $($

[1] ἄρβηλος is literally a shoemaker s knife

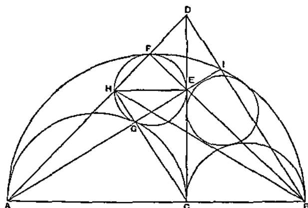
Draw the diameter $EH$ of the circle which will accordingly be perpendicular to $CD$ and therefore parallel to $AB$

Join $FH$, $HA$, and $FE$, $EB$ Then, by Prop 1, $FHA$, $FEB$ are both straight lines, since $EH$, $AB$ are parallel

For the same reason $AGE$ $CGH$ are straight lines

Let $AF$ produced meet $CD$ in $D$, and let $AE$ produced meet the outer semi circle in $I$ Join $BI$, $ID$

Then since the angles $AFB$ $ACD$ are right, the straight lines $AD$, $AB$ are such that the perpendiculars on each from the extremity of the other meet in the point $E$ Therefore by the properties of triangles $AE$ is perpendicular to the line joining $B$ to $D$




But $AE$ is perpendicular to $BI$

Therefore $BID$ is a straight line

Now, since the angles at $G$ $I$ are right $CH$ is parallel to $BD$

Therefore $$AB \quad BC = AD \quad DH$$
$$= AC \quad HE$$

so that $$AC \quad CB = AB \quad HE$$

In like manner if $d$ is the diameter of the other circle, we can prove that
$$AC \quad CB = AB \quad d$$

Therefore $d = HE$ and the circles are equal

PROPOSITION 6

*Let $AB$ the diameter of a semicircle be divided at $C$ so that $AC = \frac{3}{2}CB$ [or in any ratio] Describe semicircles within the first semicircle and on $AC$ $CB$ as diameters and suppose a circle drawn touching all three semicircles If $GH$ be the diameter of this circle to find the relation between $GH$ and $AB$*

Let $GH$ be that diameter of the circle which is parallel to $AB$, and let the circle touch the semicircles on $AB$, $AC$ $CB$ in $D$ $E$, $F$ respectively

Join $AG$, $GD$ and $BH$ $HD$ Then, by Prop 1, $AGD$, $BHD$ are straight lines

For a like reason $AEH$, $BFG$ are straight lines as also are $CEG$ $CFH$

Let $AD$ meet the semicircle on $AC$ in $I$ and let $BD$ meet the semicircle on $CB$ in $K$ Join $CI$ $CK$ meeting $AE$, $BF$ respectively in $L$ $M$, and let $GL$, $HM$ produced meet $AB$ in $N$ $P$ respectively

Now, in the triangle $AGC$, the perpendiculars from $A$, $C$ on the opposite sides meet in $L$ Therefore by the properties of triangles $GLN$ is perpendicular to $AC$

Similarly $HMP$ is perpendicular to $CB$

Again, since the angles at $I$, $K$, $D$ are right $CK$ is parallel to $AD$, and $CI$ to $BD$

Therefore $AC \; CB = AL \; LH$
$= AN \; NP$

and $BC \; CA = BM \; MG$
$= BP \; PN$

Hence $AN \; NP = NP \; PB$,
or $AN \; NP$, $PB$ are in continued proportion

Now in the case where $AC = \frac{3}{2}CB$,
$AN = \frac{3}{2}NP = \frac{9}{4}PB$,

whence $BP \; PN \; NA \; AB = 4 \; 6 \; 9 \; 19$

Therefore $GH = NP = \frac{6}{19}AB$

And similarly $GH$ can be found when $AC \; CB$ is equal to any other given ratio

PROPOSITION 7

*If circles be circumscribed about and inscribed in a square, the circumscribed circle is double of the inscribed circle*

For the ratio of the circumscribed to the inscribed circle is equal to that of the square on the diagonal to the square itself, i e to the ratio 2 1

PROPOSITION 8

*If $AB$ be any chord of a circle whose centre is $O$, and if $AB$ be produced to $C$ so that $BC$ is equal to the radius, if further $CO$ meet the circle in $D$ and be produced to meet the circle a second time in $E$, the arc $AE$ will be equal to three times the arc $BD$*

Draw the chord $EF$ parallel to $AB$, and join $OB$ $OF$

Then since the angles $OEF$, $OFE$ are equal,

$\angle COF = 2\angle OEF$
$= 2\angle BCO$, by parallels,
$= 2\angle BOD$ since $BC = BO$

Therefore
$\angle BOF = 3\angle BOD$

so that the arc $BF$ is equal to three times the arc $BD$

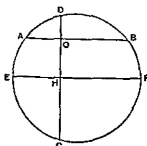
Hence the arc $AE$, which is equal to the arc $BF$, is equal to three times the arc $BD$

PROPOSITION 9

*If in a circle two chords $AB$, $CD$ which do not pass through the centre intersect at right angles then*

$$(\text{arc } AD)+(\text{arc } CB)=(\text{arc } AC)+(\text{arc } DB)$$

Let the chords intersect at $O$, and draw the diameter $EF$ parallel to $AB$ intersecting $CD$ in $H$ $EF$ will thus bisect $CD$ at right angles in $H$, and




$$(\text{arc } ED)=(\text{arc } EC)$$

Also $EDF$ $ECF$ are semicircles while

$$(\text{arc } ED)=(\text{arc } EA)+(\text{arc } AD)$$

Therefore

(sum of arcs $CF$, $EA$ $AD$) = (arc of a semicircle)

And the arcs $AE$, $BF$ are equal

Therefore

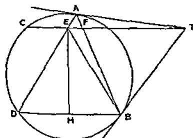
$$(\text{arc } CB)+(\text{arc } AD)=(\text{arc of a semicircle})$$

Hence the remainder of the circumference the sum of the arcs $AC$, $DB$ is also equal to a semicircle and the proposition is proved

## Proposition 10

*Suppose that $TA$ $TB$ are two tangents to a circle, while $TC$ cuts it Let $BD$ be the chord through $B$ parallel to $TC$ and let $AD$ meet $TC$ in $E$ Then, if $EH$ be drawn perpendicular to $BD$ it will bisect it in $H$*

Let $AB$ meet $TC$ in $F$, and join $BE$




Now the angle $TAB$ is equal to the angle in the alternate segment i e

$$\angle TAB=\angle ADB$$
$$=\angle AET, \text{ by parallels}$$

Hence the triangles $EAT$ $AFT$ have one angle equal and another (at $T$) common They are therefore similar and

$$FT\ AT=AT\ ET$$

Therefore $ET\ TF=TA^2$

$$=TB^2$$

It follows that the triangles $EBT$ $BFT$ are similar

Therefore $\angle TEB=\angle TBF$

$$=\angle TAB$$

But the angle $TEB$ is equal to the angle $EBD$ and the angle $TAB$ was proved equal to the angle $EDB$

Therefore $\angle EDB=\angle EBD$

And the angles at $H$ are right angles

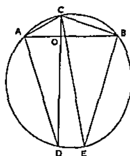
It follows that $BH=HD$

## Proposition 11

*If two chords $AB$ $CD$ in a circle intersect at right angles in a point $O$ not being the centre then*

$$AO^2+BO^2+CO^2+DO^2=(\text{diameter})^2$$

Draw the diameter $CE$ and join $AC$ $CB$ $AD$ $BE$




Then the angle $CAO$ is equal to the angle $CEB$ in the same segment and the angles $AOC$, $EBC$ are right, therefore the triangles $AOC$, $EBC$ are similar and

$$\angle ACO = \angle ECB$$

It follows that the subtended arcs and therefore the chords $AD$, $BE$ are equal

Thus $(AO^2+DO^2)+(BO^2+CO^2)=AD^2+BC^2$
$=BE^2+BC^2$
$=CE^2$

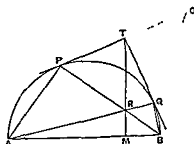
## Proposition 12

*If $AB$ be the diameter of a semicircle and $TP$, $TQ$ the tangents to it from any point $T$ and if $AQ$, $BP$ be joined meeting in $R$, then $TR$ is perpendicular to $AB$*

Let $TR$ produced meet $AB$ in $M$ and join $PA$, $QB$

Since the angle $APB$ is right,

$\angle PAB + \angle PBA =$ (a right angle)
$= \angle AQB$




Add to each side the angle $RBQ$ and

$\angle PAB + \angle QBA =$ (exterior) $\angle PRQ$

But

$\angle TPR = \angle PAB$, and $\angle TQR = \angle QBA$,

in the alternate segments

therefore $\angle TPR + \angle TQR = \angle PRQ$

It follows from this that

$$TP = TQ = TR$$

[For if $PT$ be produced to $O$ so that $TO = TQ$ we have

$\angle TOQ = \angle TQO$

And by hypothesis, $\angle PRQ = \angle TPR + TQR$

By addition $\angle POQ + \angle PRQ = \angle TPR + OQR$

It follows that, in the quadrilateral $OPRQ$ the opposite angles are together equal to two right angles Therefore a circle will go round $OPQR$ and $T$ is its centre because $TP = TO = TQ$ Therefore $TR = TP$]

Thus $\angle TRP = \angle TPR = \angle PAM$

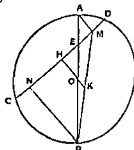
Adding to each the angle $PRM$,

$\angle PAM + \angle PRM = \angle TRP + \angle PRM$
$=$ (two right angles)

Therefore $\angle APR + \angle AMR =$ (two right angles),

whence $\angle AMR =$ (a right angle)

## Proposition 13




*If a diameter $AB$ of a circle meet any chord $CD$ not a diameter in $E$ and if $AM$ $BN$ be drawn perpendicular to $CD$ then*

$$CN = DM$$

Let $O$ be the centre of the circle and $OH$ perpendicular to $CD$ Join $BM$ and produce $HO$ to meet $BM$ in $K$

Then $CH = HD$

And by parallels, since $BO = OA$,

$BK = KM$

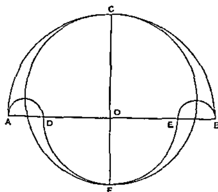
Therefore $NH = HM$

Accordingly $CN = DM$

## Proposition 14

*Let $ACB$ be a semicircle on $AB$ as diameter and let $AD$, $BE$ be equal lengths measured along $AB$ from $A$, $B$ respectively. On $AD$, $BE$ as diameters describe semicircles on the side towards $C$, and on $DE$ as diameter a semicircle on the opposite side. Let the perpendicular to $AB$ through $O$, the centre of the first semicircle, meet the opposite semicircles in $C$, $F$ respectively.*

*Then shall the area of the figure bounded by the circumferences of all the semicircles be equal to the area of the circle on $CF$ as diameter.*




By Eucl. II. 10, since $ED$ is bisected at $O$ and produced to $A$,

$$EA^2 + AD^2 = 2(EO^2 + OA^2),$$

and $CF = OA + OE = EA$.

Therefore

$$AB^2 + DE^2 = 4(EO^2 + OA^2) = 2(CF^2 + AD^2).$$

But circles (and therefore semicircles) are to one another as the squares on their radii (or diameters).

Therefore

(sum of semicircles on $AB$, $DE$)

= (circle on $CF$) + (sum of semicircles on $AD$, $BE$).

Therefore

(area of 'salinon') = (area of circle on $CF$ as diam.).

## Proposition 15

*Let $AB$ be the diameter of a circle, $AC$ a side of an inscribed regular pentagon, $D$ the middle point of the arc $AC$. Join $CD$ and produce it to meet $BA$ produced in $E$; join $AC$, $DB$ meeting in $F$, and draw $FM$ perpendicular to $AB$. Then*

$$EM = (\text{radius of circle}).$$

Let $O$ be the centre of the circle, and join $DA$, $DM$, $DO$, $CB$.

Now $\angle ABC = \frac{2}{5}(\text{right angle})$,

and $\angle ABD = \angle DBC = \frac{1}{5}(\text{right angle})$,

whence $\angle AOD = \frac{2}{5}(\text{right angle})$.

Further, the triangles $FCB$, $FMB$ are equal in all respects.

Therefore in the triangles $DCB$, $DMB$ the sides $CB$, $MB$ being equal and $BD$ common, while the angles $CBD$, $MBD$ are equal,

$$\angle BCD = \angle BMD = \frac{6}{5}(\text{right angle}).$$

But $\angle BCD + \angle BAD = (\text{two right angles})$

$= \angle BAD + \angle DAE$

$= \angle BMD + \angle DMA$,

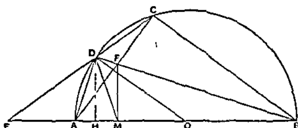
so that $\angle DAE = \angle BCD$,

and $\angle BAD = \angle AMD$.

Therefore $AD = MD$.

Now, in the triangle $DMO$,

$$\angle MOD = \tfrac{2}{5}(\text{right angle}),$$
$$\angle DMO = \tfrac{6}{5}(\text{right angle})$$




Therefore $\angle ODM = \tfrac{2}{5}(\text{right angle}) = AOD$,

whence $OM = MD$

Again $\angle EDA = (\text{supplement of } ADC)$
$$= \angle CBA$$
$$= \tfrac{2}{5}(\text{right angle})$$
$$= \angle ODM$$

Therefore, in the triangles $EDA$, $ODM$,

$$\angle EDA = \angle ODM,$$
$$\angle EAD = \angle OMD,$$

and the sides $AD$, $MD$ are equal

Hence the triangles are equal in all respects, and

$$EA = MO$$

Therefore $EM = AO$

Moreover $DE = DO$, and it follows that, since $DE$ is equal to the side of an inscribed hexagon and $DC$ is the side of an inscribed decagon, $EC$ is divided at $D$ in extreme and mean ratio [i e $EC : ED = ED : DC$], "and this is proved in the book of the *Elements*" [Eucl XIII 9 "If the side of the hexagon and the side of the decagon inscribed in the same circle be put together, the whole straight line is divided in extreme and mean ratio, and the greater segment is the side of the hexagon ]

# THE METHOD TREATING OF MECHANICAL PROBLEMS

"Archimedes to Eratosthenes greeting

"I sent you on a former occasion some of the theorems discovered by me merely writing out the enunciations and inviting you to discover the proofs which at the moment I did not give The enunciations of the theorems which I sent were as follows

1 'If in a right prism with a parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms,[1] and its sides [i e four generators] on the remaining planes (faces) of the prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is and the surface being that which is between the said planes and the segment cut off from the cylinder is one sixth part of the whole prism

2 "If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms[2] and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders which is within both cylinders is two-thirds of the whole cube

Now these theorems differ in character from those communicated before for we compared the figures then in question conoids and spheroids and segments of them in respect to size, with figures of cones and cylinders but none of those figures have yet been found to be equal to a solid figure bounded by planes whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are bounded by planes The proofs then of these theorems I have written in this book and now send to you Seeing moreover in you as I say an earnest student, a man of considerable eminence in philosophy and an admirer [of mathematical inquiry] I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics This procedure is I am persuaded no less useful even for the proof of the theorems themselves for certain things first became clear to me by a mechanical method although they had to be demonstrated by geom-

[1]The parallelograms are apparently *squares*

[2]i e squares

etry afterwards because their investigation by the said method did not furnish an actual demonstration But it is of course easier, when we have previously acquired, by the method some knowledge of the questions to supply the proof than it is to find it without any previous knowledge This is a reason why in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it and I do not want to be thought to have uttered vain words but equally because I am persuaded that it will be of no little service to mathematics for I apprehend that some either of my contemporaries or of my successors, will by means of the method when once established be able to discover other theorems in addition which have not yet occurred to me

'First then I will set out the very first theorem which became known to me by means of mechanics, namely that

*Any segment of a section of a right-angled cone (i e a parabola) is four thirds of the triangle which has the same base and equal height*

and after this I will give each of the other theorems investigated by the same method Then at the end of the book, I will give the geometrical [proofs of the propositions]

[I premise the following propositions which I shall use in the course of the work ]

1 If from [one magnitude another magnitude be subtracted which has not the same centre of gravity the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magnitude and of the subtracted part in the direction of the centre of gravity of the whole] and cutting off from it a length which has to the distance between the said centres of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder [*On the Equilibrium of Planes* I 8]

2 'If the centres of gravity of any number of magnitudes whatever be on the same straight line the centre of gravity of the magnitude made up of all of them will be on the same straight line" [Cf *Ibid* I 5]

3 The centre of gravity of any straight line is the point of bisection of the straight line [Cf *Ibid* I 4]

4 'The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the (opposite) sides cut one another [*Ibid* I 13 14]

5 The centre of gravity of any parallelogram is the point in which the diagonals meet' [*Ibid* I 10]

6 The centre of gravity of a circle is the point which is also the centre [of the circle] '

7 'The centre of gravity of any cylinder is the point of bisection of the axis'

8 The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base]

[All these propositions have already been] proved [1] [Besides these I require also the following proposition which is easily proved

If in two series of magnitudes those of the first series are in order proportional to those of the second series and further], "the magnitudes [of the first series] either all or some of them, are in any ratio whatever [to those of a third series] and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes [of a fourth series], then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series" [*On Conoids and Spheroids* Prop 1]

## Proposition 1

Let $ABC$ be a segment of a parabola bounded by the straight line $AC$ and the parabola $ABC$, and let $D$ be the middle point of $AC$ Draw the straight line $DBE$ parallel to the axis of the parabola and join $AB$ $BC$

Then shall the segment $ABC$ be $\frac{4}{3}$ of the triangle $ABC$

From $A$ draw $AKF$ parallel to $DE$ and let the tangent to the parabola at $C$ meet $DBE$ in $E$ and $AKF$ in $F$ Produce $CB$ to meet $AF$ in $K$ and again produce $CK$ to $H$ making $KH$ equal to $CK$

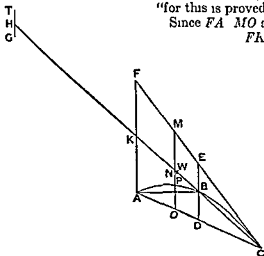
Consider $CH$ as the bar of a balance $K$ being its middle point

Let $MO$ be any straight line parallel to $ED$ and let it meet $CF$ $CK$ $AC$ in $M$ $N$, $O$ and the curve in $P$

Now since $CE$ is a tangent to the parabola and $CD$ the semi ordinate

$$EB=BD,$$

"for this is proved in the Elements [of Conics]" [2]




Since $FA$ $MO$ are parallel to $ED$ it follows that

$$FK=KA \quad MN=NO$$

Now by the property of the parabola proved in a lemma

$$MO : OP=CA : AO \text{ [Cf } \textit{Quadrature of Parabola} \text{ Prop 5]}$$

$$=CK : KN \quad \text{[Eucl vi 2]}$$

$$=HK : KN$$

Take a straight line $TG$ equal to $OP$ and place it with its centre of gravity at $H$, so that $TH=HG$ then since $N$ is the centre of gravity of the straight line $MO$ and

$$MO : TG=HK : KN$$

it follows that $TG$ at $H$ and $MO$ at $N$ will be in equilibrium about $K$ [*On the Equilibrium of Planes* I 6 7]

[1]The problem of finding the centre of gravity of a cone is not solved in any extant work of Archimedes

[2]i e the works on conics by Aristaeus and Euclid

Similarly, for all other straight lines parallel to $DE$ and meeting the arc of the parabola, (1) the portion intercepted between $FC$, $AC$ with its middle point on $KC$ and (2) a length equal to the intercept between the curve and $AC$ placed with its centre of gravity at $H$ will be in equilibrium about $K$

Therefore $K$ is the centre of gravity of the whole system consisting (1) of all the straight lines as $MO$ intercepted between $FC$, $AC$ and placed as they actually are in the figure and (2) of all the straight lines placed at $H$ equal to the straight lines as $PO$ intercepted between the curve and $AC$

And, since the triangle $CFA$ is made up of all the parallel lines like $MO$, and the segment $CBA$ is made up of all the straight lines like $PO$ within the curve

it follows that the triangle placed where it is in the figure, is in equilibrium about $K$ with the segment $CBA$ placed with its centre of gravity at $H$

Divide $KC$ at $W$ so that $CK = 3KW$,

then $W$ is the centre of gravity of the triangle $ACF$, "for this is proved in the books on equilibrium (ἐν τοῖς ἰσορροπικοῖς)

[Cf On the Equilibrium of Planes I 15]

Therefore $\triangle ACF$ (segment $ABC$) $= HK$ $KW$

$= 3$ $1$

Therefore segment $ABC = \frac{1}{3}\triangle ACF$

But $\triangle ACF = 4\triangle ABC$

Therefore segment $ABC = \frac{4}{3}\triangle ABC$

"Now the fact here stated is not actually demonstrated by the argument used but that argument has given a sort of indication that the conclusion is true Seeing then that the theorem is not demonstrated but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published"

## Proposition 2

We can investigate by the same method the propositions that

(1) *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius and*

(2) *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is $1\frac{1}{2}$ times the sphere*

(1) Let $ABCD$ be a great circle of a sphere and $AC$, $BD$ diameters at right angles to one another

Let a circle be drawn about $BD$ as diameter and in a plane perpendicular to $AC$ and on this circle as base let a cone be described with $A$ as vertex Let the surface of this cone be produced and then cut by a plane through $C$ parallel to its base the section will be a circle on $EF$ as diameter On this circle as base let a cylinder be erected with height and axis $AC$, and produce $CA$ to $H$, making $AH$ equal to $CA$

Let $CH$ be regarded as the bar of a balance $A$ being its middle point

Draw any straight line $MN$ in the plane of the circle $ABCD$ and parallel to $BD$ Let $MN$ meet the circle in $O$ $P$ the diameter $AC$ in $S$, and the straight lines $AE$, $AF$ in $Q$ $R$ respectively Join $AO$

Through $MN$ draw a plane at right angles to $AC$,

this plane will cut the cylinder in a circle with diameter $MN$, the sphere in a circle with diameter $OP$, and the cone in a circle with diameter $QR$

Now, since $MS=AC$, and $QS=AS$

$$MS\ SQ=CA\ AS$$
$$=1O$$
$$=OS\ +SQ^2$$

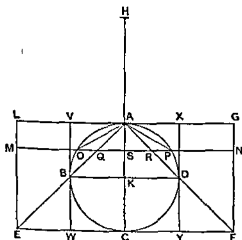
And, since $HA=AC$,

$$HA\ \ AS=CA\ \ AS$$
$$=MS\ \ SQ$$
$$=MS\ \ \ MS\ SQ$$
$$=MS\ \ \ (OS^2+SQ^2),$$
from above
$$=MN^2\ \ (OP\ +QR^2)$$
$$=(\text{circle, diam } MN)\ \ (\text{circle diam } OP + \text{circle, diam } QR)$$

That is

$HA\ \ AS=$(circle in cylinder) (circle in sphere+circle in cone)

Therefore the circle in the cylinder placed where it is is in equilibrium about $A$, with the circle in the sphere together with the circle in the cone if both the latter circles are placed with their centres of gravity at $H$




Similarly for the three corresponding sections made by a plane perpendicular to $AC$ and passing through any other straight line in the parallelogram $LF$ parallel to $EF$

If we deal in the same way with all the sets of three circles in which planes perpendicular to $AC$ cut the cylinder the sphere and the cone, and which make up those solids respectively it follows that the cylinder, in the place where it is will be in equilibrium about $A$ with the sphere and the cone together when both are placed with their centres of gravity at $H$

Therefore since $K$ is the centre of gravity of the cylinder

$$HA\ \ AK=(\text{cylinder})\ \ (\text{sphere}+\text{cone } AEF)$$

But $HA=2AK$

therefore cylinder $=2$(sphere+cone $AEF$)

Now cylinder $=3$(cone $AEF$), [Eucl XII 10]

therefore cone $AEF=2$(sphere)

But since $EF=2BD$

cone $AEF=8$(cone $ABD$)

therefore sphere $=4$(cone $ABD$)

(2) Through $B$ $D$ draw $VBW$ $XDY$ parallel to $AC$ and imagine a cylinder which has $AC$ for axis and the circles on $VX$, $WY$ as diameters for bases

Then cylinder $VY = 2$(cylinder $VD$)
$= 6$(cone $ABD$) [Eucl XII 10]
$= \frac{3}{2}$(sphere), from above

Q E D

"From this theorem to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere I conceived the notion that the surface of any sphere is four times as great as a great circle in it, for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius

## Proposition 3

By this method we can also investigate the theorem that

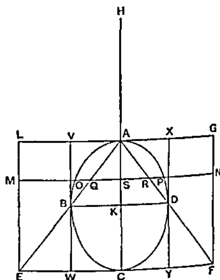
*A cylinder with base equal to the greatest circle in a spheroid and height equal to the axis of the spheroid is $1\frac{1}{2}$ times the spheroid,*

and when this is established, it is plain that

*If any spheroid be cut by a plane through the centre and at right angles to the axis the half of the spheroid is double of the cone which has the same base and the same axis as the segment (i e the half of the spheroid)*

Let a plane through the axis of a spheroid cut its surface in the ellipse $ABCD$ the diameters (i e axes) of which are $AC$ $BD$, and let $K$ be the centre

Draw a circle about $BD$ as diameter and in a plane perpendicular to $AC$ imagine a cone with this circle as base and $A$ as vertex produced and cut by a plane through $C$ parallel to its base the section will be a circle in a plane at right angles to $AC$ and about $EF$ as diameter




Imagine a cylinder with the latter circle as base and axis $AC$, produce $CA$ to $H$ making $AH$ equal to $CA$

Let $HC$ be regarded as the bar of a balance $A$ being its middle point

In the parallelogram $LF$ draw any straight line $MN$ parallel to $EF$ meeting the ellipse in $O$ $P$ and $AE$ $AF$, $AC$ in $Q$ $R$, $S$ respectively

If now a plane be drawn through $MN$ at right angles to $AC$, it will cut the cylinder in a circle with diameter $MN$ the spheroid in a circle with diameter $OP$ and the cone in a circle with diameter $QR$

Since $HA = AC$

$$HA\ \ AS = CA\ \ AS$$
$$= EA\ \ AQ$$
$$= MS\ \ SQ$$

Therefore

$$HA\ \ AS = MS\ \ \ MS\ SQ$$

But, by the property of the ellipse,

$$AS\ SC\ \ SO = AK^2\ \ KB^2$$
$$= AS^2\ \ SQ^2$$

therefore $$SQ\ \ SO^2 = AS^2\ \ AS\ \ SC$$
$$= SQ\ \ SQ\ QM,$$

and accordingly $$SO^2 = SQ\ \ QM$$

Add $SQ^2$ to each side and we have

$$SO + SQ^2 = SQ\ SM$$

Therefore from above we have

$$HA\ \ AS = MS^2\ \ (SO + SQ^2)$$
$$= MN^2\ \ (OP^2 + QR^2)$$
$$= (\text{circle diam } MN)\ \ (\text{circle diam } OP + \text{circle diam } QR)$$

That is

$HA\ \ AS =$ (circle in cylinder) (circle in spheroid + circle in cone)

Therefore the circle in the cylinder in the place where it is is in equilibrium about $A$, with the circle in the spheroid and the circle in the cone together if both the latter circles are placed with their centres of gravity at $H$

Similarly for the three corresponding sections made by a plane perpendicular to $AC$ and passing through any other straight line in the parallelogram $LF$ parallel to $EF$

If we deal in the same way with all the sets of three circles in which planes perpendicular to $AC$ cut the cylinder the spheroid and the cone and which make up those figures respectively, it follows that the cylinder in the place where it is will be in equilibrium about $A$ with the spheroid and the cone together when both are placed with their centres of gravity at $H$

Therefore since $K$ is the centre of gravity of the cylinder

$HA\ \ AK =$ (cylinder) (spheroid + cone $AEF$)

But $HA = 2AK$

therefore cylinder $= 2$(spheroid + cone $AEF$)

And cylinder $= 3$(cone $AEF$) [Eucl XII 10]

therefore cone $AEF = 2$(spheroid)

But since $EF = 2BD$

cone $AEF = 8$(cone $ABD$)

therefore spheroid $= 4$(cone $ABD$)

and half the spheroid $= 2$(cone $ABD$)

Through $B$ $D$ draw $VBW$ $XDY$ parallel to $AC$ and imagine a cylinder which has $AC$ for axis and the circles on $VX$, $WY$ as diameters for bases

Then cylinder $VY = 2$(cylinder $VD$)
$= 6$(cone $ABD$)
$= \frac{3}{2}$(spheroid) from above Q E D

Proposition 4

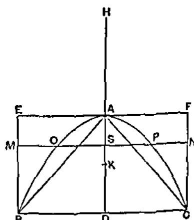
*Any segment of a right-angled conoid (i e a paraboloid of revolution) cut off by a plane at right angles to the axis is $1\frac{1}{2}$ times the cone which has the same base and the same axis as the segment*

This can be investigated by our method, as follows

Let a paraboloid of revolution be cut by a plane through the axis in the parabola $BAC$, and let it also be cut by another plane at right angles to the axis and intersecting the former plane in $BC$ Produce $DA$, the axis of the segment to $H$, making $HA$ equal to $AD$

Imagine that $HD$ is the bar of a balance, $A$ being its middle point

The base of the segment being the circle on $BC$ as diameter and in a plane perpendicular to $AD$, imagine (1) a cone drawn with the latter circle as base and $A$ as vertex and (2) a cylinder with the same circle as base and $AD$ as axis




In the parallelogram $EC$ let any straight line $MN$ be drawn parallel to $BC$, and through $MN$ let a plane be drawn at right angles to $AD$ this plane will cut the cylinder in a circle with diameter $MN$ and the paraboloid in a circle with diameter $OP$

Now $BAC$ being a parabola and $BD$, $OS$ ordinates

$$DA : AS = BD^2 : OS^2$$

or

$$HA : AS = MS^2 : SO^2$$

Therefore

$$HA : AS = (\text{circle, rad } MS) : (\text{circle rad } OS)$$
$$= (\text{circle in cylinder}) : (\text{circle in paraboloid})$$

Therefore the circle in the cylinder in the place where it is, will be in equilibrium about $A$ with the circle in the paraboloid, if the latter is placed with its centre of gravity at $H$

Similarly for the two corresponding circular sections made by a plane perpendicular to $AD$ and passing through any other straight line in the parallelogram which is parallel to $BC$

Therefore as usual if we take all the circles making up the whole cylinder and the whole segment and treat them in the same way we find that the cylinder in the place where it is is in equilibrium about $A$ with the segment placed with its centre of gravity at $H$

If $K$ is the middle point of $AD$ $K$ is the centre of gravity of the cylinder, therefore $HA : AK = (\text{cylinder}) : (\text{segment})$

Therefore cylinder $= 2(\text{segment})$

And cylinder $= 3(\text{cone } ABC)$, [Eucl XII 10]

therefore segment $= \frac{3}{2}(\text{cone } ABC)$

## Proposition 5

*The centre of gravity of a segment of a right-angled conoid (i e a paraboloid of revolution) cut off by a plane at right angles to the axis is on the straight line which is the axis of the segment and divides the said straight line in such a way that the portion of it adjacent to the vertex is double of the remaining portion*

This can be investigated by the method as follows

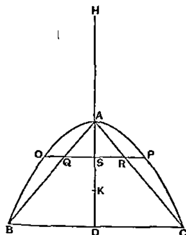
Let a paraboloid of revolution be cut by a plane through the axis in the parabola $BAC$,

and let it also be cut by another plane at right angles to the axis and intersecting the former plane in $BC$

Produce $DA$ the axis of the segment, to $H$ making $HA$ equal to $AD$, and imagine $DH$ to be the bar of a balance its middle point being $A$

The base of the segment being the circle on $BC$ as diameter and in a plane perpendicular to $AD$,

imagine a cone with this circle as base and $A$ as vertex so that $AB$ $AC$ are generators of the cone




In the parabola let any double ordinate $OP$ be drawn meeting $AB$, $AD$ $AC$ in $Q$, $S$, $R$ respectively

Now, from the property of the parabola,

$$BD^2 \quad OS = DA \quad AS$$
$$= BD \quad QS$$
$$= BD^2 \quad BD \quad QS$$

Therefore $OS^2 = BD \quad QS$,

or $BD \quad OS = OS \quad QS$,

whence $BD \quad QS = OS^2 \quad QS$

But $BD \quad QS = AD \quad AS$

$= HA \quad AS$

Therefore $HA \quad AS = OS^2 \quad QS^2$

$= OP^2 \quad QR^2$

If now through $OP$ a plane be drawn at right angles to $AD$, this plane cuts the paraboloid in a circle with diameter $OP$ and the cone in a circle with diameter $QR$

We see therefore that $HA \quad AS =$ (circle, diam $OP$) (circle diam $QR$)

= (circle in paraboloid) (circle in cone),

and the circle in the paraboloid in the place where it is is in equilibrium about $A$ with the circle in the cone placed with its centre of gravity at $H$

Similarly for the two corresponding circular sections made by a plane perpendicular to $AD$ and passing through any other ordinate of the parabola

Dealing therefore in the same way with all the circular sections which make up the whole of the segment of the paraboloid and the cone respectively we see that the segment of the paraboloid in the place where it is is in equilibrium about $A$ with the cone placed with its centre of gravity at $H$

Now since $A$ is the centre of gravity of the whole system as placed and the centre of gravity of part of it namely the cone as placed is at $H$, the centre of gravity of the rest namely the segment is at a point $K$ on $HA$ produced such that

$HA \quad AK =$ (segment) (cone)

But segment $= \frac{3}{2}$(cone) [Prop 4]

Therefore $HA = \frac{3}{2} AK$,

that is $K$ divides $AD$ in such a way that $AK = 2KD$

### Proposition 6

*The centre of gravity of any hemisphere [is on the straight line which] is its axis, and divides the said straight line in such a way that the portion of it adjacent to the surface of the hemisphere has to the remaining portion the ratio which* 5 *has to* 3

Let a sphere be cut by a plane through its centre in the circle $ABCD$, let $AC$, $BD$ be perpendicular diameters of this circle, and through $BD$ let a plane be drawn at right angles to $AC$

The latter plane will cut the sphere in a circle on $BD$ as diameter

Imagine a cone with the latter circle as base and $A$ as vertex

Produce $CA$ to $H$ making $AH$ equal to $CA$, and let $HC$ be regarded as the bar of a balance, $A$ being its middle point

In the semicircle $BAD$, let any straight line $OP$ be drawn parallel to $BD$ and cutting $AC$ in $E$ and the two generators $AB$ $AD$ of the cone in $Q$ $R$ respectively Join $AO$

Through $OP$ let a plane be drawn at right angles to $AC$

this plane will cut the hemisphere in a circle with diameter $OP$ and the cone in a circle with diameter $QR$

Now

$HA$ $AE = AC$ $AE$

$= AO$ $AE^2$

$= (OE^2 + AE^2)$ $AE^2$

$= (OE^2 + QE^2)$ $QE^2$

$=$ (circle diam $OP$ + circle diam $QR$)

(circle diam $QR$)

Therefore the circles with diameters $OP$, $QR$, in the places where they are are in equilibrium about $A$ with the circle with diameter $QR$ if the latter is placed with its centre of gravity at $H$

And since the centre of gravity of the two circles with diameters $OP$, $QR$ taken together in the place where they are is

[There is a lacuna here but the proof can easily be completed on the lines of the corresponding but more difficult case in Prop 8

We proceed thus from the point where the circles with diameters $OP$, $QR$ in the place where they are balance about $A$, the circle with diameter $QR$ placed with its centre of gravity at $H$

A similar relation holds for all the other sets of circular sections made by other planes passing through points on $AG$ and at right angles to $AG$

Taking then all the circles which fill up the hemisphere $BAD$ and the cone $ABD$ respectively, we find that

the hemisphere $BAD$ and the cone $ABD$ in the places where they are, together balance about $A$ a cone equal to $ABD$ placed with its centre of gravity at $H$

Let the cylinder $M+N$ be equal to the cone $ABD$

Then since the cylinder $M+N$ placed with its centre of gravity at $H$ balances the hemisphere $BAD$ and the cone $ABD$ in the places where they are suppose that the portion $M$ of the cylinder placed with its centre of gravity at

$H$, balances the cone $ABD$ (alone) in the place where it is therefore the portion $N$ of the cylinder placed with its centre of gravity at $H$ balances the hemisphere (alone) in the place where it is

Now the centre of gravity of the cone is at a point $V$ such that $AG=4GV$, therefore, since $M$ at $H$ is in equilibrium with the cone

$$M \quad (\text{cone}) = \tfrac{3}{4}AG \quad HA = \tfrac{3}{8}AC \quad AC,$$

whence $$M=\tfrac{3}{8}(\text{cone})$$

But $M+N=(\text{cone})$ therefore $N=\tfrac{5}{8}(\text{cone})$

Now let the centre of gravity of the hemisphere be at $W$ which is somewhere on $AG$

Then, since $N$ at $H$ balances the hemisphere alone,

$$(\text{hemisphere}) \quad N = HA \quad AW$$

But the hemisphere $BAD$ = twice the cone $ABD$,

[*On the Sphere and Cylinder* I 34 and Prop 2 above]

and $N=\tfrac{5}{8}(\text{cone})$, from above

Therefore $$2 \quad \tfrac{5}{8} = HA \quad AW$$
$$= 2AG \quad AW$$

whence $AW=\tfrac{5}{8}AG$, so that $W$ divides $AG$ in such a way that

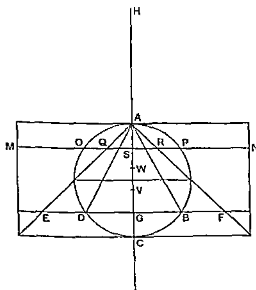
$$AW \quad WG = 5 \quad 3]$$

## Proposition 7

We can also investigate by the same method the theorem that

[*Any segment of a sphere has*] *to the cone* [*with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment*]

[There is a lacuna here but all that is missing is the construction and the construction is easily understood by means of the figure $BAD$ is of course the segment of the sphere the volume of which is to be compared with the volume of a cone with the same base and height]




The plane drawn through $MN$ and at right angles to $AC$ will cut the cylinder in a circle with diameter $MN$ the segment of the sphere in a circle with diameter $OP$ and the cone on the base $EF$ in a circle with diameter $QR$

In the same way as before [cf Prop 2] we can prove that the circle with diameter $MN$ in the place where it is is in equilibrium about $A$ with the two circles with diameters $OP$ $QR$ if these circles are both moved and placed with their centres of gravity at $H$

The same thing can be proved of all sets of three circles in which the cylin

der, the segment of the sphere, and the cone with the common height $AG$ are all cut by any plane perpendicular to $AC$

Since then the sets of circles make up the whole cylinder, the whole segment of the sphere and the whole cone respectively, it follows that the cylinder in the place where it is, is in equilibrium about $A$ with the sum of the segment of the sphere and the cone if both are placed with their centres of gravity at $H$

Divide $AG$ at $W$, $V$ in such a way that

$$AW = WG,\ AV = 3VG$$

Therefore $W$ will be the centre of gravity of the cylinder, and $V$ will be the centre of gravity of the cone

Since, now the bodies are in equilibrium as described,

$$(\text{cylinder}) : (\text{cone } AEF + \text{segment } BAD \text{ of sphere}) = HA : AW$$

[The rest of the proof is lost, but it can easily be supplied thus

We have

$$(\text{cone } AEF + \text{segmt } BAD) : (\text{cylinder}) = AW : AC$$
$$= AW \cdot AC : AC^2$$

But

$$(\text{cylinder}) : (\text{cone } AEF) = AC^2 : \tfrac{1}{3}EG^2$$
$$= AC^2 : \tfrac{1}{3}AG^2$$

Therefore, *ex aequali*

$$(\text{cone } AEF + \text{segmt } BAD) : (\text{cone } AEF) = AW \cdot AC : \tfrac{1}{3}AG^2$$
$$= \tfrac{1}{2}AC : \tfrac{1}{3}AG,$$

whence

$$(\text{segmt } BAD) : (\text{cone } AEF) = (\tfrac{1}{2}AC - \tfrac{1}{3}AG) : \tfrac{1}{3}AG$$

Again

$$(\text{cone } AEF) : (\text{cone } ABD) = EG^2 : DG^2$$
$$= AG^2 : AG \cdot GC$$
$$= AG : GC$$
$$= \tfrac{1}{3}AG : \tfrac{1}{3}GC$$

Therefore, *ex aequali*

$$(\text{segment } BAD) : (\text{cone } ABD) = (\tfrac{1}{2}AC - \tfrac{1}{3}AG) : \tfrac{1}{3}GC$$
$$= (\tfrac{3}{2}AC - AG) : GC$$
$$= (\tfrac{1}{2}AC + GC) : GC \qquad \text{Q E D}]$$

## Proposition 8

[*The enunciation the setting out, and a few words of the construction are missing*

The enunciation however can be supplied from that of Prop 9 with which it must be identical except that it cannot refer to *any* segment, and the presumption therefore is that the proposition was enunciated with reference to one kind of segment only i e either a segment greater than a hemisphere or a segment less than a hemisphere

Heiberg's figure corresponds to the case of a segment greater than a hemisphere The segment investigated is of course the segment $BAD$ The setting-out and construction are self-evident from the figure ]

Produce $AC$ to $H$ $O$, making $HA$ equal to $AC$ and $CO$ equal to the radius of the sphere

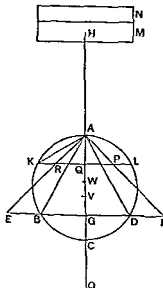
*and let $HC$ be regarded as the bar of a balance the middle point being $A$*

In the plane cutting off the segment describe a circle with $G$ as centre and radius ($GE$) equal to $AG$ and on this circle as base and with $A$ as vertex, let a cone be described $AE$ $AF$ are generators of this cone

Draw $KL$ through any point $Q$ on $AG$, parallel to $EF$ and cutting the segment in $K$, $L$ and $AE$ $AF$ in $R$, $P$ respectively Join $AK$

Now

$$HA \quad AQ = CA \quad AQ$$
$$= AK^2 \quad AQ^2$$
$$= (KQ^2+QA^2) \quad QA^2$$
$$= (KQ^2+PQ^2) \quad PQ^2$$
$$= (\text{circle, diam } KL + \text{circle diam } PR) \quad (\text{circle diam } PR)$$




Imagine a circle equal to the circle with diameter $PR$ placed with its centre of gravity at $H$ therefore the circles on diameters $KL$ $PR$ in the places where they are are in equilibrium about $A$ with the circle with diameter $PR$ placed with its centre of gravity at $H$

Similarly for the corresponding circular sections made by any other plane perpendicular to $AG$

Therefore, taking all the circular sections which make up the segment $ABD$ of the sphere and the cone $AEF$ respectively, we find that the segment $ABD$ of the sphere and the cone $AEF$ in the places where they are are in equilibrium with the cone $AEF$ assumed to be placed with its centre of gravity at $H$

Let the cylinder $M+N$ be equal to the cone $AEF$ which has $A$ for vertex and the circle on $EF$ as diameter for base

Divide $AG$ at $V$ so that

$$AG = 4VG,$$

therefore $V$ is the centre of gravity of the cone $AEF$, for this has been proved before

Let the cylinder $M+N$ be cut by a plane perpendicular to the axis in such a way that the cylinder $M$ (alone) placed with its centre of gravity at $H$ is in equilibrium with the cone $AEF$

Since $M+N$ suspended at $H$ is in equilibrium with the segment $ABD$ of the sphere and the cone $AEF$ in the places where they are

while $M$, also at $H$ is in equilibrium with the cone $AEF$ in the place where it is, it follows that

$N$ at $H$ is in equilibrium with the segment $ABD$ of the sphere in the place where it is

Now (segment $ABD$ of sphere) (cone $ABD$) $= OG$ $GC$

for this is already proved [Cf *On the Sphere and Cylinder* II 2 Cor as well as Prop 7 *ante*]

And (cone $ABD$) (cone $AEF$)

$$= (\text{circle diam } BD) \quad (\text{circle diam } EF)$$
$$= BD^2 \quad EF^2$$
$$= BG^2 \quad GE^2$$
$$= CG \quad GA \quad GA^2$$
$$= CG \quad GA$$

Therefore, *ex aequali*,

$$(\text{segment } ABD \text{ of sphere}) : (\text{cone } AEF) = OG : GA.$$

Take a point $W$ on $AG$ such that

$$AW : WG = (GA+4GC) : (GA+2GC).$$

We have then inversely,

$$GW : WA = (2GC+GA) : (4GC+GA),$$

and *componendo*,

$$GA : AW = (6GC+2GA) : (4GC+GA).$$

But $GO = \frac{1}{4}(6GC+2GA)$, [for $GO-GC = \frac{1}{2}(CG+GA)$]

and $CV = \frac{1}{4}(4GC+GA)$;

therefore $GA : AW = OG : CV$,

and alternately and inversely,

$$OG : GA = CV : WA.$$

It follows, from above, that

$$(\text{segment } ABD \text{ of sphere}) : (\text{cone } AEF) = CV : WA.$$

Now, since the cylinder $M$ with its centre of gravity at $H$ is in equilibrium about $A$ with the cone $AEF$ with its centre of gravity at $V$,

$$(\text{cone } AEF) : (\text{cylinder } M) = HA : AV$$
$$= CA : AV;$$

and, since the cone $AEF$ = the cylinder $M+N$, we have, *dividendo* and *invertendo*,

$$(\text{cylinder } M) : (\text{cylinder } N) = AV : CV.$$

Hence, *componendo*,

$$(\text{cone } AEF) : (\text{cylinder } N) = CA : CV$$
$$= HA : CV.$$

But it was proved that

$$(\text{segment } ABD \text{ of sphere}) : (\text{cone } AEF) = CV : WA;$$

therefore, *ex aequali*,

$$(\text{segment } ABD \text{ of sphere}) : (\text{cylinder } N) = HA : AW.$$

And it was above proved that the cylinder $N$ at $H$ is in equilibrium about $A$ with the segment $ABD$ in the place where it is;

therefore, since $H$ is the centre of gravity of the cylinder $N$, $W$ is the centre of gravity of the segment $ABD$ of the sphere.

## Proposition 9

In the same way we can investigate the theorem that

*The centre of gravity of any segment of a sphere is on the straight line which is the axis of the segment, and divides this straight line in such a way that the part of it adjacent to the vertex of the segment has to the remaining part the ratio which the sum of the axis of the segment and four times the axis of the complementary segment has to the sum of the axis of the segment and double the axis of the complementary segment.*

[As this theorem relates to "*any* segment" but states the same result as that proved in the preceding proposition, it follows that Prop. 8 must have related to one kind of segment, either a segment greater than a semicircle (as in Heiberg's figure of Prop. 8) or a segment less than a semicircle; and the present proposition completed the proof for both kinds of segments. It would only require a slight change in the figure in any case.]

PROPOSITION 10

By this method too we can investigate the theorem that

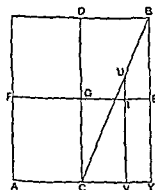
*[A segment of an obtuse angled conoid (i e a hyperboloid of revolution) has to the cone which has] the same base [as the segment and equal height the same ratio as the sum of the axis of the segment and three times] the "annex to the axis" (i e half the transverse axis of the hyperbolic section through the axis of the hyperboloid, or in other words the distance between the vertex of the segment and the vertex of the enveloping cone) has to the sum of the axis of the segment and double of the 'annex'* [this is the theorem proved in *On Conoids and Spheroids*, Prop 25] "and also many other theorems which as the method has been made clear by means of the foregoing examples I will omit in order that I may now proceed to compass the proofs of the theorems mentioned above "

PROPOSITION 11

*If in a right prism with square bases a cylinder be inscribed having its bases in opposite square faces and touching with its surface the remaining four parallelogrammic faces and if through the centre of the circle which is the base of the cylinder and one side of the opposite square face a plane be drawn, the figure cut off by the plane so drawn is one sixth part of the whole prism*

'This can be investigated by the method and when it is set out I will go back to the proof of it by geometrical considerations '

[The investigation by the mechanical method is contained in the two Propositions, 11, 12 Prop 13 gives another solution which, although it contains no mechanics is still of the character which Archimedes regards as inconclusive since it assumes that the solid is actually *made up* of parallel plane sections and that an auxiliary parabola is actually *made up* of parallel straight lines in it Prop 14 added the conclusive geometrical proof ]




Let there be a right prism with a cylinder inscribed as stated

Let the prism be cut through the axis of the prism and cylinder by a plane perpendicular to the plane which cuts off the portion of the cylinder let this plane make as section the parallelogram $AB$, and let it cut the plane cutting off the portion of the cylinder (which plane is perpendicular to $AB$) in the straight line $BC$

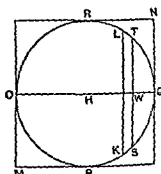
Let $CD$ be the axis of the prism and cylinder let $EF$ bisect it at right angles and through $EF$ let a plane be drawn at right angles to $CD$ this plane will cut the prism in a square and the cylinder in a circle

Let $MN$ be the square and $OPQR$ the circle and let the circle touch the sides of the square in $O$ $P$ $Q$ $R$ [$F$ $E$ in the first figure are identical with $O$ $Q$ respectively] Let $H$ be the centre of the circle

Let $KL$ be the intersection of the plane through $EF$ perpendicular to the axis of the cylinder and the plane cutting off the portion of the cylinder $KL$ is bisected by $OHQ$ [and passes through the middle point of $HQ$]

Let any chord of the circle, as $ST$, be drawn perpendicular to $HQ$, meeting $HQ$ in $W$;

and through $ST$ let a plane be drawn at right angles to $OQ$ and produced on both sides of the plane of the circle $OPQR$.




The plane so drawn will cut the half cylinder having the semicircle $PQR$ for section and the axis of the prism for height in a parallelogram, one side of which is equal to $ST$ and another is a generator of the cylinder; and it will also cut the portion of the cylinder cut off in a parallelogram, one side of which is equal to $ST$ and the other is equal and parallel to $UV$ (in the first figure).

$UV$ will be parallel to $BY$ and will cut off, along $EG$ in the parallelogram $DE$, the segment $EI$ equal to $QW$.

Now, since $EC$ is a parallelogram, and $VI$ is parallel to $GC$,

$$EG : GI = YC : CV$$
$$= BY : UV$$
$$= (\square \text{ in half cyl.}) : (\square \text{ in portion of cyl.}).$$

And $EG = HQ$, $GI = HW$, $QH = OH$;

therefore $\qquad OH : HW = (\square \text{ in half cyl.}) : (\square \text{ in portion}).$

Imagine that the parallelogram in the portion of the cylinder is moved and placed at $O$ so that $O$ is the centre of gravity, and that $OQ$ is the bar of a balance, $H$ being its middle point.

Then, since $W$ is the centre of gravity of the parallelogram in the half cylinder, it follows from the above that the parallelogram in the half cylinder, in the place where it is, with its centre of gravity at $W$, is in equilibrium about $H$ with the parallelogram in the portion of the cylinder when placed with its centre of gravity at $O$.

Similarly for the other parallelogrammic sections made by any plane perpendicular to $OQ$ and passing through any other chord in the semicircle $PQR$ perpendicular to $OQ$.

If then we take all the parallelograms making up the half cylinder and the portion of the cylinder respectively, it follows that the half cylinder, in the place where it is, is in equilibrium about $H$ with the portion of the cylinder cut off when the latter is placed with its centre of gravity at $O$.

## Proposition 12.

Let the parallelogram (square) $MN$ perpendicular to the axis, with the circle $OPQR$ and its diameters $OQ$, $PR$, be drawn separately.

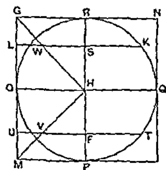
Join $HG$, $HM$, and through them draw planes at right angles to the plane of the circle, producing them on both sides of that plane.

This produces a prism with triangular section $GHM$ and height equal to the axis of the cylinder; this prism is $\frac{1}{4}$ of the original prism circumscribing the cylinder.

Let $LK$, $UT$ be drawn parallel to $OQ$ and equidistant from it, cutting the circle in $K$, $T$, $RP$ in $S$, $F$, and $GH$, $HM$ in $W$, $V$ respectively.

Through $LK$, $UT$ draw planes at right angles to $PR$, producing them on both sides of the plane of the circle;

these planes produce as sections in the half cylinder $PQR$ and in the prism $GHM$ four parallelograms in which the heights are equal to the axis of the cylinder, and the other sides are equal to $KS$ $TF$, $LW$, $UV$ respectively




[The rest of the proof is missing but, as Zeuthen says the result obtained and the method of arriving at it are plainly indicated by the above

Archimedes wishes to prove that the half cylinder $PQR$, in the place where it is balances the prism $GHM$, in the place where it is, about $H$ as fixed point

He has first to prove that the elements (1) the parallelogram with side $=KS$ and (2) the parallelogram with side $=LW$, in the places where they are, balance about $S$, or, in other words that the straight lines $SK$ $LW$, in the places where they are, balance about $S$

Now (radius of circle $OPQR)^2=SK^2+SH^2$,

or $SL^2=SK^2+SW^2$

Therefore $LS^2-SW^2=SK^2$,

and accordingly $(LS+SW)\ LW=SK^2$,

whence $\frac{1}{2}(LS+SW)\ \frac{1}{2}SK=SK\ \ LW$

And $\frac{1}{2}(LS+SW)$ is the distance of the centre of gravity of $LW$ from $S$ while $\frac{1}{2}SK$ is the distance of the centre of gravity of $SK$ from $S$

Therefore $SK$ and $LW$, in the places where they are, balance about $S$

Similarly for the corresponding parallelograms

Taking *all* the parallelogrammic elements in the half cylinder and prism respectively we find that

the half cylinder $PQR$ and the prism $GHM$ in the places where they are respectively, balance about $H$

From this result and that of Prop 11 we can at once deduce the volume of the portion cut off from the cylinder For in Prop 11 the portion of the cylinder, placed with its centre of gravity at $O$, is shown to balance (about $H$) the half cylinder in the place where it is By Prop 12 we may substitute for the half-cylinder in the place where it is the prism $GHM$ of that proposition turned the opposite way relatively to $RP$ The centre of gravity of the prism as thus placed is at a point (say $Z$) on $HQ$ such that $HZ=\frac{2}{3}HQ$

Therefore assuming the prism to be applied at its centre of gravity we have

(portion of cylinder) (prism) $=\frac{2}{3}HQ$ $OH$

$=2$ $3$

therefore (portion of cylinder) $=\frac{2}{3}$(prism $GHM$)

$=\frac{1}{6}$(original prism)

PROPOSITION 13

Let there be a right prism with square bases one of which is $ABCD$

in the prism let a cylinder be inscribed the base of which is the circle $EFGH$ touching the sides of the square $ABCD$ in $E$ $F$ $G$, $H$

Through the centre and through the side corresponding to $CD$ in the square

face *opposite* to $ABCD$ let a plane be drawn, this will cut off a prism equal to $\frac{1}{6}$ of the original prism and formed by three parallelograms and two triangles, the triangles forming opposite faces

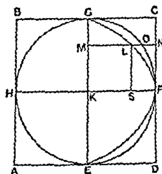
In the semicircle $EFG$ describe the parabola which has $FK$ for axis and passes through $E$, $G$, draw $MN$ parallel to $KF$ meeting $GE$ in $M$, the parabola in $L$, the semicircle in $O$ and $CD$ in $N$

Then $$MN : NL = NF^2,$$
"for this is clear" [Cf Apollonius, *Conics* I 11]

[The parameter is of course equal to $GK$ or $KF$]

Therefore $$MN : NL = GK^2 : LS^2$$

Through $MN$ draw a plane at right angles to $EG$, this will produce as sections (1) in the prism cut off from the whole prism a right angled triangle, the base of which is $MN$, while the perpendicular is perpendicular at $N$ to the plane $ABCD$ and equal to the axis of the cylinder, and the hypotenuse is in the plane cutting the cylinder and (2) in the portion of the cylinder cut off a right angled triangle the base of which is $MO$ while the perpendicular is the generator of the cylinder perpendicular at $O$ to the plane $KN$ and the hypotenuse is




[There is a lacuna here to be supplied as follows

Since $$MN : NL = GK^2 : LS^2$$
$$= MN^2 : LS^2,$$
it follows that $$MN : ML = MN^2 : (MN^2 - LS^2)$$
$$= MN^2 : (MN^2 - MK^2)$$
$$= MN^2 : MO^2$$

But the triangle (1) in the prism is to the triangle (2) in the portion of the cylinder in the ratio of $MN^2 : MO^2$

Therefore (△ in prism) : (△ in portion of cylinder)
$= MN : ML$
= (straight line in rect $DG$) : (straight line in parabola)

We now take all the corresponding elements in the prism, the portion of the cylinder the rectangle $DG$ and the parabola $EFG$ respectively],

and it will follow that

(all the △s in prism) : (all the △s in portion of cylinder)
= (all the str lines in ▱$DG$) : (all the straight lines between parabola and $EG$)

But the prism is made up of the triangles in the prism [the portion of the cylinder is made up of the triangles in it] the parallelogram $DG$ of the straight lines in it parallel to $KF$ and the parabolic segment of the straight lines parallel to $KF$ intercepted between its circumference and $EG$,

therefore (prism) : (portion of cylinder)
= (▱$GD$) : (parabolic segment $EFG$)

But ▱$GD = \frac{3}{2}$(parabolic segment $EFG$),

"for this is proved in my earlier treatise [*Quadrature of Parabola*]

Therefore prism $= \frac{3}{2}$(portion of cylinder)

If then we denote the portion of the cylinder by 2 the prism is 3 and the original prism circumscribing the cylinder is 12 (being 4 times the other prism),

therefore the portion of the cylinder $= \frac{1}{6}$(original prism) Q E D

[The above proposition and the next are peculiarly interesting for the fact that the parabola is an auxiliary curve introduced for the sole purpose of analytically reducing the required cubature to the known quadrature of the parabola ]

Proposition 14

Let there be a right prism with square bases [and a cylinder inscribed therein having its base in the square *ABCD* and touching its sides at *E*, *F*, *G*, *H* let the cylinder be cut by a plane through *EG* and the side corresponding to *CD* in the square face opposite to *ABCD* ]

This plane cuts off from the prism a prism, and from the cylinder a portion of it

It can be proved that the portion of the cylinder cut off by the plane is $\frac{1}{6}$ of the whole prism

But we will first prove that it is possible to inscribe in the portion cut off from the cylinder, and to circumscribe about it solid figures made up of prisms which have equal height and similar triangular bases in such a way that the circumscribed figure exceeds the inscribed by less than any assigned magnitude

But it was proved that

(prism cut off by oblique plane) $< \frac{3}{2}$(figure inscribed in portion of cylinder)

Now (prism cut off) (inscribed figure)

$= \square DG$ ($\square$s inscribed in parabolic segment)

therefore $\square DG < \frac{3}{2}$($\square$s in parabolic segment)

which is impossible since it has been proved elsewhere' that the parallelogram *DG* is $\frac{3}{2}$ of the parabolic segment

Consequently

not greater

And (all the prisms in prism cut off)

(all prisms in circumscr figure)

$=$ (all $\square$s in $\square DG$) (all $\square$s in fig circumscr about parabolic segmt ),

therefore

(prism cut off) (figure circumscr about portion of cylinder)

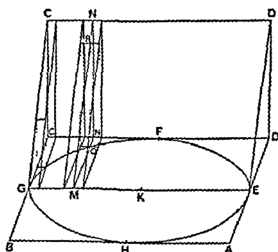
$= (\square DG)$ (figure circumscr about parabolic segment)

But the prism cut off by the oblique plane is $> \frac{3}{2}$ of the solid figure circumscribed about the portion of the cylinder

[There are large gaps in the exposition of this geometrical proof but the way in which the method of exhaustion was applied and the parallelism between this and other applications of it are clear The first fragment shows that solid figures made up of prisms were circumscribed and inscribed to the portion of the cylinder The parallel triangular faces of these prisms were perpendicular to *GE* in the figure of Prop 13 they divided *GE* into equal portions of the requisite smallness each section of the portion of the cylinder by such a

plane was a triangular face common to an inscribed and a circumscribed right prism The planes also produced prisms in the prism cut off by the same oblique plane as cuts off the portion of the cylinder and standing on $GD$ as base

The number of parts into which the parallel planes divided $GE$ was made great enough to secure that the circumscribed figure exceeded the inscribed figure by less than a small assigned magnitude

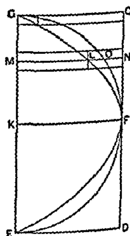



The second part of the proof began with the assumption that the portion of the cylinder is $> \frac{2}{3}$ of the prism cut off, and this was proved to be impossible, by means of the use of the auxiliary parabola and the proportion

$$MN : ML = MN^2 : MO^2$$

which are employed in Prop 13

We may supply the missing proof as follows




In the accompanying figure are represented (1) the first element-prism circumscribed to the portion of the cylinder (2) two element prisms adjacent to the ordinate $OM$ of which that on the left is circumscribed and that on the right (equal to the other) inscribed (3) the corresponding element-prisms forming part of the prism cut off ($CC'GEDD'$) which is $\frac{1}{4}$ of the original prism

In the second figure are shown element rectangles circumscribed and inscribed to the auxiliary parabola, which rectangles correspond exactly to the circumscribed and inscribed element prisms represented in the first figure (the length of $GM$ is the same in both figures and the breadths of the element-rectangles are the same as the heights of the element-prisms), the corresponding element-rectangles forming part of the rectangle $GD$ are similarly shown

For convenience we suppose that $GE$ is divided into an even number of equal parts so that $GK$ contains an integral number of these parts

For the sake of brevity we will call each of the two element prisms of which $OM$ is an edge "el prism $(O)$" and each of the element-prisms of which $MNN'$ is a common face "el prism $(N)$" Similarly we will use the corresponding abbreviations "el rect $(L)$" and "el rect $(N)$" for the corresponding elements in relation to the auxiliary parabola as shown in the second figure

Now it is easy to see that the figure made up of all the inscribed prisms is less than the figure made up of the circumscribed prisms by twice the final circumscribed prism adjacent to $FK$, i e by twice "el prism $(N)$" and as the height of this prism may be made as small as we please by dividing $GK$ into sufficiently small parts it follows that inscribed and circumscribed solid figures made up of element prisms can be drawn differing by less than any assigned solid figure

(1) Suppose if possible that

$$(\text{portion of cylinder}) > \tfrac{2}{3}(\text{prism cut off}),$$

or

$$(\text{prism cut off}) < \tfrac{3}{2}(\text{portion of cylinder})$$

Let $\quad (\text{prism cut off}) = \tfrac{3}{2}(\text{portion of cylinder} - X)$, say

Construct circumscribed and inscribed figures made up of element-prisms such that

$$(\text{circumscr fig}) - (\text{inscr fig}) < X$$

Therefore $\quad (\text{inscr fig}) > (\text{circumscr fig} - X),$

and *a fortiori* $\quad > (\text{portion of cyl} - X)$

It follows that

$$(\text{prism cut off}) < \tfrac{3}{2}(\text{inscribed figure})$$

Considering now the element-prisms in the prism cut off and those in the inscribed figure respectively we have

$$\text{el prism }(N) : \text{el prism }(O) = MN^2 : MO^2$$
$$= MN : ML \qquad \text{[as in Prop 13]}$$
$$= \text{el rect }(N) : \text{el rect }(L)$$

It follows that

$$\Sigma\{\text{el prism }(N)\} : \Sigma\{\text{el prism }(O)\} = \Sigma\{\text{el rect }(N)\} : \Sigma\{\text{el rect }(L)\}$$

(There are really two more prisms and rectangles in the first and third than there are in the second and fourth terms respectively but this makes no difference because the first and third terms may be multiplied by a common factor as $n/(n-2)$ without affecting the truth of the proportion Cf the proposition from *On Conoids and Spheroids* quoted on p 571 above)

Therefore

$$(\text{prism cut off}) : (\text{figure inscr in portion of cyl})$$
$$= (\text{rect } GD) : (\text{fig inscr in parabola})$$

But it was proved above that

$$(\text{prism cut off}) < \tfrac{3}{2}(\text{fig inscr in portion of cyl})$$

therefore $\quad (\text{rect } GD) < \tfrac{3}{2}(\text{fig inscr in parabola}),$

and *a fortiori* $\quad (\text{rect } GD) < \tfrac{3}{2}(\text{parabolic segmt})$

which is impossible since

$$(\text{rect } GD) = \tfrac{3}{2}(\text{parabolic segmt})$$

Therefore (portion of cyl) is *not* greater than $\tfrac{2}{3}$(prism cut off)

(2) In the second lacuna must have come the beginning of the next *reductio ad absurdum* demolishing the other possible assumption that the portion of the cylinder is $< \tfrac{2}{3}$ of the prism cut off

In this case our assumption is that

(prism cut off) $>\frac{3}{2}$(portion of cylinder),

and we circumscribe and inscribe figures made up of element prisms, such that

(prism cut off) $>\frac{3}{2}$(fig circumscr about portion of cyl )

We now consider the element prisms in the prism cut off and in the circumscribed figure respectively, and the same argument as above gives

(prism cut off)  (fig circumscr about portion of cyl )
= (rect $GD$)  (fig circumscr about parabola),

whence it follows that

(rect $GD$) $>\frac{2}{3}$(fig circumscribed about parabola),

and, *a fortiori*,

(rect $GD$) $>\frac{3}{2}$(parabolic segment)

which is impossible since

(rect $GD$) $=\frac{3}{2}$(parabolic segmt )

Therefore

(portion of cyl ) is *not* less than $\frac{2}{3}$(prism cut off)

But it was also proved that neither is it greater,

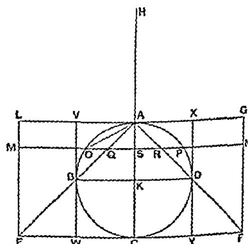
therefore (portion of cyl ) $=\frac{2}{3}$(prism cut off)
$=\frac{1}{6}$(original prism) ]

## [PROPOSITION 15]

[This proposition, which is lost would be the mechanical investigation of the second of the two special problems mentioned in the preface to the treatise, namely that of the cubature of the figure included between two cylinders, each of which is inscribed in one and the same cube so that its opposite bases are in two opposite faces of the cube and its surface touches the other four faces

Zeuthen has shown how the mechanical method can be applied to this case

In the accompanying figure $VWYX$ is a section of the cube by a plane (that of the paper) passing through the axis $BD$ of one of the cylinders inscribed in the cube and parallel to two opposite faces




The same plane gives the circle $ABCD$ as the section of the other inscribed cylinder with axis perpendicular to the plane of the paper and extending on each side of the plane to a distance equal to the radius of the circle or half the side of the cube

$AC$ is the diameter of the circle which is perpendicular to $BD$

Join $AB$ $AD$ and produce them to meet the tangent at $C$ to the circle in $E$ $F$

Then $EC = CF = CA$

Let $LG$ be the tangent at $A$ and complete the rectangle $EFGL$

Draw straight lines from $A$ to the four corners of the section in which the plane through $BD$ perpendicular to $AK$ cuts the cube These straight lines if

produced, will meet the plane of the face of the cube opposite to $A$ in four points forming the four corners of a square in that plane with sides equal to $EF$ or double of the side of the cube, and we thus have a pyramid with $A$ for vertex and the latter square for base

Complete the prism (parallelepiped) with the same base and height as the pyramid

Draw in the parallelogram $LF$ any straight line $MN$ parallel to $EF$ and through $MN$ draw a plane at right angles to $AC$

This plane cuts—

(1) the solid included by the two cylinders in a square with side equal to $OP$,
(2) the prism in a square with side equal to $MN$, and
(3) the pyramid in a square with side equal to $QR$

Produce $CA$ to $H$, making $HA$ equal to $AC$, and imagine $HC$ to be the bar of a balance

Now, as in Prop 2 since $MS=AC$, $QS=AS$,

$$\begin{aligned} MS \cdot SQ &= CA \cdot AS \\ &= AO^2 \\ &= OS^2+SQ^2 \end{aligned}$$

Also
$$\begin{aligned} HA : AS &= CA : AS \\ &= MS : SQ \\ &= MS^2 : MS \cdot SQ \\ &= MS^2 : (OS^2+SQ^2) \text{ from above,} \\ &= MN^2 : (OP^2+QR^2) \\ &= (\text{square side } MN) : (\text{sq , side } OP + \text{sq , side } QR) \end{aligned}$$

Therefore the square with side equal to $MN$, in the place where it is is in equilibrium about $A$ with the squares with sides equal to $OP$, $QR$ respectively placed with their centres of gravity at $H$

Proceeding in the same way with the square sections produced by other planes perpendicular to $AC$ we finally prove that the prism in the place where it is is in equilibrium about $A$ with the solid included by the two cylinders and the pyramid, both placed with their centres of gravity at $H$

Now the centre of gravity of the prism is at $K$

Therefore $HA : AK = (\text{prism}) : (\text{solid} + \text{pyramid})$

or $2 : 1 = (\text{prism}) : (\text{solid} + \tfrac{1}{3} \text{ prism})$

Therefore $2 (\text{solid}) + \tfrac{2}{3}(\text{prism}) = (\text{prism})$

It follows that

$$\begin{aligned} (\text{solid included by cylinders}) &= \tfrac{1}{6}(\text{prism}) \\ &= \tfrac{2}{3}(\text{cube}) \end{aligned}$$

Q E D

There is no doubt that Archimedes proceeded to and completed the rigorous geometrical proof by the method of exhaustion

As observed by Prof C Juel (Zeuthen *l c*) the solid in the present proposition is made up of 8 pieces of cylinders of the type of that treated in the preceding proposition As however the two propositions are separately stated, there is no doubt that Archimedes proofs of them were distinct

In this case $AC$ would be divided into a very large number of equal parts and planes would be drawn through the points of division perpendicular to $AC$ These planes cut the solid and also the cube $VY$ in square sections Thus we can inscribe and circumscribe to the solid the requisite solid figures made up of element prisms and differing by less than any assigned solid magnitude the

prisms have square bases and their heights are the small segments of $AC$ The element prism in the inscribed and circumscribed figures which has the square equal to $OP^2$ for base corresponds to an element prism in the cube which has for base a square with side equal to that of the cube, and as the ratio of the element prisms is the ratio $OS^2 \; BK^2$, we can use the same auxiliary parabola, and work out the proof in exactly the same way, as in Prop 14 ]

# CONICS

# BIOGRAPHICAL NOTE

## Apollonius, *c* 262–*c* 200 B C

Apollonius was born at Perga in Pamphylia Asia Minor some twenty five years after the birth of Archimedes which would place his birth around the year 262 B C He seems to have gone when quite young to Alexandria where, according to Pappus the fourth century mathematician he was attracted by the reputation of the astronomer, Aristarchus of Samos Apollonius studied under the successors of Euclid at Alexandria and continued to reside there during the reigns of Ptolemy Euergetes and of Ptolemy Philopator (247–203 B C ) He was also for some time in Pergamum, where he made the acquaintance of the mathematician, Eudemus to whom he dedicated the first three books of his *Conics* and of King Attalus I (269–197 B C ), to whom the remaining five books of the *Conics* were dedicated

Apollonius appears to have been associated with the leading mathematicians of his day In the dedicatory epistles of the *Conics* he records that he met Philonides while on a trip to Ephesus and that he undertook the composition of this work in the first instance for Naucrates, who was staying in Alexandria Speaking in the same place of the preceding writers on conics Apollonius points out their limitations and inadequacies in such a way that some of his readers such as Pappus have considered him boastful and envious but it would seem that Apollonius is only trying to explain the appearance of a new text book on the elements of conics (Books I IV) and the publication of his own original and more advanced investigations (Books V VIII)

The *Conics* were at once recognized as the authoritative treatise on the subject winning for their author the name of the great geometer They are regularly cited by later writers Pappus added a group of lemmas and Eutocius (*fl* 500 A D ) edited and commented on the first four books These books are extant in the original Greek the fifth sixth and seventh books exist in an Arabic translation, the eighth book is known only indirectly

Although the titles and a general indication of the contents of other works by Apollonius are given by later writers especially by Pappus only one the *Cutting of a Ratio* has survived and that like parts of the *Conics* only in an Arabic version All of the original work with the exception of the second half of the *Conics* has perished Books not extant but known through Pappus are *Cutting of an Area Determinate Section Tangencies Inclinations* and *Plane Loci* He wrote on irrationals and like Archimedes devised a system of multiplication for counting large numbers and calculated an approximate value for the ratio of the circumference of a circle to the diameter The ancient writers also record that Apollonius wrote *On the Burning Glass* in which he probably treated the properties of the parabola a work comparing the dodecahedron and the icosahedron inscribed in the same sphere and a book perhaps on the general principles of mathematics in which he criticized and suggested improvements for Euclid s *Elements* Lastly in astronomy he is credited by Ptol-

emy with an explanation of the motion of the planets by means of epicycles and eccentric circles He seems to have been especially interested in the theory of the moon, and the Alexandrians are said to have called him Epsilon from the resemblance of that Greek letter to the lunar crescent

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# TRANSLATOR'S NOTE

If on first appearance this treatise should seem to the reader a jumble of propositions rigorous indeed, but without much rhyme or reason in their sequence, then he can be sure he has not read aright, and as with the planets, he must look further to save the appearances. There are one or two hypotheses at least that can order the apparent wanderings of parabolas, hyperbolas, and ellipses through the first four books. Such hypotheses are the analogies between the three sections, and especially the development of the analogy between the hyperbola and the ellipse reaching its culmination, in the first book, with the final theorem, the construction of conjugate opposite sections.

In First Definitions I 5, Apollonius innocently defines two kinds of diameters, the transverse and the upright. Each one, in a conic section, bisects all the straight lines parallel to the other. But the upright diameter, defined here only as to position, has, in the case of the ellipse, natural bounds fixed by the section itself, and in Proposition I 15 we find it is the mean proportional between the corresponding transverse diameter (or conjugate diameter) and its parameter. The transverse diameter, in turn, is the mean proportional between the upright diameter (or conjugate) and its parameter, so "upright" and "transverse" become meaningless terms in the case of the ellipse, for something better expressed by the symmetrical relation "conjugate" (First Def. I 6). Immediately in Proposition I 16, as if arbitrarily, the upright diameter of the hyperbola is bounded in the same way, given a definite magnitude and becomes "the second diameter." But so far transverse and upright diameters, or transverse and second diameters, are distinct things in the case of the hyperbola, and there seems to be little reason for giving this second diameter in magnitude. νόμος has not yet become φύσις. That the upright diameter should be given even in position for the hyperbola becomes only very significant with two pairs of propositions—Propositions I 37 and 38 and I 39 and 40—where it is shown that certain properties holding for ordinates to the transverse diameter of the hyperbola and ellipse hold also for the ordinates to their conjugates. But it is only with the final proposition of the first book (I 60) that the magnitude of the hyperbola's second diameter is justified in magnitude as well as position. It is the corresponding diameter of the opposite sections conjugate to the first. And this analogy between the hyperbol-

and ellipse now stands on the threshold of a vast development For this theorem, coming as a climax to the first book, makes possible the main theme of the second book the asymptotes, those strange lines all but touching each opposite section (II 2, 13, 14) and forming a single bound between each adjacent pair (II 15, 17), so making the hyperbola an all but closed section, a puckered ellipse, a mouth turned inside out And in the third book, the fruits of this analogy are gathered as in the especially nice case of Proposition III 15

Although this translation is literal, we have not hesitated to use such symbols and abbreviations as, without prejudicing any Greek number theory or introducing any modern theory of symbols, would yet make the reading and the mechanic of study easier and at the same time preserve all the rigor of Greek mathematics

As for the Greek text, we have used Heiberg, and have constantly referred to the *editio princeps* of Halley In certain instances we have been glad to consult the very excellent French translation of Paul Ver Eecke (Desclée de Brouwer, Bruges, 1923) We have also deferred, at all relevant points, to the English usage of T L Heath's translation of Euclid's *Elements*

## *EXAMPLES OF ABBREVIATIONS AND SYMBOLS USED*

$A=B$ *for* $A$ is equal to $B$
$A+B$ *for* $A$ added to $B$
$A-B$ *for* $B$ subtracted from $A$
$A \quad B \quad C \quad D$ *for* $A$ is to $B$ as $C$ is to $D$
rect $AB, BC$ *for* rectangle $AB, BC$
sq $AB$ *for* square on $AB$
ar *for* area
pllg *for* parallelogram
trgl *for* triangle
quadr *for* quadrilateral
rect $AB\ BC$ rect $CD\ DE$ comp $AB \quad CD\ BC \quad DE$ *for* ratio of rectangle $AB\ BC$ to rectangle $CD, DE$ is compounded of the ratio of $AB$ to $CD$ and of $BC$ to $DE$
ratio comp $AB \quad BC, CD \quad DE$ = ratio comp $XY \quad YZ, ZW \quad WV$ *for* ratio compounded of $AB$ to $BC$ and of $CD$ to $DE$ is the same as the ratio compounded of $XY$ to $YZ$ and of $ZW$ to $WV$
$A>B$ *for* $A$ is greater than $B$
$A<B$ *for* $A$ is less than $B$
rt angle *for* right angle

# BOOK ONE

Apollonius to Eudemus greetings

If you are restored in body and other things go with you to your mind well and good and we too fare pretty well At the time I was with you in Pergamum I observed you were quite eager to be kept informed of the work I was doing in conics And so I have sent you this first book revised and we shall dispatch the others when we are satisfied with them For I don t believe you have forgotten hearing from me how I worked out the plan for these conics at the request of Naucrates the geometer at the time he was with us in Alexandria lecturing and how on arranging them in eight books we immediately communicated them in great haste because of his near departure not revising them but putting down whatever came to us with the intention of a final going over And so finding now the occasion of correcting them one book after another we publish them And since it happened that some others among those frequenting us got acquainted with the first and second books before the revision don t be surprised if you come upon them in a different form

Of the eight books the first four belong to a course in the elements The first book contains the generation of the three sections and of the opposite branches and the principal properties (τα αρχικα συμπτωματα) in them worked out more fully and universally than in the writings of others The second book contains the properties (τα συμβαινοντα) having to do with the diameters and axes and also the asymptotes and other things of a general and necessary use for limits of possibility (πρὸς τους διορισμους) And what I call diameters and what I call axes you will know from this book The third book contains many incredible theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful are new And when we had grasped these we knew that the three line and four line locus had not been constructed by Euclid but only a chance part of it and that not very happily For it was not possible for this construction to be completed without the additional things found by us The fourth book shows in how many ways the sections of a cone intersect with each other and with the circumference of a circle and contains other things in addition none of which has been written up by our predecessors that is in how many points the section of a cone or the circumference of a circle and the opposite branches meet the opposite branches The rest of the books are fuller in treatment For there is one dealing more fully with maxima and minima and one with equal and similar sections of a cone and one with limiting theorems and one with determinate conic problems And so indeed with all of them published those happening upon them can judge them as they see fit Good bye

## First Definitions

1 If from a point a straight line is joined to the circumference of a circle which is not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generating straight line is produced indefinitely, I call a conic surface, and I call the fixed point the vertex, and the straight line drawn from the vertex to the center of the circle the axis

2 And the figure contained by the circle and by the conic surface between the vertex and the circumference of the circle I call a cone, and the point which is also the vertex of the surface I call the vertex of the cone, and the straight line drawn from the vertex to the center of the circle the axis and the circle the base of the cone

3 I call right cones those having axes perpendicular to their bases and oblique those not having axes perpendicular to their bases

4 Of any curved line which is in one plane I call that straight line the diameter which drawn from the curved line, bisects all straight lines drawn to this curved line parallel to some straight line and I call the end of that straight line (the diameter) situated on the curved line the vertex of the curved line, and I say that each of these parallels is drawn ordinatewise to the diameter (τεταγμένως ἐπὶ τὴν διάμετρον κατῆχθαι)[1]

5 Likewise of any two curved lines lying in one plane I call that straight line the transverse diameter (διάμετρος πλαγία) which cuts the two curved lines and bisects all the straight lines drawn to either of the curved lines parallel to some straight line and I call the ends of the diameter situated on the curved lines the vertices of the curved lines and I call that straight line the upright diameter (διάμετρος ὀρθία) which lying between the two curved lines, bisects all the straight lines intercepted between the curved lines and drawn parallel to some straight line and I say that each of the parallels is drawn ordinatewise to the diameter

6 The two straight lines each of which being a diameter bisects the straight lines parallel to the other I call the conjugate diameters (συζυγεῖς διάμετροι) of a curved line and of two curved lines

7 And I call that straight line the axis of a curved line and of two curved lines which being a diameter of the curved line or lines cuts the parallel straight lines at right angles

8 And I call those straight lines the conjugate axes of a curved line and of two curved lines which being conjugate diameters cut the straight lines parallel to each other at right angles

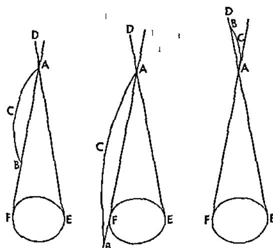
## Proposition 1

*The straight lines drawn from the vertex of the conic surface to points on the surface are on that surface*

Let there be a conic surface whose vertex is the point A and let there be

[1] We shall follow modern usage and generally call these parallels ordinates

taken some point $B$ on the conic surface and let a straight line $ACB$ be joined I say that the straight line $ACB$ is on the conic surface




For if possible, let it not be, and let the straight line $DE$ be the line generating the surface, and $EF$ be the circle along which $ED$ is moved Then if, the point $A$ remaining fixed, the straight line $DE$ is moved along the circumference of the circle $EF$ it will also go through the point $B$ (Def 1), and two straight lines will have the same ends And this is absurd

Therefore the straight line joined from $A$ to $B$ cannot not be on the surface Therefore it is on the surface

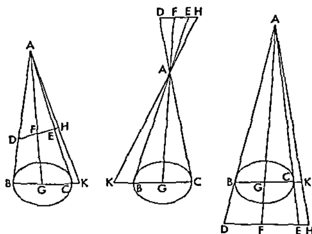
### Porism

It is also evident that if a straight line is joined from the vertex to some point among those within the surface it will fall within the conic surface, and if it is joined to some point among those without it will be outside the surface

### Proposition 2

*If on either one of the two vertically opposite surfaces two points are taken and the straight line joining the points does not verge to the vertex then it will fall within the surface and produced it will fall outside*

Let there be a conic surface whose vertex is the point $A$, and a circle $BC$




along whose circumference the generating straight line is moved and let two points $D$ and $E$ be taken on either one of the two vertically opposite

and let the joining straight line $DE$ not verge to the point $A$

I say that the straight line $DE$ will be within the surface, and produced will be without

Let $AF$ and $AD$ be joined and produced Then they will fall on the circumference of the circle (I 1) Let them fall to the points $B$ and $C$, and let $BC$ be joined Therefore the straight line $BC$ will be within the circle, and so too within the conic surface

Then let a point $F$ be taken at random on $DE$, and let the straight line $AF$ be joined and produced Then it will fall on the straight line $BC$, for the triangle $BCA$ is in one plane (Eucl XI 2) Let it fall to the point $G$ Since then the point $G$ is within the conic surface, therefore the straight line $AG$ is also within the conic surface (I 1, porism), and so too the point $F$ is within the conic surface Then likewise it will be shown that all the points on the straight line $DE$ are within the surface Therefore the straight line $DE$ is within the surface

Then let $DE$ be produced to $H$ I say then it will fall outside the conic surface

For if possible let there be some point $H$ of it not outside the conic surface and let $AH$ be joined and produced Then it will fall either on the circumference of the circle or within (I 1 and porism) And this is impossible, for it falls on $BC$ produced as for example to the point $K$ Therefore the straight line $EH$ is outside the surface

Therefore the straight line $DE$ is within the conic surface and produced is outside

## Proposition 3

*If a cone is cut by a plane through the vertex the section is a triangle*




Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$ and let it be cut by some plane through the point $A$ and let it make as sections lines $AB$ and $AC$ on the surface and the straight line $BC$ in the base

I say that $ABC$ is a triangle

For since the line joined from $A$ to $B$ is the common section of the cutting plane and of the surface of the cone therefore $AB$ is a straight line And likewise also $AC$ And $BC$ is also a straight line Therefore $ABC$ is a triangle

If then a cone is cut by some plane through the vertex, the section is a triangle

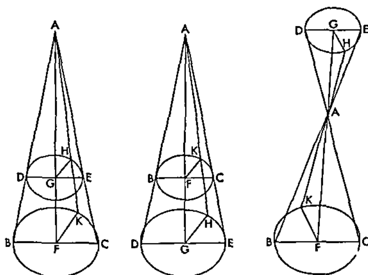
## Proposition 4

*If either one of the vertically opposite surfaces is cut by some plane parallel to the circle along which the straight line generating the surface is moved, the plane cut off within the surface will be a circle having its center on the axis, and the figure contained by the circle and the conic surface intercepted by the cutting plane on the side of the vertex will be a cone*

Let there be a conic surface whose vertex is the point $A$ and whose circle along which the straight line generating the surface is moved is $BC$ and let it be cut by some plane parallel to the circle $BC$, and let it make on the surface as a section the line $DE$

I say that the line $DE$ is a circle having its center on the axis

For let the point $F$ be taken as the center of the circle $BC$, and let $AF$ be joined Therefore $AF$ is the axis (Def 1) and meets the cutting plane Let it meet it at the point $G$, and let some plane be produced through $AF$ Then the




section will be the triangle $ABC$ (I 3) And since the points $D$ $G$, $E$ are points in the cutting plane, and are also in the plane of the triangle $ABC$, therefore $DGE$ is a straight line (Eucl XI 3)

Then let some point $H$ be taken on the line $DE$ and let $AH$ be joined and produced Then it falls on the circumference $BC$ (I 1) Let it meet it at $K$ and let $GH$ and $FK$ be joined And since two parallel planes $DE$ and $BC$ are cut by a plane $ABC$ their common sections are parallel (Eucl XI 16) Therefore the straight line $DE$ is parallel to the straight line $BC$ Then for the same reason the straight line $GH$ is also parallel to the straight line $KF$ Therefore

$FA\ AG \quad FB\ DG \quad FC\ GE \quad FK\ GH$ (Eucl VI 4)

And $BF = KF = FC$

Therefore also $DG = GH = GE$ (Eucl V 9)

Then likewise we could show also that all the straight lines falling from the point $G$ on the line $DE$ are equal to each other

Therefore the line $DE$ is a circle having its center on the axis

And it is evident that the figure contained by the circle $DE$ and the conic surface cut off by it on the side of the point $A$ is a cone

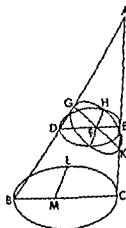
And it is therewith proved that the common section of the cutting plane and of the axial triangle (triangle through the axis) is a diameter of the circle

## Proposition 5

*If an oblique cone is cut by a plane through the axis at right angles to the base, and is also cut by another plane on the one hand at right angles to the axial and on the other cutting off on the side of the vertex a triangle similar to the*

*triangle and lying subcontrariwise then the section is a circle, and let such a section be called subcontrary*

Let there be an oblique cone whose vertex is the point $A$ and whose base is the circle $BC$, and let it be cut by a plane through the axis perpendicular to the circle $BC$, and let it make as a section the triangle $ABC$ (i 3) Then let it also be cut by another plane perpendicular to the triangle $ABC$ and cutting off on the side of the point $A$ the triangle $AKG$ similar to the triangle $ABC$ and lying subcontrariwise, that is, so that the angle $AKG$ is equal to the angle $ABC$ And let it make as a section on the surface, the line $GHK$




I say that the line $GHK$ is a circle

For let any points $H$ and $L$ be taken on the lines $GHK$ and $BC$, and from the points $H$ and $L$ let perpendiculars be dropped to the plane through the triangle $ABC$ Then they will fall to the common sections of the planes (Eucl xi def 6) Let them fall as for example $FH$ and $LM$ Therefore $FH$ is parallel to $LM$ (Eucl xi 6)

Then let the straight line $DFE$ be drawn through $F$ parallel to $BC$, and $FH$ is also parallel to $LM$ Therefore the plane through $FH$ and $DE$ is parallel to the base of the cone (Eucl xi 15) Therefore it is a circle whose diameter is the straight line $DE$ (i 4)

Therefore

$$\text{rect } DF,\ FE = \text{sq } FH \text{ (Eucl iii 31 and vi 8, porism)}$$

And since $ED$ is parallel to $BC$ angle $ADE$ is equal to angle $ABC$ And angle $AKG$ is supposed equal to angle $ABC$ And therefore angle $AKG$ is equal to angle $ADE$ And the vertical angles at the point $F$ are also equal Therefore triangle $DFG$ is similar to triangle $KFE$ and therefore

$$EF : FK = GF : FD \text{ (Eucl vi 4)}$$

Therefore

$$\text{rect } EF,\ FD = \text{rect } KF,\ FG \text{ (Eucl vi 16)}$$

But it has been shown that

$$\text{sq } FH = \text{rect } EF,\ FD,$$

and therefore

$$\text{rect } KF,\ FG = \text{sq } FH$$

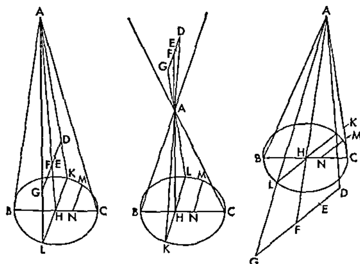
Likewise then all the perpendiculars drawn from the line $GHK$ to the straight line $GK$ could also be shown to be equal in square to the rectangle, in each case contained by the segments of the straight line $GK$

Therefore the section is a circle whose diameter is the straight line $GK$

## Proposition 6

*If a cone is cut by a plane through the axis and some point is taken on the surface of the cone which is not on a side of the axial triangle and from it is drawn a straight line parallel to some straight line which is a perpendicular from the circumference of the circle to the base of the triangle then it meets the axial triangle, and on being produced to the other side of the surface it will be bisected by the triangle*

Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$, and let the cone be cut by a plane through the axis and let it make a common section the triangle $ABC$ (I 3), and from some point $M$ of those on the




circumference let the straight line $MN$ be drawn perpendicular to the straight line $BC$ Then let some point $D$ be taken on the surface of the cone and through $D$ let the straight line $DE$ be drawn parallel to $MN$

I say that the straight line $DE$ produced will meet the plane of the triangle $ABC$ and, if further produced toward the other side of the cone until it meet its surface will be bisected by the triangle $ABC$

Let the straight line $AD$ be joined and be produced Therefore it will meet the circumference of the circle $BC$ (I 1) Let it meet it at $K$ and from $K$ let the straight line $KHL$ be drawn perpendicular to the straight line $BC$ Therefore $KH$ is parallel to $MN$, and therefore to $DE$ (Eucl XI 9)

Let the straight line $AH$ be joined from $A$ to $H$ Since then in the triangle $AHK$ the straight line $DE$ is parallel to the straight line $HK$, therefore $DE$ produced will meet $AH$ But $AH$ is in the plane of $ABC$ therefore $DE$ will meet the plane of the triangle $ABC$

For the same reasons it also meets $AH$ let it meet it at $F$ and let $DF$ be produced in a straight line until it meet the surface of the cone Let it meet it at $G$

I say that $DF$ is equal to $FG$

For since $A, G$ $L$ are points on the surface of the cone but also in the plane extended through the straight lines $AH$ $AK$ $DG$ $KL$ which is a triangle through the vertex of the cone (I 3) therefore $A$ $G$ $L$ are points on the common section of the cone s surface and of the triangle Therefore the line through $A, G$ $L$ is a straight line Since then in the triangle $ALK$ the straight line $DG$ has been drawn parallel to the base $KHL$ and some straight line $AFH$ has been drawn across them from the point $A$ therefore

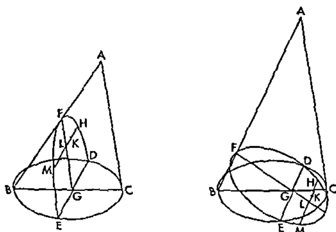
$$KH \quad HL \quad DF \quad FG \text{ (Eucl VI 2)}$$

But $KH$ is equal to $HL$ since $KL$ is a chord in circle $BC$ perpendicular to the diameter (Eucl III 3) Therefore $DF$ is equal to $FG$

PROPOSITION 7

*If a cone is cut by a plane through the axis and if it is also cut by another plane cutting the plane the base of the cone is in, in a straight line perpendicular either to the base of the axial triangle or to it produced, then the straight lines drawn from the resulting section on the cone's surface, made by the cutting plane, parallel to the straight line perpendicular to the base of the triangle will fall on the common section of the cutting plane and of the axial triangle and further produced to the other side of the section, are bisected by the common section, and if it is a right cone the straight line in the base will be perpendicular to the common section of the cutting plane and of the axial triangle and if oblique, it will not always be perpendicular, but whenever the plane through the axis is perpendicular to the base of the cone*

Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$ and let it be cut by a plane through the axis and let it make as a section

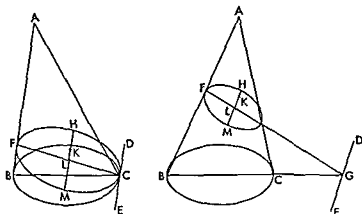



the triangle $ABC$ (I 3) And let it also be cut by another plane cutting the plane the circle $BC$ is in in the straight line $DE$ perpendicular either to the straight line $BC$ or to it produced and let it make as a section on the surface of the cone the line $DFE$ Then the straight line $FG$ is the common section of the cutting plane and of the triangle $ABC$ And let any point $H$ be taken on the section $DFE$ and let the straight line $HK$ be drawn through $H$ parallel to the straight line $DE$

I say that the straight line $HK$ meets the straight line $FG$, and on being produced to the other side of the section $DFE$, will be bisected by $FG$

For since a cone whose vertex is the point $A$ and whose base is the circle $BC$ has been cut by a plane through its axis, and makes as a section the triangle $ABC$ and some point $H$ on the surface, not on a side of the triangle $ABC$ has been taken and since the straight line $DG$ is perpendicular to the straight line $BC$ therefore the straight line drawn through $H$ parallel to $DG$ that is $HK$, meets the triangle $ABC$ and if further produced to the other side of the surface, will be bisected by the triangle (I 6)

Then since the straight line drawn through $H$ parallel to the straight line $DE$ meets the triangle $ABC$ and is in the plane of the section $DFE$, therefore




it will fall on the common section of the cutting plane and of the triangle $ABC$ But the straight line $FG$ is the common section of the planes Therefore the straight line drawn through $H$ parallel to $DE$ will fall on $FG$, and, if further produced to the other side of the section $DFE$, will be bisected by the straight line $FG$

Then either the cone is a right cone or the axial triangle $ABC$ is perpendicular to the circle $BC$, or neither

First let the cone be a right cone Then the triangle $ABC$ would be perpendicular to the circle $BC$ (Def 3 Eucl XI 18) Since then the plane $ABC$ is perpendicular to the plane $BC$ and the straight line $DE$ has been drawn in one of the planes $BC$ perpendicular to their common section the straight line $BC$ therefore the straight line $DE$ is perpendicular to the triangle $ABC$ (Eucl XI def 4) and therefore to all the straight lines touching it and in the triangle $ABC$ (Eucl XI def 3) And so it is also perpendicular to the straight line $FG$

Then let the cone not be a right cone If now the axial triangle is perpendicular to the circle $BC$, we could likewise show that $DE$ is perpendicular to $FG$

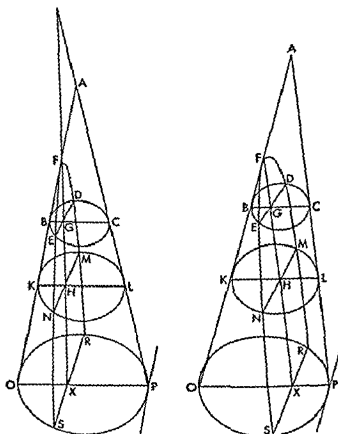
Then let the axial triangle $ABC$ not be perpendicular to the circle $BC$ —I say that $DE$ is not perpendicular to $FG$ For if possible let it be And it is also perpendicular to the straight line $BC$ Therefore $DE$ is perpendicular to both $BC$ and $FG$, and therefore it will be perpendicular to the plane through $BC$ and $FG$ But the plane through $BC$ and $GF$ is the triangle $ABC$ and therefore $DE$ is perpendicular to the triangle $ABC$ And therefore all the planes through it are perpendicular to the triangle $ABC$ But one of the planes through $DE$ is the circle $BC$ therefore the circle $BC$ is perpendicular to the triangle $ABC$ And so the triangle $ABC$ will also be perpendicular to the circle $BC$ And this is not supposed Therefore the straight line $DE$ is not perpendicular to the straight line $FG$

PORISM

Then from this it is evident that the straight line $FG$ is the diameter of the section $DFE$, since it bisects the straight lines drawn parallel to some straight line $DE$ and that it is possible for some parallels to be bisected by the diameter $FG$ and not be perpendicular

PROPOSITION 8

*If a cone is cut by a plane through its axis and is cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the resulting section on the surface is either parallel to one of the sides of the triangle or meets one of them beyond the vertex of the cone and the surface of the cone and the cutting plane are produced indefinitely then the section will also increase indefinitely, and some straight line drawn from the section of the cone parallel to the straight line in the base of the cone will cut off from the diameter on the side of the vertex a straight line equal to any given straight line*




Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$ and let it be cut by a plane through its axis and let it make as a section the triangle $ABC$ (I 3) And let it be cut also by another plane cutting the

circle $BC$ in a straight line $DE$ perpendicular to the straight line $BC$, and let it make as a section on the surface the line $DFE$ And let the diameter $FG$ of the section $DFE$ be either parallel to the straight line $AC$ or on being produced meet it beyond the point $A$ (I 7 and porism)

I say that, if both the surface of the cone and the cutting plane are produced indefinitely, the section $DFE$ also will increase indefinitely

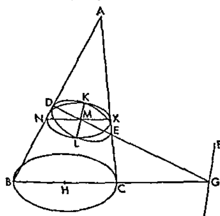
For let both the surface of the cone and the cutting plane be produced Then it is evident that also the straight lines $AB$ $AC$, $FG$ will be therewith produced Since the straight line $FG$ is either parallel to $AC$ or produced meets it beyond the point $A$ therefore the straight lines $FG$ and $AC$ on being produced in the direction of $C$ and $G$ will never meet Then let them be produced and let some point $H$ be taken at random on the straight line $FG$, and let the straight line $KHL$ be drawn through the point $H$ parallel to the straight line $BC$ and $MHN$ parallel to $DE$ Therefore the plane through $KL$ and $MN$ is parallel to the plane through $BC$ and $DE$ (Eucl XI 15) Therefore the plane $KLMN$ is a circle (I 4)

And since the points $D$ $E$, $M$, $N$ are in the cutting plane and also on the surface of the cone therefore they are on the common section Therefore the section $DFE$ has increased to the points $M$ and $N$ Therefore, with the surface of the cone and the cutting plane increased to the circle $KLMN$ the section $DFE$ has also increased to the points $M$ and $N$ Then likewise we could show also that if the surface of the cone and the cutting plane are extended indefinitely the section $MDFEN$ will also increase indefinitely

And it is evident that some straight line will cut off on straight line $FH$ on the side of point $F$ a straight line equal to any given straight line For if we lay down the straight line $FX$ equal to the given straight line and draw a parallel to $DE$ through $X$, it will meet the section just as the straight line through $H$ was also proved to meet the section in the points $M$ and $N$ And so some straight line is drawn meeting the section parallel to $DE$, and cutting off on $FG$ on the side of point $H$ a straight line equal to the given straight line

PROPOSITION 9

*If a cone is cut by a plane meeting both sides of the axial triangle and neither parallel to the base nor situated subcontrariwise then the section will not be a circle*




Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$ and let it be cut by some plane neither parallel to the base nor situated subcontrariwise, and let it make as a section on the surface the line $DKE$

I say that the line $DKE$ will not be a circle

For if possible let it be and let the cutting plane meet the base and let the straight line $FG$ be the common section of the planes, and let the point $H$ be the center of the circle $BC$ and let the straight line $HG$ be

drawn from it perpendicular to the straight line $FG$ And let a plane be extended through $GH$ and the axis and let it make as sections on the conic surface the straight lines $BA$ and $AC$ (I 1) Since then $D$, $E$, $G$ are points in the plane through the line $DKE$ and also in the plane through the points $A$, $B$ $C$, therefore $D$, $E$, $G$ are points on the common section of the planes Therefore $GED$ is a straight line (Eucl XI 3)

Then let some point $K$ be taken on the line $DKE$, and through $K$ let the straight line $KL$ be drawn parallel to the straight line $FG$, then $KM$ will be equal to $ML$ (I 7) Therefore the straight line $DE$ is the diameter of the circle $DKLE$ (Def 4) Then let the straight line $NMX$ be drawn through $M$ parallel to the straight line $BC$ But $KL$ is also parallel to $FG$ And so the plane through the straight lines $NX$ and $KM$ is parallel to the plane through the straight lines $BC$ and $FG$, that is to the base (Eucl XI 15), and the section will be a circle (I 4) Let it be the circle $NKX$

And since the straight line $FG$ is perpendicular to the straight line $BG$, the straight line $KM$ is also perpendicular to the straight line $NX$ (Eucl XI 10) And so

$$\text{rect } NM\ MX = \text{sq } KM \text{ (Eucl III 31 VI 8, porism)}$$

But

$$\text{rect } DM\ ME = \text{sq } KM,$$

for the line $DKEL$ is supposed a circle, and the straight line $DE$ is its diameter Therefore

$$\text{rect } NM\ MX = \text{rect } DM, ME$$

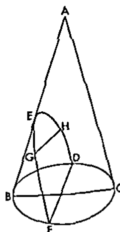
Therefore

$$MN\ MD \quad EM\ MX \text{ (Eucl VI 16)}$$

Therefore triangle $DMN$ is similar to triangle $XME$ (Eucl VI 6, VI def 1) and angle $DNM$ is equal to angle $MEX$ But angle $DNM$ is equal to angle $ABC$, for the straight line $NX$ is parallel to the straight line $BC$ And therefore angle $ABC$ is equal to angle $MEX$ Therefore the section is subcontrary (I 5) And this is not supposed Therefore the line $DKE$ is not a circle

## Proposition 10

*If two points are taken on the section of a cone the straight line joining the two points will fall within the section, and produced in a straight line it will fall outside*




Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$, and let it be cut by a plane through the axis and let it make as a section the triangle $ABC$ (I 3) Then let it also be cut by another plane, and let it make as a section on the surface of the cone the line $DEF$, and let two points $G$ and $H$ be taken on the line $DEF$

I say that the straight line joining the two points $G$ and $H$ will fall within the line $DEF$, and produced in a straight line it will fall outside

For since a cone, whose vertex is the point $A$ and whose base is the circle $BC$, has been cut by a plane through the axis and some points $G$ and $H$ have been taken on its surface which are not on a side of the axial

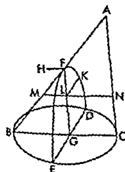
triangle and since the straight line joining $G$ and $H$ does not verge to the point $A$ therefore the straight line joining $G$ and $H$ will fall within the cone and produced in a straight line it will fall outside (I 2), consequently also outside the section $DFE$

Proposition 11

*If a cone is cut by a plane through its axis and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one side of the axial triangle then any straight line which is drawn from the section of the cone to its diameter parallel to the common section of the cutting plane and of the cone s base will equal in square the rectangle contained by the straight line cut off by it on the diameter beginning from the section s vertex and by another straight line which has the ratio to the straight line between the angle of the cone and the vertex of the section that the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle And let such a section be called a parabola* (παραβολή)

Let there be a cone whose vertex is the point $A$, and whose base is the circle $BC$, and let it be cut by a plane through its axis and let it make as a section the triangle $ABC$ (I 3) And let it also be cut by another plane cutting the base of the cone in the straight line $DE$ perpendicular to the straight line $BC$ and let it make as a section on the surface of the cone the line $DFE$, and let the diameter of the section $FG$ (I 7 and def 4) be parallel to one side $AC$ of the axial triangle And let the straight line $FH$ be drawn from the point $F$ perpendicular to the straight line $FG$ and let it be contrived that

$$\text{sq } BC \quad \text{rect } BA\ AC \quad FH \quad FA$$




And let some point $K$ be taken at random on the section and through $K$ let the straight line $KL$ be drawn parallel to the straight line $DE$

I say that sq $KL$ = rect $HF$ $FL$

For let the straight line $MN$ be drawn through $L$ parallel to the straight line $BC$ And the straight line $DE$ is also parallel to the straight line $KL$ Therefore the plane through $KL$ and $MN$ is parallel to the plane through $BC$ and $DE$ (Eucl XI 15) that is to the base of the cone Therefore the plane through $KL$ and $MN$ is a circle whose diameter is $MN$ (I 4) And $KL$ is perpendicular to $MN$ since $DE$ is also perpendicular to $BC$ (Eucl XI 10) Therefore

$$\text{rect } ML\ LN = \text{sq } KL \text{ (Eucl III 31 VI 8 porism)}$$

And since

$$\text{sq } BC \quad \text{rect } BA\ AC \quad HF \quad FA$$

and

$$\text{sq } BC \quad \text{rect } BA\ AC \text{ comp } BC \quad CA \quad BC \quad BA \text{ (Eucl VI 23)},$$

therefore

$$HF \quad FA \text{ comp } BC \quad CA \quad BC \quad BA$$

But

$$BC \quad CA \quad MN \quad NA \quad ML \quad LF \text{ (Eucl VI 4)},$$

and

$$BC \quad BA \quad MN \quad MA \quad LM \quad MF \quad NL \quad FA \text{ (Eucl VI 2)}$$

Therefore

$$H\Gamma \quad FA \text{ comp } ML \quad LF, NL \quad FA$$

But

rect $ML, LN$ rect $LF, \Gamma A$ comp $ML$ $L\Gamma, LN$ $FA$ (Eucl VI 23)

Therefore

$$HF \quad \Gamma A \quad \text{rect } ML, LN \quad \text{rect } LF, \Gamma A$$

But with the straight line $FL$ taken as common height

$$HF \quad FA \quad \text{rect } HF, \Gamma L \quad \text{rect } L\Gamma, FA \text{ (Eucl VI 1)},$$

therefore

rect $ML, LN$ rect $LF, FA$ rect $HF, FL$ rect $L\Gamma, \Gamma A$ (Eucl V 11)

Therefore

$$\text{rect } ML, LN = \text{rect } HF, FL \text{ (Eucl V 9)}$$

But

$$\text{rect } ML \; LN = \text{sq } KL$$

therefore also

$$\text{sq } KL = \text{rect } HF \; FL$$

And let such a section be called a parabola, and let $HF$ be called the straight line to which the straight lines drawn ordinatewise to the diameter $FG$ are applied in square (παρ ἣν δυναται αἱ καταγόμεναι τεταγμένως ἐπὶ την ZH διάμετρον), and let it also be called the upright side (ὀρθια) [1]

## Proposition 12

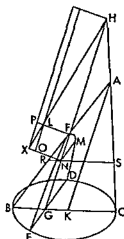
*If a cone is cut by a plane through its axis and also by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the section produced meets one side of the axial triangle beyond the vertex of the cone then any straight line which is drawn from the section to its diameter parallel to the common section of the cutting plane and of the cone s base will equal in square some area applied to a straight line to which the straight line added along the diameter of the section and subtending the exterior angle of the triangle has the ratio that the square on the straight line drawn from the cone's vertex to the triangle s base parallel to the section s diameter has to the rectangle contained by the sections of the base which this straight line makes when drawn, this area having as breadth the straight line cut off on the diameter beginning from the section s vertex by this straight line from the section to the diameter and exceeding* (ὑπερβάλλον) *by a figure* (εἶδος) *similar and similarly situated to the rectangle contained by the straight line subtending the exterior angle of the triangle and by the parameter And let such a section be called an hyperbola* (ὑπερβολη)

Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$, and let it be cut by a plane through its axis, and let it make as a section the triangle $ABC$ (I 3) And let it also be cut by another plane cutting the base of the cone in the straight line $DE$ perpendicular to $BC$ the base of the triangle $ABC$, and let it make as a section on the surface of the cone the line $DFE$, and

[1] The Greek of the phrase the straight line to which the straight lines drawn ordinatewise to the diameter are applied in square that is ἡ παῤ ἣ δ αναι αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν δ άμετρο soon becomes abbreviated to ἡ παῤ ἣν δύ αντα αἱ καταγόμεναι and to ἡ παῤ ἣν δύ αντα We shall translate these abbreviations by the word parameter And we shall later on after proposition XIV shorten the long expression to the parameter of the ordinates to the diameter

The Latin translation of ὀρθία (πλευρά) is *latus rectum* which has become an English term too

let $FG$ the diameter of the section (I 7 and def 4) when produced meet $AC$ one side of the triangle $ABC$ beyond the vertex of the cone at the point $H$ And let the straight line $AK$ be drawn through $A$ parallel to the diameter of the section $FG$, and let it cut $BC$ And let the straight line $FL$ be drawn from $F$ perpendicular to $FG$, and let it be contrived that




$$\text{sq } KA \quad \text{rect } BK\ KC \quad FH \quad FL$$

And let some point $M$ be taken at random on the section, and through $M$ let the straight line $MN$ be drawn parallel to $DE$, and through $N$ let the straight line $NOX$ be drawn parallel to $FL$ And let the straight line $HL$ be joined and produced to $X$, and let the straight lines $LO$ and $XP$ be drawn through $L$ and $X$ parallel to $FN$

I say that $MN$ is equal in square to the parallelogram $FX$ which is applied to $FL$ having $FN$ as breadth and exceeding by a figure $LX$ similar to the rectangle contained by $HF$ and $FL$

For let the straight line $RNS$ be drawn through $N$ parallel to $BC$ and $NM$ is also parallel to $DE$ Therefore the plane through $MN$ and $RS$ is parallel to the plane through $BC$ and $DE$ that is to the base of the cone (Eucl XI 15) Therefore if the plane is produced through $MN$ and $RS$, the section will be a circle whose diameter is the straight line $RNS$ (I 4) And $MN$ is perpendicular to it Therefore

$$\text{rect } RN, NS = \text{sq } MN$$

And since

$$\text{sq } AK \quad \text{rect } BK\ KC \quad FH \quad FL,$$

and

$$\text{sq } AK \quad \text{rect } BK\ KC \text{ comp } AK\ KC\ AK\ KB \text{ (Eucl VI 23)},$$

therefore also

$$FH\ \ FL \text{ comp } AK\ \ KC\ AK\ \ KB$$

But

$$AK\ \ KC \quad HG\ \ GC \quad HN\ \ NS \text{ (Eucl VI 4)},$$

and

$$AK\ \ KB \quad FG\ \ GB \quad FN\ \ NR$$

Therefore

$$HF\ \ FL \text{ comp } HN\ \ NS\ FN\ \ NR$$

And

$$\text{rect } HN\ NF \quad \text{rect } SN\ NR \text{ comp } HN\ \ NS\ FN\ \ NR \text{ (Eucl VI 23)}$$

Therefore also

$$\text{rect } HN\ NF \quad \text{rect } SN\ NR \quad HF\ \ FL \quad HN\ \ NX \text{ (Eucl VI 4)}$$

But, with the straight line $FN$ taken as common height

$$HN\ \ NX \quad \text{rect } HN\ NF \quad \text{rect } FN\ NX \text{ (Eucl VI 1)}$$

Therefore also

$$\text{rect } HN\ NF \quad \text{rect } SN\ NR \quad \text{rect } HN\ NF \quad \text{rect } XN\ NF \text{ (Eucl V 11)}$$

Therefore

$$\text{rect } SN\ NR = \text{rect } XN\ NF \text{ (Eucl V 9)}$$

But it was shown

$$\text{sq } MN = \text{rect } SN\ NR,$$

therefore also

sq $MN$ = rect $XN$ $NF$

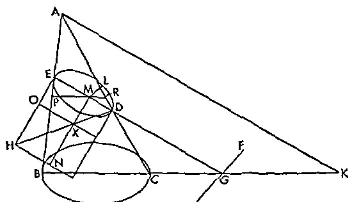
But the rectangle contained by $XN$ and $NF$ is the parallelogram $XF$ Therefore the straight line $MN$ is equal in square to $XF$ which is applied to the straight line $FL$, having $FN$ as breadth and exceeding by the parallelogram $LX$ similar to the rectangle contained by $HF$ and $FL$ (Eucl vi 24)

And let such a section be called an hyperbola, and let $LF$ be called the straight line to which the straight lines drawn ordinatewise to $FG$ are applied in square, and let the same straight line also be called the upright side, and the straight line $FH$ the transverse side

PROPOSITION 13

*If a cone is cut by a plane through its axis, and is also cut by another plane on the one hand meeting both sides of the axial triangle and on the other extended neither parallel to the base nor subcontrariwise, and if the plane the base of the cone is in and the cutting plane meet in a straight line perpendicular either to the base of the axial triangle or to it produced then any straight line which is drawn from the section of the cone to the diameter of the section parallel to the common section of the planes will equal in square some area applied to a straight line to which the diameter of the section has the ratio that the square on the straight line drawn from the cone's vertex to the triangle's base parallel to the section's diameter has to the rectangle contained by the intercepts of this straight line (on the base) from the sides of the triangle an area having as breadth the straight line cut off on the diameter beginning from the section's vertex by this straight line from the section to the diameter and deficient* (ἐλλεῖπον) *by a figure similar and similarly situated to the rectangle contained by the diameter and parameter And let such a section be called an ellipse* (ἔλλειψις)

Let there be a cone whose vertex is the point $A$ and whose base is the circle $BC$ and let it be cut by a plane through its axis, and let it make as a section




the triangle $ABC$ And let it also be cut by another plane on the one hand meeting both sides of the axial triangle and on the other extended neither parallel to the base of the cone nor subcontrariwise, and let it make as a section on the surface of the cone the line $DE$ And let the common section of the cut

ting plane and of the plane the base of the cone is in, be the straight line $FG$ perpendicular to the straight line $BC$, and let the diameter of the section be the straight line $ED$ (I 7 and Def 4) And let the straight line $EH$ be drawn from $E$ perpendicular to $ED$ and let the straight line $AK$ be drawn through $A$ parallel to $ED$ and let it be contrived that

$$\text{sq } AK \quad \text{rect } BK\ KC \quad DE \quad EH$$

And let some point $L$ be taken on the section and let the straight line $LM$ be drawn through $L$ parallel to $FG$

I say that the straight line $LM$ is equal in square to some area which is applied to $EH$, having $EM$ as breadth and deficient by a figure similar to the rectangle contained by $DE$ and $EH$

For let the straight line $DH$ be joined and on the one hand let the straight line $MXN$ be drawn through $M$ parallel to $HE$ and on the other let the straight lines $HN$ and $XO$ be drawn through $H$ and $X$ parallel to $EM$ and let the straight line $PMR$ be drawn through $M$ parallel to $BC$

Since then $PR$ is parallel to $BC$ and $LM$ is also parallel to $FG$ therefore the plane through $LM$ and $PR$ is parallel to the plane through $FG$ and $BC$ that is to the base of the cone (Eucl XI 15) If therefore a plane is extended through $LM$ and $PR$ the section will be a circle whose diameter is $PR$ (I 4) And $LM$ is perpendicular to it Therefore

$$\text{rect } PM\ MR = \text{sq } LM$$

And since

$$\text{sq } AK \quad \text{rect } BK\ KC \quad ED \quad EH$$

and

$$\text{sq } AK \quad \text{rect } BK\ KC \text{ comp } AK \quad KB\ AK \quad KC \text{ (Eucl VI 23)},$$

but

$$AK \quad KB \quad EG \quad GB \quad EM \quad MP \text{ (Eucl VI 4)}$$

and

$$AK \quad KC \quad DG \quad GC \quad DM \quad MR$$

therefore

$$DE \quad EH \text{ comp } EM \quad MP\ DM \quad MR$$

But

rect $EM\ MD$ rect $PM\ MR$ comp $EM$ $MP$ $DM$ $MR$ (Eucl VI 23)

Therefore

rect $EM\ MD$ rect $PM\ MR$ $DE$ $EH$ $DM$ $MX$ (Eucl VI 4)

And with the straight line $ME$ taken as common height

$DM$ $MX$ rect $DM\ ME$ rect $XM\ ME$ (Eucl VI 1)

Therefore also

rect $DM, ME$ rect $PM\ MR$ rect $DM\ ME$ rect $XM\ ME$ (Eucl V 11)

Therefore

$$\text{rect } PM\ MR = \text{rect } XM\ ME \text{ (Eucl V 9)}$$

But it was shown

$$\text{rect } PM\ MR = \text{sq } LM$$

therefore also

$$\text{rect } XM\ ME = \text{sq } LM$$

Therefore the straight line $LM$ is equal in square to the parallelogram $MO$ which is applied to the straight line $HE$ having $EM$ as breadth and deficient by the figure $ON$ similar to the rectangle contained by $DE$ and $EH$ (Eucl VI 24)

And let such a section be called an ellipse, and let $EH$ be called the straight line to which the straight lines drawn ordinatewise to $DE$ are applied in square, and let the same straight line also be called the upright side, and the straight line $ED$ the transverse side

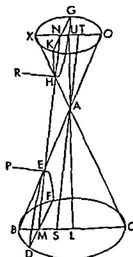
PROPOSITION 14

*If the vertically opposite surfaces are cut by a plane not through the vertex the section on each of the two surfaces will be that which is called the hyperbola and the diameter of the two sections will be the same straight line and the straight lines to which the straight lines drawn to the diameter parallel to the straight line in the cone s base are applied in square, are equal and the transverse side of the figure, that between the vertices of the sections, is common And let such sections be called opposite* (ἀντικείμεναι)

Let there be the vertically opposite surfaces whose vertex is the point $A$ and let them be cut by a plane not through the vertex and let it make as sections on the surface the lines $DEF$ and $GHK$

I say that each of the two sections $DEF$ and $GHK$ is the so-called hyperbola

For let there be the circle $BDCF$ along which the line generating the surface moves and let the plane $XGOK$ be extended parallel to it on the vertically opposite surface and the straight lines $FD$ and $GK$ are common sections of the sections $GHK$ and $FED$, and of the circles (I 4) Then they will be parallel (Eucl XI 16) And let the straight line $LAU$ be the axis of the conic surface and the points $L$ and $U$ be the centers of the circles and let a straight line drawn from $L$ perpendicular to the straight line $FD$ be produced to the points $B$ and $C$ and let a plane be produced through the straight line $BC$ and the axis Then it will make as sections in the circles the parallel straight lines $XO$ and $BC$ (Eucl XI 16) and on the surface the straight lines $BAO$ and $CAX$ (I 1 and Def 4)




Then the straight line $XO$ will be perpendicular to the straight line $GK$, since the straight line $BC$ is also perpendicular to the straight line $FD$ and each of the two is parallel to the other (Eucl XI 10) And since the plane through the axis meets the sections in the points $M$ and $N$ within the lines it is clear that the plane also cuts the lines Let it cut them at $H$ and $E$, therefore $M$ $E$ $H$ and $N$ are points on the plane through the axis and in the plane the lines are in therefore the line $MEHN$ is a straight line (Eucl XI 3) It is also evident both that $X$ $H$ $A$, and $C$ are in a straight line and $B, E, A$, and $O$ also For they are both on the conic surface and in the plane through the axis (I 1)

Let then the straight lines $HR$ and $EP$ be drawn from $H$ and $E$ perpendicular to $HE$ and let the straight line $SAT$ be drawn through $A$ parallel to $MEHN$ and let it be contrived that

$$HE : EP :: \text{sq } AS : \text{rect } BS, SC,$$

and

$$EH : HR :: \text{sq } AT : \text{rect } OT, TX$$

Since then a cone whose vertex is the point $A$ and whose base is the circle $BC$, has been cut by a plane through its axis and it has made as a section the triangle $ABC$ and it has also been cut by another plane cutting the base of the cone in the straight line $DMF$ perpendicular to the straight line $BC$ and it has made as a section on the surface the line $DEF$, and the diameter $ME$ produced has met one side of the axial triangle beyond the vertex of the cone and through the point $A$ the straight line $AS$ has been drawn parallel to the diameter of the section $EM$, and from $E$ the straight line $EP$ has been drawn perpendicular to the straight line $EM$, and

$$EH \quad EP \quad \text{sq } AS \quad \text{rect } BS, SC,$$

therefore the section $DEF$ is an hyperbola (I 12) and $EP$ is the straight line to which the straight lines drawn ordinatewise to $EM$ are applied in square and the straight line $HE$ is the transverse side of the figure And likewise $GHK$ is also an hyperbola whose diameter is the straight line $HN$ and whose straight line to which the straight lines drawn ordinatewise to $HN$ are applied is $HR$, and the transverse side of whose figure is $HE$

I say that the straight line $HR$ is equal to the straight line $EP$

For since $BC$ is parallel to $XO$,

$$AS \quad SC \quad AT \quad TX$$

and

$$AS \quad SB \quad AT \quad TO$$

But

$$\text{sq } AS \quad \text{rect } BS, SC \text{ comp } AS \quad SC\ AS \quad SB \text{ (Eucl VI 23)}$$

and

$$\text{sq } AT \quad \text{rect } XT, TO \text{ comp } AT \quad TX, AT \quad TO,$$

therefore

$$\text{sq } AS \quad \text{rect } BS\ SC \quad \text{sq } AT \quad \text{rect } XT, TO$$

Also

$$\text{sq } AS \quad \text{rect } BS\ SC \quad HE \quad EP,$$

and

$$\text{sq } AT \quad \text{rect } XT\ TO \quad HE \quad HR$$

Therefore also

$$HE \quad EP \quad EH \quad HR \text{ (Eucl V 11)}$$

Therefore

$$EP = HR \text{ (Eucl V 9)}$$

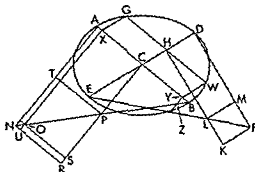
PROPOSITION 15

*If in an ellipse a straight line drawn ordinatewise from the midpoint of the diameter, is produced both ways to the section and if it is contrived that as the straight line so produced is to the diameter so is the diameter to some straight line then any straight line which is drawn, from the section to the straight line produced parallel to the diameter, will equal in square the area applied to this third proportional and having as breadth the straight line cut off by it beginning from the section and deficient by a figure similar to the rectangle contained by the straight line to which the straight lines are drawn and by the parameter and if further produced to the other side of the section will be bisected by the straight line to which it has been drawn*

Let there be an ellipse whose diameter is the straight line $AB$, and let $AB$ be bisected at the point $C$ and through $C$ let the straight line $DCE$ be drawn

ordinatewise and produced both ways to the section and from the point $D$ let the straight line $DF$ be drawn perpendicular to $DE$ And let it be contrived that

$$DE \quad AB \quad AB \quad DF$$




And let some point $G$ be taken on the section, and through $G$ let the straight line $GH$ be drawn parallel to $AB$, and let $FF$ be joined and through $H$ let the straight line $HL$ be drawn parallel to $DF$, and through $F$ and $L$ let the straight lines $FK$ and $LM$ be drawn parallel to $HD$

I say that the straight line $GH$ is equal in square to the area $DL$ which is applied to the straight line $DF$ having as breadth the straight line $DH$ and deficient by a figure $LF$ similar to the rectangle contained by $FD$ and $DF$

For let $AN$ be the parameter of the ordinates to $AB$, and let $BN$ be joined, and through $G$ let the straight line $GY$ be drawn parallel to $DE$, and through $Y$ and $C$ let the straight lines $YO$ and $CP$ be drawn parallel to $AN$, and through $N$, $O$ and $P$ let the straight lines $NU$ $OS$ and $TP$ be drawn parallel to $AB$ Therefore

$$\text{sq } DC = \text{ar } AP, \text{ sq } GY = \text{ar } AO \text{ (I 13)}$$

And since

$$BA \quad AN \quad BC \quad CP \quad PT \quad TN \text{ (Eucl vi 4)},$$

and

$$BC = CA = TP,$$

and

$$CP = TA,$$

therefore

$$\text{ar } AP = \text{ar } TR,$$

and

$$\text{ar } YT = \text{ar } TU$$

Since also

$$\text{ar } OT = \text{ar } OR \text{ (Eucl i 43)},$$

and area $NO$ is common therefore

$$\text{ar } TU = \text{ar } NS$$

But

$$\text{ar } TU = \text{ar } TY$$

and $TS$ is common Therefore

$$\text{ar } NP = \text{ar } PA = \text{ar } AO + \text{ar } PO$$

and so

$$\text{ar } PA - \text{ar } AO = \text{ar } PO$$

Also

$$\text{ar } AP = \text{sq } CD, \text{ ar } AO = \text{sq } YG,$$

and

$$\text{ar } OP = \text{rect } OS, SP$$

therefore

$$\text{sq } CD - \text{sq } GY = \text{rect } OS \; SP$$

Since also the straight line $DE$ has been cut into equal parts at $C$, and into unequal parts at $H$ therefore

$$\text{rect } EH, HD + \text{sq } CH = \text{sq } CD \text{ (Eucl ii 5)},$$

or

$$\text{rect } EH, HD + \text{sq } XG = \text{sq } CD$$

Therefore

$$\text{sq } CD - \text{sq } XG = \text{rect } EH, HD,$$

but

$$\text{sq } CD - \text{sq } XG = \text{rect } OS, SP,$$

therefore

$$\text{rect } EH\ HD = \text{rect } OS, SP$$

And since

$$DE \quad AB \qquad AB \quad DF,$$

therefore

$$DE \quad DF \qquad \text{sq } DE \quad \text{sq } AB \text{ (Eucl vi 20)},$$

that is

$$DE \quad DF \qquad \text{sq } CD \quad \text{sq } CB \text{ (Eucl v 15)},$$

And

$$\text{rect } PC, CA = \text{rect } PC\ CB = \text{sq } CD \text{ (i 13)},$$

and since

$$DE \quad DF \qquad EH \quad HL \text{ (Eucl vi 4)},$$

or

$$DE \quad DF \qquad \text{rect } EH\ HD \quad \text{rect } DH\ HL \text{ (Eucl vi 1)},$$

and since

$$DE \quad DF \qquad \text{rect } PC\ CB \quad \text{sq } CB,$$

and

$$\text{rect } PC, CB \quad \text{sq } CB \qquad \text{rect } OS\ SP \quad \text{sq } OS\,^{[1]}$$

therefore also

$$\text{rect } EH, HD \quad \text{rect } DH\ HL \qquad \text{rect } OS\ SP \quad \text{sq } OS$$

And

$$\text{rect } EH, HD = \text{rect } OS\ SP$$

therefore

$$\text{rect } DH\ HL = \text{sq } OS = \text{sq } GH$$

Therefore the straight line $GH$ is equal in square to the area $DL$ which is applied to the straight line $DF$ deficient by a figure $FL$ similar to the rectangle contained by $ED$ and $DF$ (Eucl vi 24)

I say then that also if produced to the other side of the section the straight line $GH$ will be bisected by the straight line $DE$

For let it be produced and let it meet the section at $W$, and let the straight line $WY$ be drawn through $Y$ parallel to $GX$, and through $Y$ let the straight line $YZ$ be drawn parallel to $AN$ And since

$$GX = WY,$$

therefore also

$$\text{sq } GX = \text{sq } WY$$

[1]This follows from the proportions

$$PC \quad CB \qquad PS \quad OS \text{ (Eucl vi 4)}$$

and

$$PC \quad CB \qquad \text{rect } PC\ CB \quad \text{sq } CB$$

and

$$PS \quad OS \qquad \text{rect } PS\ OS \quad \text{sq } OS \text{ (Eucl. vi. 1)}$$

But

$$\text{sq } GX = \text{rect } AX, XO \text{ (I 13)},$$

and

$$\text{sq } WY = \text{rect } AY, YZ \text{ (I 13)}$$

Therefore

$$OX \quad ZY \quad YA \quad AX \text{ (Eucl VI 16)}$$

And

$$OX \quad ZY \quad XB \quad BY \text{ (Eucl VI 4)},$$

therefore also

$$YA \quad AX \quad XB \quad BY$$

And *separando*

$$YX \quad AX \quad YX \quad BY \text{ (Eucl V 17)}$$

Therefore

$$AX = YB$$

And also

$$AC = CB,$$

therefore also the remainders

$$XC = CY,$$

and so also

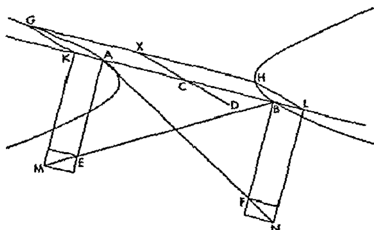
$$GH = HW$$

Therefore the straight line $HG$, produced to the other side of the section, is bisected by the straight line $DH$

PROPOSITION 16

*If through the midpoint of the transverse side of the opposite sections a straight line be drawn parallel to a straight line drawn ordinatewise it will be a diameter of the opposite sections conjugate to the diameter just mentioned*

Let there be the opposite sections whose diameter is the straight line $AB$,




and let $AB$ be bisected at $C$ and through $C$ let the straight line $CD$ be drawn parallel to a straight line drawn ordinatewise

I say that the straight line $CD$ is a diameter conjugate to $AB$

For let the straight lines $AE$ and $BF$ be the parameters and let the straight

lines $AF$ and $BL$ be joined and produced, and let some point $G$ be taken at random on either section and through $G$ let the straight line $GH$ be drawn parallel to $AB$, and from $G$ and $H$ let the straight lines $GK$ and $HL$ be drawn ordinatewise, and through $K$ and $L$ let the straight lines $KM$ and $LN$ be drawn parallel to $AE$ and $BF$ Since then

$$GK = HL \text{ (Eucl I 34)},$$

therefore also

$$\text{sq } GK = \text{sq } HL,$$

But

$$\text{sq } GK = \text{rect } AK, KM \text{ (I 12)},$$

and

$$\text{sq } HL = \text{rect } BL, LN \text{ (I 12)},$$

therefore

$$\text{rect } AK, KM = \text{rect } BL, LN$$

And since

$$AE = BF,$$

therefore

$$AE \ \ AB \quad BF \ \ BA \text{ (Eucl V 7)}$$

But

$$AE \ \ AB \quad MK \ \ KB \text{ (Eucl VI 4)},$$

and as

$$BF \ \ BA \quad NL \ \ LA \text{ (Eucl VI 4)}$$

And therefore

$$MK \ \ KB \quad NL \ \ LA$$

But, with $KA$ taken as common height,

$$MK \ \ KB \quad \text{rect } MK\ KA \quad \text{rect } BK, KA,$$

and, with $BL$ taken as common height

$$NL \ \ LA \quad \text{rect } NL\ LB \quad \text{rect } AL\ LB$$

And therefore

$$\text{rect } MK\ KA \quad \text{rect } BK, KA \quad \text{rect } NL\ LB \quad \text{rect } AL, LB$$

And alternately

$$\text{rect } MK\ KA \quad \text{rect } NL\ LB \quad \text{rect } BK, KA \quad \text{rect } AL, LB \text{ (Eucl V 16)}$$

And

$$\text{rect } AK, KM = \text{rect } BL\ LN,$$

therefore

$$\text{rect } BK\ KA = \text{rect } AL, LB,$$

therefore

$$AK = LB^{1}$$

But also

$$AC = CB,$$

and therefore

$$KC = CL,$$

and so also

$$GX = XH$$

[1]The intermediary steps to this conclusion are as follows If

| | |
|---|---|
| | $\text{rect } BK\ KA = \text{rect } AL\ LB$ |
| then | $BK \ \ LB \quad AL \ \ KA$ |
| or | $BA+AK \ \ LB \quad BA+LB \ \ AK$ |
| and *componendo* | $BA+AK+LB \ \ LB \quad BA+LB+AK \ \ AK$ |

Therefore the straight line $GH$ has been bisected by the straight line $XCD$ and is parallel to the straight line $AB$ Therefore the straight line $XCD$ is a diameter and *conjugate to* the straight line $AB$ (Defs 4, 6)

## SECOND DEFINITIONS

9 Let the midpoint of the diameter of both the hyperbola and the ellipse be called the center of the section, and let the straight line drawn from the center to meet the section be called the radius of the section

10 And likewise let the midpoint of the transverse side of the opposite sections be called the center

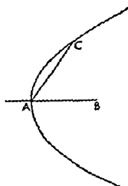
11 And let the straight line drawn from the center parallel to an ordinate, being a mean proportional to the sides of the figure (τὸ εἶδος) and bisected by the center, be called the second diameter

## PROPOSITION 17

*If in a section of a cone a straight line is drawn from the vertex of the line, and parallel to an ordinate, it will fall outside the section* (Cf Eucl III 16)

Let there be a section of a cone whose diameter is the straight line $AB$

I say that the straight line drawn from the vertex that is from the point $A$ parallel to an ordinate will fall outside the section

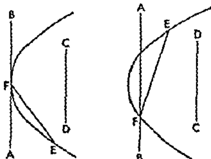



For if possible let it fall within as $AC$ Since then a point $C$ has been taken at random on a section of a cone therefore the straight line drawn from the point $C$ within the section parallel to an ordinate will meet the diameter $AB$ and will be bisected by it (I 7) Therefore the straight line $AC$ produced will be bisected by the straight line $AB$ And this is absurd For the straight line $AC$ if produced will fall outside the section (I 10) Therefore the straight line drawn from the point $A$ parallel to an ordinate will not fall within the line therefore it will fall outside and therefore it is tangent to the section

## PROPOSITION 18

*If a straight line meeting a section of a cone and produced both ways falls outside the section and some point is taken within the section and through it a parallel to the straight line meeting the section is drawn the parallel so drawn if produced both ways will meet the section*

Let there be a section of a cone and the straight line $AFB$ meeting it and let it fall when produced both ways outside the section And let some point $C$ be taken within the section and through $C$ let the straight line $CD$ be drawn parallel to the straight line $AB$




I say that the straight line $CD$ produced both ways will meet the section

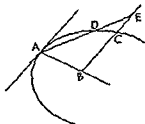
For let some point $E$ be taken on the

ection and let the straight line $EF$ be joined And since the straight line $AB$ is parallel to $CD$ and some straight line $EF$ meets $AB$, therefore $CD$ produced will also meet $EF$ And if it meets $EF$ between the points $E$ and $F$, it is evident that it also meets the section, but if beyond the point $E$, that it will first meet the section Therefore $CD$ produced to the side of points $D$ and $E$ meets the section Then likewise we could show that, produced to the side of points $F$ and $B$ it also meets it Therefore the straight line $CD$ produced both ways will meet the section

## Proposition 19

*In every section of a cone, any straight line drawn from the diameter parallel to an ordinate, will meet the section*

Let there be a section of a cone whose diameter is the straight line $AB$, and let some point $B$ be taken on the diameter, and through $B$ let the straight line $BC$ be drawn parallel to an ordinate



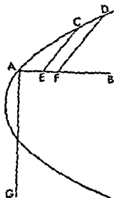

I say that the straight line $BC$ produced will meet the section

For let some point $D$ be taken on the section But $A$ is also on the section, therefore the straight line joined from $A$ to $D$ will fall within the section (I 10) And since the straight line drawn from $A$ parallel to an ordinate falls outside the section (I 17), and the straight line $AD$ meets it and the straight line $BC$ is parallel to the ordinate therefore $BC$ will also meet $AD$ And if it meets $AD$ between the points $A$ and $D$ it is evident that it will also meet the section, but if beyond point $D$ as at $E$ that it will first meet the section Therefore the straight line drawn from $B$ parallel to an ordinate will meet the section

## Proposition 20

*If in a parabola two straight lines are dropped ordinatewise to the diameter, the squares on them will be to each other as the straight lines cut off by them on the diameter beginning from the vertex*[1] *are to each other*

Let there be a parabola whose diameter is the straight line $AB$ and let some points $C$ and $D$ be taken on it and from the points $C$ and $D$ let the straight lines $CE$ and $DF$ be dropped ordinatewise to $AB$




I say that

sq $DF$ sq $CE$ $FA$ $AE$

For let $AG$ be the parameter therefore

sq $DF$ = rect $FA, AG$,

and

sq $CE$ = rect $EA$ $AG$ (I 11)

Therefore

sq $DF$ sq $CE$ rect $FA$ $AG$ rect $EA$ $AG$

But

rect $FA$ $AG$ rect $EA$ $AG$ $FA$ $AE$ (Eucl VI 1),

[1] These are usually called abscissas from the Latin *abscindere*, to cut off

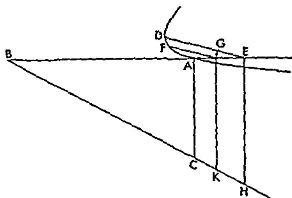
and therefore

sq $DF$ sq $CE$ $FA$ $AE$

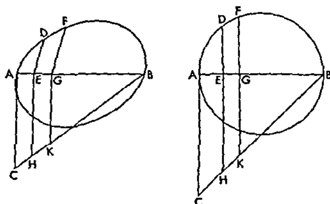
Proposition 21

*If in an hyperbola or ellipse or in the circumference of a circle straight lines are dropped ordinatewise to the diameter the squares on them will be to the areas contained by the straight lines cut off by them beginning from the ends of the transverse side of the figure, as the upright side of the figure is to the transverse and to each other as the areas contained by the straight lines cut off (abscissas), as we have said*

Let there be an hyperbola or ellipse or circumference of a circle whose diam




eter is $AB$ and whose parameter is the straight line $AC$ and let the straight lines $DE$ and $FG$ be dropped ordinatewise to the diameter




I say that

sq $FG$ rect $AG$ $GB$ $AC$ $AB$

and

sq $FG$ sq $DE$ rect $AG$ $GB$ rect $AE$ $EB$

For let the straight line $BC$ determining the figure be joined and through $E$ and $G$ let the straight lines $EH$ and $GK$ be drawn parallel to the straight line $AC$ Therefore

$$\text{sq } FG = \text{rect } KG\ GA$$
$$\text{sq } DE = \text{rect } HE\ EA \ (\text{I } 12, 13)$$

And since

$$KG \quad GB \qquad CA \quad AB,$$

and with $AG$ taken as common height

$$KG \quad GB \qquad \text{rect } KG\ GA \quad \text{rect } BG, GA,$$

therefore

$$CA \quad AB \qquad \text{rect } KG, GA \quad \text{rect } BG\ GA$$

or

$$CA \quad AB \qquad \text{sq } FG \quad \text{rect } BG, GA$$

Then also for the same reasons

$$CA \quad AB \qquad \text{sq } DE \quad \text{rect } BE, EA$$

And therefore

$$\text{sq } FG \quad \text{rect } BG, GA \qquad \text{sq } DE \quad \text{rect } BE\ EA$$

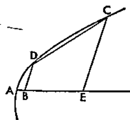
alternately

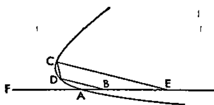
$$\text{sq } FG \quad \text{sq } DE \qquad \text{rect } BG\ GA \quad \text{rect } BE, EA\ [1]$$

## Proposition 22

*If a straight line cuts a parabola or hyperbola in two points not meeting the diameter inside it will, if produced meet the diameter of the section outside the section*

Let there be a parabola or hyperbola whose diameter is the straight line $AB$, and let some straight line cut the section in two points $C$ and $D$




I say that the straight line $DC$, if produced, will meet the straight line $AB$ outside the section

For let the straight lines $CE$ and $DB$ be dropped ordinatewise from $C$ and $D$, and first let the section be a parabola Since then in the parabola

$$\text{sq } CE \quad \text{sq } DB \qquad EA \quad AB \ (\text{I } 20)$$

and

$$EA > AB,$$

[1]Eutocius commenting says It is to be noted that the parameter that is the upright side in the case of the circle is equal to the diameter For if

$$\text{sq } DE \quad \text{rect } AE\ EB \qquad CA \quad AB$$

and only in the case of the circle

$$\text{sq } DE = \text{rect. } AE\ EB$$

therefore also

$$CA = AB$$

And this must also be noted that the ordinates on the circumference of the circle are in every case perpendicular to the diameter and are in a straight line with the parallels to $AC$ (Eucl. III 3 4)

therefore also

$$\text{sq } CE > \text{sq } DB \text{ (Eucl v 14)}$$

And so also

$$CE > DB$$

And they are parallel, therefore $CD$ produced will meet the diameter $AB$ outside the section (I 10, Eucl I 33)

But then let it be an hyperbola Since then in the hyperbola

$$\text{sq } CE \quad \text{sq } DB \quad \text{rect } FE,EA \quad \text{rect } FB,BA \text{ (I 21)},$$

therefore also

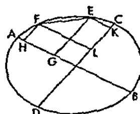
$$\text{sq } CE > \text{sq } DB$$

And they are parallel, therefore the straight line $CD$ produced will meet the diameter of the section outside the section

## Proposition 23

*If a straight line lying between the two (conjugate) diameters[1] cuts the ellipse, it will when produced, meet each of the diameters outside the section*

Let there be an ellipse whose diameters are the straight lines $AB$ and $CD$ (I 15), and let some straight line $EF$ lying between the diameters $AB$ and $CD$ cut the section




I say that the straight line $EF$, when produced will meet each of the straight lines $AB$ and $CD$ outside the section

For let the straight lines $GE$ and $FH$ be dropped ordinatewise from $E$ and $F$ to $AB$ and the straight lines $EK$ and $FL$ ordinatewise to $CD$ Therefore

$$\text{sq } EG \quad \text{sq } FH \quad \text{rect } BG,GA \quad \text{rect } BH\ HA \text{ (I 21)}$$

and

$$\text{sq } FL \quad \text{sq } EK \quad \text{rect } DL\ LC \quad \text{rect } DK\ KC \text{ (I 21)}$$

And

$$\text{rect } BG\ GA > \text{rect } BH\ HA$$

for the point $G$ is nearer the midpoint (Eucl VI 27 II 5), and

$$\text{rect } DL,LC > \text{rect } DK\ KC,$$

therefore also

$$\text{sq } GE > \text{sq } FH$$

and

$$\text{sq } FL > \text{sq } EK,$$

therefore also

$$GE > FH,$$

and

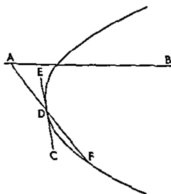
$$FL > EK$$

And $GE$ is parallel to $FH$ and $FL$ to $EK$ therefore the straight line $EF$ produced will meet each of the diameters $AB$ and $CD$ outside the section (I 10 Eucl I 33)

[1] So far Apollonius by theorems I 6 13 15 has shown for every ellipse the existence of at least one diameter and of one set of conjugate diameters but of no more He can therefore now speak of the two diameters Later on he will show the existence of an infinite number of such sets The same is true of hyperbolas

PROPOSITION 24

*If a straight line, meeting a parabola or hyperbola at a point when produced both ways, falls outside the section, then it will meet the diameter*




Let there be a parabola or hyperbola whose diameter is the straight line $AB$, and let the straight line $CDE$ meet it at $D$ and when produced both ways, let it fall outside the section

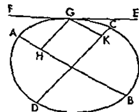
I say that it will meet the diameter $AB$

For let some point $F$ be taken on the section, and let the straight line $DF$ be joined therefore $DF$ produced will meet the diameter of the section (I 22) Let it meet it at $A$ and the straight line $CDE$ lies between the section and the straight line $FDA$ And therefore the line $CDE$ produced will meet the diameter outside the section

PROPOSITION 25

*If a straight line, meeting an ellipse between the two (conjugate) diameters and produced both ways, falls outside the section, it will meet each of the diameters*

Let there be an ellipse whose diameters are the straight lines $AB$ and $CD$ (I 15) and let $EF$, some straight line between the two diameters meet it at $G$, and produced both ways fall outside the section

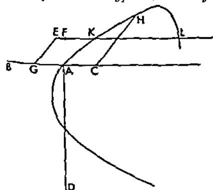



I say that the straight line $EF$ will meet each of the straight lines $AB$ and $CD$

Let the straight lines $GH$ and $GK$ be dropped ordinatewise to the straight lines $AB$ and $CD$ respectively Since $GK$ is parallel to $AB$ (I 15) and some straight line $GF$ has met $GK$ therefore it will also meet $AB$ Then likewise $EF$ will also meet $CD$

PROPOSITION 26

*If in a parabola or hyperbola a straight line is drawn parallel to the diameter of the section it will meet the section in one point only*




Let there first be a parabola whose diameter is the straight line $ABC$, and whose upright side is the straight line $AD$ and let the straight line $EF$ be drawn parallel to $AB$

I say that the straight line $EF$ produced will meet the section

For let some point $E$ be taken on $EF$ and from $E$ let the straight line $EG$ be drawn parallel to an ordinate, and let

$$\text{rect } DA, AC > \text{sq } GE,$$

and from $C$ let $CH$ be erected ordinatewise (I 19) Therefore

$$\text{sq } HC = \text{rect } DA, AC \text{ (I 11)}$$

But

$$\text{rect } DA, AC > \text{sq } EG,$$

therefore

$$\text{sq } HC > \text{sq } EG,$$

therefore

$$HC > EG$$

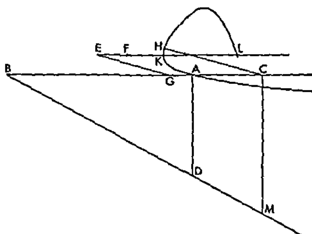
And they are parallel, therefore the straight line $EF$ produced cuts the straight line $HC$, and so it will also meet the section

Let it meet it at the point $K$

Then I say also that it will meet it in the one point $K$ only

For if possible let it also meet it in the point $L$ Since then a straight line cuts a parabola in two points, if produced it will meet the diameter of the section (I 22) And this is absurd for it is supposed parallel Therefore the straight line $EF$ produced meets the section in only one point

Next let the section be an hyperbola and the straight line $AB$ the transverse




side of the figure and the straight line $AD$ the upright side and let the straight line $DB$ be joined and produced Then with the same things being constructed, let the straight line $CM$ be drawn from $C$ parallel to $AD$ Since then

$$\text{rect } MC, CA > \text{rect } DA, AC,$$

and

$$\text{sq } CH = \text{rect } MC, CA,$$

and

$$\text{rect } DA, AC > \text{sq } GE,$$

therefore also

$$\text{sq } CH > \text{sq } GE$$

And so also

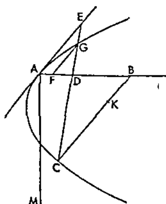
$$CH > GE$$

and the same things as in the first case will come to pass

### Proposition 27

*If a straight line cuts the diameter of a parabola, then produced both ways it will meet the section*

Let there be a parabola whose diameter is the straight line $AB$, and let some straight line $CD$ cut it within the section




I say that the straight line $CD$ produced both ways will meet the section

For let some straight line $AE$ be drawn from $A$ parallel to an ordinate, therefore the straight line $AE$ will fall outside the section (I 17)

Then either the straight line $CD$ is parallel to the straight line $AE$ or not

If now it is parallel to it it has been dropped ordinatewise so that produced both ways it will meet the section (I 18)

Next let it not be parallel to $AE$ but produced let it meet $AE$ at $E$ Then it is evident that it meets the section the side the point $E$ is on, for if it meets $AE$, *a fortiori* it cuts the section

I say that, produced the other way it also meets the section For let the straight line $MA$ be the parameter and the straight line $GF$ an ordinate and let

sq $AD$ = rect $BA$ $AF$ (Eucl VI 11)

and let the straight line $BK$, parallel to the ordinate, meet the straight line $DC$ at $C$ Since

rect $BA$ $AF$ = sq $AD$

hence

$AB$ $AD$ $AD$ $AF$

and therefore

$BD$ $DF$ $AB$ $AD$ (Eucl V 19)

Therefore also

sq $BD$ sq $DF$ sq $AB$ sq $AD$

But since

sq $AD$ = rect $BA$ $AF$

hence

$AB$ $AF$ sq $AB$ sq $AD$ sq $BD$ sq $FD$

But

sq $BD$ sq $DF$ sq $BC$ sq $FG$

and

$AB$ $AF$ rect $BA$,$AM$ rect $FA$ $AM$

Therefore

sq $BC$ sq $FG$ rect $BA$ $AM$ rect $FA$ $AM$

and alternately

sq $BC$ rect $BA$ $AM$ sq $FG$ rect $FA$ $AM$

But

sq $FG$ = rect $FA$ $AM$

because of the section (I 11) Therefore also

$$\text{sq } BC = \text{rect } BA, AM$$

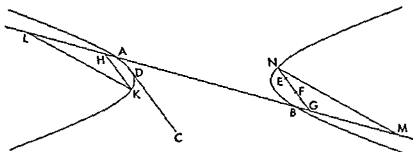
But the straight line $AM$ is the upright side,[1] and the straight line $BC$ is parallel to an ordinate Therefore the section passes through the point $C$ (I 20), and the straight line $CD$ meets the section at the point $C$

## Proposition 28

*If a straight line touches one of the opposite sections, and some point is taken within the other section and through it a straight line is drawn parallel to the tangent then produced both ways it will meet the section*

Let there be opposite sections whose diameter is the straight line $AB$, and let some straight line $CD$ touch the section $A$, and let some point $E$ be taken within the other section and through $E$ let the straight line $EF$ be drawn parallel to the straight line $CD$

I say that the straight line $EF$ produced both ways will meet the section




Since then it has been proved that the straight line $CD$ produced will meet the diameter $AB$ (I 24), and $EF$ is parallel to it therefore $EF$ produced will meet the diameter Let it meet it at $G$, and let $AH$ be made equal to $GB$, and through $H$ let $HK$ (I 18) be drawn parallel to $EF$, and let the straight line $KL$ be dropped ordinatewise and let $GM$ be made equal to $LH$, and let the straight line $MN$ be drawn parallel to an ordinate and let $GN$ be further produced in the same straight line And since $KL$ is parallel to $MN$ and $KH$ to $GN$ and $LM$ is one straight line triangle $KHL$ is similar to triangle $HMN$ And

$$LH = GM$$

therefore

$$KL = MN$$

And so also

$$\text{sq } KL = \text{sq } MN$$

And since

$$LH = GM$$

and

$$AH = BG,$$

and $AB$ is common therefore

$$BL = AM$$

[1]The text reads πλαγία which is impossible I have corrected to ὀρθία

therefore

$$\text{rect } BL, LA = \text{rect } AM\ MB$$

Therefore

$$\text{rect } BL, LA \quad \text{sq } LK \qquad \text{rect } AM\ MB \quad \text{sq } MN$$

And

rect $BL, LA$ sq $LK$ the transverse the upright (I 21),

therefore also

rect $AM\ MB$ sq $MN$ the transverse the upright

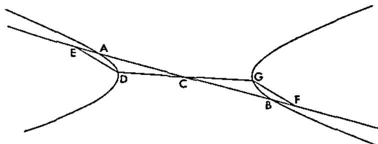
Therefore the point $N$ is on the section Therefore the straight line $EF$ produced will meet the section at the point $N$ (I 21)

Likewise then it could be shown that produced to the other side it will meet the section

## Proposition 29

*If in opposite sections a straight line is drawn through the center to meet either of the sections, then produced it will cut the other section*

Let there be opposite sections whose diameter is the straight line $AB$, and whose center is the point $C$, and let the straight line $CD$ cut the section $AD$




I say that it will also cut the other section

For let the straight line $ED$ be dropped ordinatewise and let the straight line $BF$ be made equal to the straight line $AE$ and let the straight line $FG$ be drawn ordinatewise (I 19) And since

$$EA = BF$$

and $AB$ is common therefore

$$\text{rect } BE\ EA = \text{rect } BF\ FA$$

And since

rect $BE, EA$ sq $DE$ the transverse the upright (I 21)

but also

rect $BF\ FA$ sq $FG$ the transverse the upright (I 21),

therefore also

rect $BE, EA$ sq $DE$ rect $BF\ FA$ sq $FG$ (I 14)

But

$$\text{rect } BE\ EA = \text{rect } BF\ FA$$

therefore also

$$\text{sq } DE = \text{sq } FG$$

Since then

$$EC = CF,$$

and

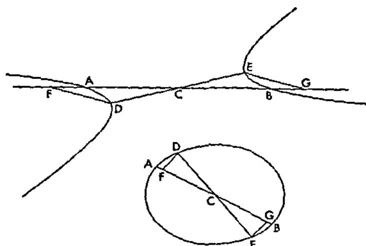
$$DF = FG,$$

and $EF$ is a straight line, and $ED$ is parallel to $FG$, therefore $DG$ is also a straight line (Eucl VI 32) And therefore $CD$ will also cut the other section

PROPOSITION 30

*If in an ellipse or in opposite sections a straight line is drawn in both directions from the center meeting the section, it will be bisected at the center*

Let there be an ellipse or opposite sections and their diameter the straight line $AB$, and their center $C$, and through $C$ let some straight line $DCE$ be drawn (I 29)




I say that the straight line $CD$ is equal to the straight line $CE$

For let the straight lines $DF$ and $EG$ be drawn ordinatewise And since

rect $BF,FA$ sq $FD$ the transverse the upright (I 21),

but also

rect $AG,GB$ sq $GE$ the transverse the upright (I 21),

therefore also

rect $BF\ FA$ sq $FD$ rect $AG\ GB$ sq $GE$ (I 14)

And alternately

rect $BF\ FA$ rect $AG,GB$ sq $FD$ sq $GE$

But

sq $FD$ sq $GE$ sq $FC$ sq $CG$ (Eucl VI 4),

therefore alternately

rect $BF\ FA$ sq $FC$ rect $AG\ GB$ sq $CG$

Therefore also *componendo* in the case of the ellipse and inversely and *convertendo* in the case of the opposite sections (Eucl V Defs 14,13,16),

sq $AC$ sq $CF$ sq $BC$ sq $CG$ (Eucl II 5 6),

and alternately But

$$\text{sq } CB = \text{sq } AC$$

therefore also

$$\text{sq } CG = \text{sq } CF$$

Therefore

$$CG=CF$$

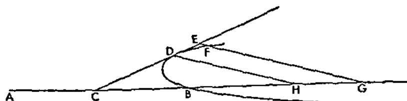
And the straight lines $DF$ and $GE$ are parallel, therefore also

$$DC=CE$$

### Proposition 31

*If on the transverse side of the figure of an hyperbola some point be taken cutting off from the vertex of the section not less than half of the transverse side of the figure and a straight line be drawn from it to meet the section, then, when further produced it will fall within the section on the near side of the section*

Let there be an hyperbola whose diameter is the straight line $AB$ and let $C$ some point on the diameter be taken cutting off the straight line $CB$ not less




than half of $AB$ and let some straight line $CD$ be drawn to meet the section

I say that the straight line $CD$ produced will fall within the section

For if possible let it fall outside the section as the line $CDF$ (I 24), and from $E$ a point at random let the straight line $EG$ be dropped ordinatewise also $DH$ and first let

$$AC=CB$$

And since

$$\text{sq } EG \;\; \text{sq } DH > \text{sq } FG \;\; \text{sq } DH \text{ (Eucl V 8)},$$

but

$$\text{sq } EG \;\; \text{sq } DH \quad \text{sq } CG \;\; \text{sq } CH$$

because of $EG$ s being parallel to $DH$ and

$$\text{sq } FG \;\; \text{sq } DH \quad \text{rect } AG\,GB \;\; \text{rect } AH\,HB$$

because of the section (I 21),

therefore

$$\text{sq } CG \;\; \text{sq } CH > \text{rect } AG\,GB \;\; \text{rect } AH\,HB$$ [1]

Alternately therefore

$$\text{sq } CG \;\; \text{rect } AG\,GB > \text{sq } CH \;\; \text{rect } AH, HB$$

Therefore *separando*

$$\text{sq } CB \;\; \text{rect } AG\,GB > \text{sq } CB \;\; \text{rect } AH\,HB$$

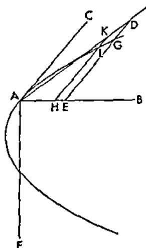
and this is impossible (Eucl V 8) Therefore the straight line $CDE$ will not fall outside the section therefore inside And for this reason the straight line from some one of the points on the straight line $AC$ will *a fortiori* fall inside, since it will also fall inside $CD$

[1]The rules governing operations on inequalities in proportions are not developed by Euclid in Book V of the *Elements* But they can be deduced on Euclid s principles

### Proposition 32

*If a straight line is drawn through the vertex of a section of a cone parallel to an ordinate, then it touches the section, and another straight line will not fall into the space between the conic section and this straight line*

Let there be a section of a cone, first the so called parabola whose diameter is the straight line $AB$, and from $A$ let the straight line $AC$ be drawn parallel to an ordinate




Now it has been shown that it falls outside the section (I 17)

Then I say that also another straight line will not fall into the space between the straight line $AC$ and the section

For if possible, let it fall in, as the straight line $AD$ and let some point $D$ be taken on it at random and let the straight line $DE$ be dropped ordinatewise, and let the straight line $AF$ be the parameter of the ordinates And since

sq $DE$ sq $EA >$ sq $GE$ sq $EA$ (Eucl v 8),

and

sq $GE =$ rect $FA$ $AE$ (I 11),

therefore also

sq $DE$ sq $EA >$ rect $FA$ $AE$ sq $EA$

or

$> FA$ $EA$

Let it be contrived then that

sq $DE$ sq $EA$ $FA$ $HA$ (Eucl VI 20 11),

and through the point $H$ let the straight line $HLK$ be drawn parallel to $ED$ Since then

sq $DE$ sq $EA$ $FA$ $AH$ rect $FA$ $AH$ sq $AH$

and

sq $DE$ sq $EA$ sq $KH$ sq $HA$ (Eucl VI 22)

and

sq $HL =$ rect $FA$ $AH$ (I 11)

therefore also

sq $KH$ sq $HA$ sq $LH$ sq $HA$

Therefore

$KH = HL$

and this is absurd Therefore another straight line will not fall into the space between the straight line $AC$ and the section

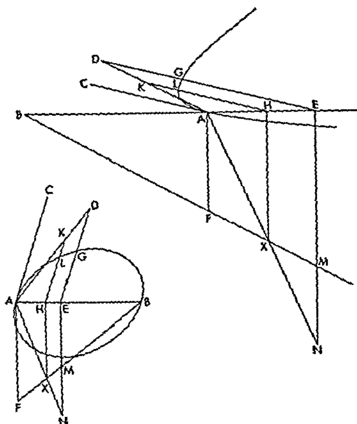
Next let the section be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line $AB$ and whose upright side is the straight line $AF$ and let the straight line $BF$ be joined and produced, and from the point $A$ let the straight line $AC$ be drawn parallel to an ordinate

Now it has been shown that it falls outside the section (I 17)

Then I say that also another straight line will not fall into the space between the straight line $AC$ and the section

For if possible let it fall as the straight line $AD$ and let some point $D$ be taken at random on it and from it let the straight line $DE$ be dropped ordi

natewise and through $E$ let the straight line $EM$ be drawn parallel to the straight line $AF$




And since

$$\text{sq } GE = \text{rect } AE\ EV \text{ (I 12 13)}$$

let it be contrived that

$$\text{rect } AE\ EN = \text{sq } DE,$$

and let the straight line joining $AN$ cut the straight line $FM$ at $X$ and through $X$ let the straight line $\backslash H$ be drawn parallel to $FA$ and through $H$ $HLK$ parallel to $AC$ Since then

$$\text{sq } DE = \text{rect } AM\ EN,$$

hence

$$NE\quad ED\qquad DE\quad EA$$

and therefore

$$NE\quad EA\qquad \text{sq } DE\quad \text{sq } EA \text{ (Eucl VI 20)}$$

But

$$NE\quad EA\qquad XH\quad HA$$

and

$$\text{sq } DE\quad \text{sq } EA\qquad \text{sq } KH\quad \text{sq } HA$$

Therefore

$$\backslash H\quad HA\qquad \text{sq } KH\quad \text{sq } HA$$

therefore

$$\Lambda H \quad HK \quad KH \quad HA$$ (Eucl $\text{vi}$ 20)

Therefore

$$\text{sq } \Lambda H = \text{rect } AH\ HX,$$

but also

$$\text{sq } LH = \text{rect } AH, HX$$

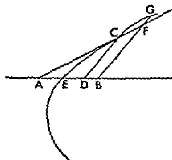
because of the section (I 12, 13),
therefore

$$\text{sq } \Lambda H = \text{sq } HL$$

and this is absurd Therefore another straight line will not fall into the space between the straight line $AC$ and the section

Proposition 33

*If in a parabola some point is taken, and from it an ordinate is dropped to the diameter and to the straight line cut off by it on the diameter from the vertex a straight line in the same straight line from its extremity is made equal then the straight line joined from the point thus resulting to the point taken will touch the section*




Let there be a parabola whose diameter is the straight line $AB$ and let the straight line $CD$ be dropped ordinatewise and let the straight line $AE$ be made equal to the straight line $ED$ and let the straight line $AC$ be joined

I say that the straight line $AC$ produced will fall outside the section

For if possible let it fall within as the straight line $CF$ and let the straight line $GB$ be dropped ordinatewise And since

$$\text{sq } BG \quad \text{sq } CD > \text{sq } FB \quad \text{sq } CD$$

but

$$\text{sq } FB \quad \text{sq } CD \quad \text{sq } BA \quad \text{sq } AD$$

and

$$\text{sq } BG \quad \text{sq } CD \quad BE \quad DE$$ (I 20),

therefore

$$BE \quad DE > \text{sq } BA \quad \text{sq } AD$$

But

$$BE \quad DE \quad 4 \text{ rect } BE\ EA \quad 4 \text{ rect } DE\ EA$$

therefore also

$$4 \text{ rect } BE\ EA \quad 4 \text{ rect } DE\ EA > \text{sq } AB \quad \text{sq } AD$$

Therefore alternately

$$4 \text{ rect } BE\ EA \quad \text{sq } AB > 4 \text{ rect} \quad \text{sq } AD$$

and this is absurd for since $DE,$

hence

But

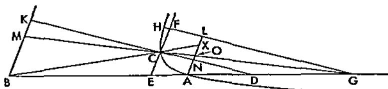
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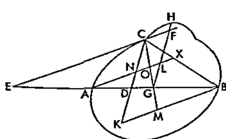
for $E$ is not the m of $A$ line
$AC$ does not fal

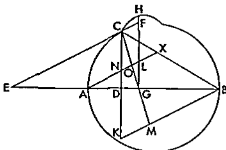
PROPOSITION 34

*If on an hyperbola or ellipse or circumference of a circle some point is taken and from it a straight line is dropped ordinatewise to the diameter, and whatever ratio the straight lines cut off by the ordinate from the ends of the figure's transverse side have to each other, that ratio have the segments of the transverse side to each other so that the segments from the vertex are corresponding then the straight line joining the point taken on the transverse side and that taken on the section will touch the section*

Let there be an hyperbola or ellipse or circumference of a circle whose diam-




eter is the straight line $AB$ and let some point $C$ be taken on the section and from $C$ let the straight line $CD$ be drawn ordinatewise and let it be contrived that

$$BD \quad DA \qquad BE \quad EA$$ [1]

and let the straight line $EC$ be joined

I say that the straight line $CE$ touches the section

[1]This construction is easy In the case of the hyperbola *componendo*

$$BD+DA \quad DA \qquad BA \quad EA$$

and in the case of the ellipse *separando*

$$BD-DA \quad DA \qquad BA \quad EA$$

This proportion is the same as the harmonic proportion defined by Nicomachus in his *Introduction to Arithmetic* For if

$$BD \quad DA \qquad BE \quad EA$$

then $\quad BD+DA \quad BD \qquad BA \quad BE$

and $\quad BA \quad BD \qquad BE-EA \quad BE$

Hence $\quad BD+DA \quad BA \qquad BA \quad BE-EA$

But $\quad DA=BD-BA \quad EA=BA-BE$

Therefore $\quad 2BD-BA \quad BA \qquad BA \quad 2BE-BA$

And so $BA$ is the harmonic mean between $BD$ and $BE$

For if possible, let it cut it, as the straight line $ECF$, and let some point $F$ be taken on it, and let the straight line $GFH$ be dropped ordinatewise and let the straight lines $AL$ and $BK$ be drawn through $A$ and $B$ parallel to the straight line $EC$, and let the straight lines $DC$, $BC$, and $GC$ be joined and produced to the points $M$, $X$, and $K$. And since

$$BD \; DA \quad BE \; EA$$

but

$$BD \; DA \quad BK \; AN,$$

and

$$BE \; AE \quad BC \; CX \quad BK \; XN \text{ (Eucl. vi. 4)},$$

therefore

$$BK \; AN \quad BK \; XN,$$

therefore

$$AN = NX$$

Therefore

$$\text{rect. } AN, NX > \text{rect. } AO, OX \text{ (Eucl. vi. 27, ii. 5)}$$

Therefore

$$NX \; XO > OA \; AN.^{1}$$

But

$$NX \; XO \quad KB \; BM \text{ (Eucl. vi. 4)}$$

therefore

$$KB \; BM > OA \; AN$$

Therefore

$$\text{rect. } KB, AN > \text{rect. } BM, OA$$

And so

$$\text{rect. } KB, AN \; \text{sq. } CE > \text{rect. } BM, OA \; \text{sq. } CE \text{ (Eucl. v. 8)}$$

But

$$\text{rect. } KB, AN \; \text{sq. } CE \quad \text{rect. } BD, DA \; \text{sq. } DE$$

through the similarity of the triangles $BKD$, $ECD$ and $NAD$, and

[1] Eutocius commenting says: For since

$$\text{rect. } AN, NX > \text{rect. } AO, OX$$

let $\quad \text{rect. } AN, NX = \text{rect. } AO, XP$

where $XP$ is some line such that $\quad XP > XC$

therefore $\quad OA \; AN \quad NX \; XP$

But $\quad NX \; XO > NX \; XP$ (Eucl. v. 8)

and therefore $\quad NX \; XO > OA \; AN$

Then the converse is also evident that if

$$NX \; XO > OA \; AN$$

then $\quad \text{rect. } XN, NA > \text{rect. } AO, OX$

For let it be that $\quad OA \; AN \quad NX \; XP$

where $\quad XP > XO$

therefore $\quad \text{rect. } XN, NA = \text{rect. } AO, XP$

and so $\quad \text{rect. } XN, NA > \text{rect. } AO, OX$

[2] Eutocius commenting, says: Since then because $AN$, $EC$ and $KB$ are parallel

$$AN \; EC \quad AD \; DE$$

and $\quad EC \; KB \quad ED \; DB$

therefore *ex aequali* $\quad AN \; KB \quad AD \; DB$

therefore also $\quad \text{sq. } AN \; \text{rect. } AN, KB \quad \text{sq. } AD \; \text{rect. } AD, DB$

But $\quad \text{sq. } EC \; \text{sq. } AN \quad \text{sq. } ED \; \text{sq. } AD$

therefore *ex aequali* $\quad \text{sq. } EC \; \text{rect. } AN, KB \quad \text{sq. } ED \; \text{rect. } AD, DB$

and inversely $\quad \text{rect. } KB, AN \; \text{sq. } EC \quad \text{rect. } AD, DB \; \text{sq. } ED$

A similar proof holds for the proportion following

rect $BM,OA$ sq $CE$ rect $BG,GA$ sq $GE$,

therefore

rect $BD$ $DA$ sq $DE>$rect $BG,GA$ sq $GE$

Therefore alternately

rect $BD,DA$ rect $BG,GA>$sq $DE$ sq $GE$

But

rect $BD$ $DA$ rect $AG,GB$ sq $CD$ sq $GH$ (I 21),

and

sq $DE$ sq $EG$ sq $CD$ sq $FG$ (Eucl vi 4),

therefore also

sq $CD$ sq $HG>$sq $CD$ sq $FG$

Therefore

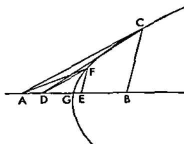
$HG<FG$ (Eucl v 10)

and this is impossible Therefore the straight line $EC$ does not cut the section therefore it touches it

## Proposition 35

*If a straight line touches a parabola meeting the diameter outside the section, the straight line drawn from the point of contact ordinatewise to the diameter will cut off on the diameter beginning from the vertex of the section a straight line equal to the straight line between the vertex and the tangent, and no straight line will fall into the space between the tangent and the section*

Let there be a parabola whose diameter is the straight line $AB$, and let the straight line $BC$ be erected ordinatewise and let the straight line $AC$ be tangent to the section




I say that the straight line $AG$ is equal to the straight line $GB$

For if possible, let it be unequal to it, and let the straight line $GE$ be made equal to $AG$ and let the straight line $EF$ be erected ordinatewise and let the straight line $AF$ be joined Therefore $AF$ produced will meet the straight line $AC$ (I 33) and this is impossible For two straight lines will have the same ends Therefore the straight line $AG$ is not unequal to the straight line $GB$, therefore it is equal

Then I say that no straight line will fall into the space between the straight line $AC$ and the section

For if possible let the straight line $CD$ fall in between and let $GE$ be made equal to $GD$ and let the straight line $EF$ be erected ordinatewise Therefore the straight line joined from $D$ to $F$ touches the section (I 33) therefore produced it will fall outside it And so it will meet $DC$ and two straight lines will have the same ends and this is impossible Therefore a straight line will not fall into the space between the section and the straight line $AC$

## Proposition 36

*If some straight line meeting the transverse side of the figure touches an hyperb or ellipse or circumference of a circle and a straight line is dropped from the*

*of contact ordinatewise to the diameter then as the straight line cut off by the tangent from the end of the transverse side is to the straight line cut off by the tangent from the other end of that side so will the straight line cut off by the ordinate from the end of the side be to the straight line cut off by the ordinate from the other end of the side in such a way that the corresponding straight lines are continuous and another straight line will not fall into the space between the tangent and the section of the cone*

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line $AB$ and let the straight line $CD$ be tangent, and let the straight line $CE$ be dropped ordinatewise

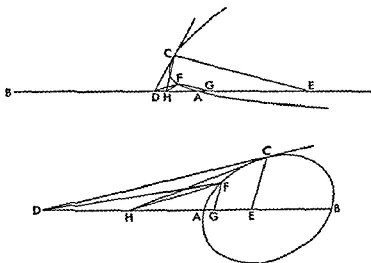
I say that

$$BE\ \ EA \quad BD\ \ DA$$

For if it is not, let it be

$$BD\ \ DA \quad BG\ \ GA,$$

and let the straight line $GF$ be erected ordinatewise therefore the straight line




joined from $D$ to $F$ will touch the section (I 34) therefore produced it will meet $CD$ Therefore two straight lines will have the same ends, and this is impossible

I say that no straight line will fall between the section and the straight line $CD$

For if possible let it fall between as the straight line $CH$, and let it be contrived that

$$BH\ \ HA \quad BG\ \ GA,$$

and let the straight line $GF$ be erected ordinatewise therefore the straight line joined from $H$ to $F$ when produced will meet $HC$ (I 34) Therefore two straight lines will have the same ends and this is impossible Therefore a straight line will not fall into the space between the section and the straight line $CD$

## Proposition 37

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and from the point of contact to the diameter a straight line is dropped ordinatewise, then the straight line cut off by the ordinate from the center of the section with the straight line cut off by the tangent from the center of the section will contain an area equal to the square on the radius of the section, and with the straight line between the ordinate and the tangent will contain an area having the ratio to the square on the ordinate which the transverse has to the upright*

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line $AB$, and let the straight line $CD$ be drawn tangent and let the straight line $CE$ be dropped ordinatewise, and let the point $F$ be the center

I say that

$$\text{rect } DF, FE = \text{sq } FB,$$

and

$$\text{rect } DE, EF \quad \text{sq } EC \quad \text{the transverse} \quad \text{the upright}$$

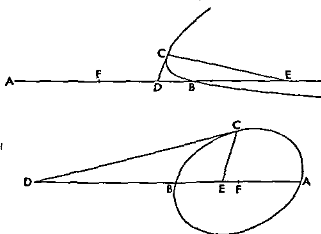
For since $CD$ touches the section, and $CE$ has been dropped ordinatewise hence

$$AD \quad DB \qquad AE \quad EB \text{ (I 36)}$$

Therefore *componendo*

$$AD+DB \quad DB \qquad AE+EB \quad EB$$

And let the halves of the antecedents be taken (Eucl v 15) in the case of the




hyperbola we shall say but

$$\text{half } (AE+EB) = FE,$$

and

$$\text{half } AB = FB,$$

therefore

$$FE \quad EB \qquad FB \quad BD$$

Therefore *convertendo*

$$FE \quad FB \qquad FB \quad FD,$$

therefore

$$\text{rect } EF, FD = \text{sq } FB$$

And since

$$FE \quad EB \qquad FB \quad BD \qquad AF \quad BD,$$

alternately

$$AF \quad FE \qquad DB \quad BE,$$

*componendo*

$$AE \quad EF \qquad DE \quad EB$$

and so

$$\text{rect } AE\ EB = \text{rect } FE, ED$$

But

$$\text{rect } AE\ EB \quad \text{sq } CE \qquad \text{the transverse} \quad \text{the upright (I 21)},$$

therefore also

$$\text{rect } FE\ ED \quad \text{sq } CE \qquad \text{the transverse} \quad \text{the upright}$$

And in the case of the ellipse and of the circle we shall say but

$$\text{half } (AD+DB) = DF,$$

and

$$\text{half } AB = FB$$

therefore

$$FD \quad DB \qquad FB \quad BE$$

Therefore *convertendo*

$$DF \quad FB \qquad BF \quad FE$$

Therefore

$$\text{rect } DF\ FE = \text{sq } BF$$

But

$$\text{rect } DF\ FE = \text{rect } DE\ EF + \text{sq } FE \text{ (Eucl II 3)},$$

and

$$\text{sq } BF = \text{rect } AE\ EB + \text{sq } FE \text{ (Eucl II 5)}$$

Let the common square on $EF$ be subtracted therefore

$$\text{rect } DE\ EF = \text{rect } AE\ EB$$

Therefore

$$\text{rect } DE\ EF \quad \text{sq } CE \qquad \text{rect } AE\ EB \quad \text{sq } CE$$

But

$$\text{rect } AE\ EB \quad \text{sq } CE \qquad \text{the transverse} \quad \text{the upright (I 21)}$$

Therefore

$$\text{rect } DE\ EF \quad \text{sq } CE \qquad \text{the transverse} \quad \text{the upright}$$

## Proposition 38

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter and from the point of contact a straight line is dropped to the same diameter parallel to the other diameter, then the straight line cut off by the dropped straight* (κατηγμένη)[1] *line from the center of the section with the straight*

[1] When this word κατηγμένη is used in connection with the first diameter we translate it as ordinate but we have preferred to stick more closely to the original when it is referred to the second diameter For although it is certainly an ordinate in the case of the ellipse yet in the case of the hyperbola it is only analogically an ordinate This analogy however becomes stronger and stronger as the treatise moves on It is therefore no accident that κατηγμένη is used in both cases On the other hand in First Definitions I 5 Apollonius definitely calls both cases ordinates as if announcing the culmination of an analogy to be worked out in the course of the treatise

*line cut off by the tangent from the center of the section will contain an area equal to the square on the half of the second diameter and with the straight line between the dropped straight line and the tangent will contain an area having a ratio to the square on the dropped straight line which the upright side of the figure has to the transverse*

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line $AGB$ and whose second diameter is the straight line $CGD$ and let the straight line $ELF$ meeting $CD$ at $F$, be a tangent to the section and let the straight line $HE$ be parallel to $AB$

I say that

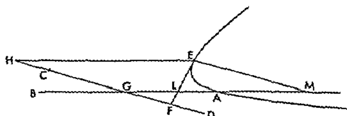
rect $FG$ $GH$ = sq $GC$

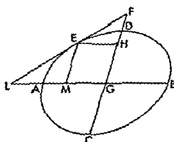
and

rect $GH,HF$ sq $HE$ the upright the transverse

Let the straight line $ME$ be drawn *ordinatewise therefore*

rect $GM$ $ML$ sq $ME$ the transverse the upright (I 37)




But

*the transverse* $BA$ $CD$ $CD$ the upright (see *Def* 11)

and therefore

the transverse the upright sq $BA$ sq $CD$ (Eucl VI 20)

and as the quarters of them that is

the transverse the upright sq $GA$ sq $GC$

therefore also

rect $GM$ $ML$ sq $ME$ sq $GA$ sq $GC$

But

rect $GM$ $ML$ sq $ME$ comp $GM$ $ME$ $LM$ $MF$

or

rect $GM$ $ML$ sq $ME$ comp $GM$ $GH$ $LM$

Therefore inversely

$$\text{sq } CG \quad \text{sq } GA \text{ comp } EM \quad MG \text{ or } HG \quad GM \; EM \quad ML \text{ or } FG \quad GL$$

Therefore

$$\text{sq } GC \quad \text{sq } GA \text{ comp } HG \quad GM, FG \quad GL,$$

which is the same as

$$\text{rect } FG\,GH \quad \text{rect } MG\,GL$$

Therefore

$$\text{rect } FG,GH \quad \text{rect } MG,GL \quad \text{sq } CG \quad \text{sq } GA$$

And alternately therefore

$$\text{rect } FG\,GH \quad \text{sq } CG \quad \text{rect } MG\,GL \quad \text{sq } GA$$

But

$$\text{rect } MG,GL = \text{sq } GA \text{ (I 37)},$$

therefore also

$$\text{rect } FG,GH = \text{sq } CG$$

Again since

$$\text{the upright} \quad \text{the transverse} \quad \text{sq } EM \quad \text{rect } CM\;ML \text{ (I 37)},$$

and

$$\text{sq } EM \quad \text{rect } GM\,ML \text{ comp } EM \quad GM\;EM \quad ML$$

or

$$\text{sq } EM \quad \text{rect } GM\,ML \text{ comp } HG \quad HE\;FG \quad GL \text{ or } FH \quad HE$$

which is the same as

$$\text{rect } FH\,HG \quad \text{sq } HE$$

therefore

$$\text{rect } FH\,HG \quad \text{sq } HE \quad \text{the upright} \quad \text{the transverse}$$

With the same things supposed it remains to be shown that as the straight line between the tangent and the end of the (second) diameter on the same side with the dropped straight line is to the straight line between the tangent and the second diameter so is the straight line between the other end and the dropped straight line to the straight line between the first end and the dropped straight line

*For since*

$$\text{rect } FG\,GH = \text{sq } GC = \text{rect } CG\,GD \text{ (2 para above)},$$

for

$$CG = GD$$

therefore

$$\text{rect } FG\,GH = \text{rect } CG\,GD$$

therefore

$$FG \quad GD \quad CG \quad GH$$

And *convertendo*

$$GF \quad FD \quad GC \quad CH$$

And let the doubles of the antecedents be taken, but

$$2GF = CF + FD$$

because

$$CG = GD$$

and

$$2GC = CD$$

therefore

$$CF + FD \quad FD \quad DC \quad CH$$

And *separando*

*CF FD DH HC*,

and this was to be shown

Then it is clear from what has been said that the straight line *EF* touches the section either if

rect *FG,GH*=sq *GC*,

or if

rect *FH HG* sq *GC*

in the ratio we said, for it could be shown conversely

PROPOSITION 39

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and from the point of contact a straight line is dropped ordinatewise to the diameter, then whichever of the two straight lines is taken of which one is the straight line between the ordinate and the center of the section and the other is between the ordinate and the tangent then the ordinate will have to it the ratio compounded of the ratio of the other of the two straight lines to the ordinate and of the ratio of the upright side of the figure to the transverse*

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line *AB* and let the center of it be the point *F*, and let the straight line *CD* be drawn tangent to the section, and the straight line *CE* be dropped ordinatewise

I say that

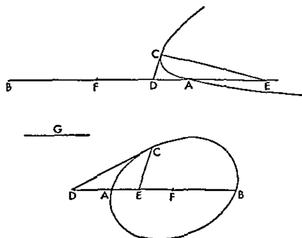
*CE FE* comp the upright the transverse *ED EC*,

and

*CE ED* comp the upright the transverse *FE EC*

For let

rect *FE ED*=rect *EC G*




And since

rect *FE ED* sq *CE* the transverse the upright (I 37),

and

$$\text{rect } FE, ED = \text{rect } CE, G,$$

therefore

$$\text{rect } CE, G : \text{sq } CE = G : CE = \text{the transverse} : \text{the upright}$$

And since

$$\text{rect } FE, ED = \text{rect } CE, G,$$

hence

$$FE : EC = G : ED$$

And since

$$CE : ED \text{ comp } CE : G, G : ED,$$

but

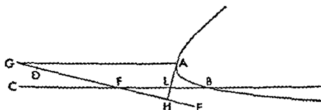
$$CE : G = \text{the upright} : \text{the transverse}$$

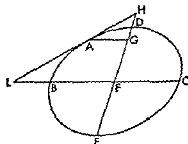
therefore

$$CE : ED \text{ comp the upright} : \text{the transverse}, FE : EC$$

PROPOSITION 40

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter, and from the point of contact a straight line is dropped to the same diameter parallel to the other diameter, then whichever of the two straight lines is taken of which one is the straight line between the dropped straight line and the center of the section and the other is between the dropped straight line and the*




*tangent the dropped straight line will have to it the ratio compounded of the ratio of the transverse side to the upright and of the ratio of the other of the two straight lines to the dropped straight line*

Let there be an hyperbola or ellipse or circumference of a circle $AB$ and its diameter the straight line $BFC$ and its second diameter the straight line $DFE$ and let the straight line $HLA$ be drawn tangent and the straight line $AG$ parallel to the straight line $BC$

I say that

$AG : HG$ comp the transverse : the upright, $FG : GA$,

and

$AG : FG$ comp the transverse : the upright, $HG : GA$

Let

rect $GA, K$ = rect $HG, GF$

And since

the upright : the transverse :: rect $HG, GF$ : sq $GA$ (I 38),

and

rect $GA, K$ = rect $HG, GF$,

therefore also

rect $GA, K$ : sq $GA$ :: $K : AG$ :: the upright : the transverse

And since

$AG : GF$ comp $AG : K, K : GF$,

but

$AG : K$ :: the transverse : the upright,

and

$K : GF$ :: $HG : GA$

because

rect $HG, GF$ = rect $AG, K$

therefore

$AG : GF$ comp the transverse : the upright, $GH : GA$

PROPOSITION 41

*If in an hyperbola or ellipse or circumference of a circle a straight line is dropped ordinatewise to the diameter, and equiangular parallelogrammic figures are described both on the ordinate and on the radius, and the ordinate side has to the remaining side of the figure the ratio compounded of the ratio of the radius to the remaining side of its figure, and of the ratio of the upright side of the section's figure to the transverse, then the figure on the straight line between the center and the ordinate similar to the figure on the radius is in the case of the hyperbola greater than the figure on the ordinate by the figure on the radius, and, in the case of the ellipse and circumference of a circle, together with the figure on the ordinate is equal to the figure on the radius*

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line $AB$ and center the point $E$, and let the straight line $CD$ be dropped ordinatewise, and on the straight lines $EA$ and $CD$ let the equiangular figures $AF$ and $DG$ be described, and let

$CD : CG$ comp $AE : EF$, the upright : the transverse

I say that, with the figure on $ED$ similar to $AF$, in the case of the hyperbola

figure on $ED = AF + GD$

and in the case of the ellipse and circle

figure on $ED + GD = AF$

For let it be contrived that

the upright : the transverse :: $DC : CH$

And since

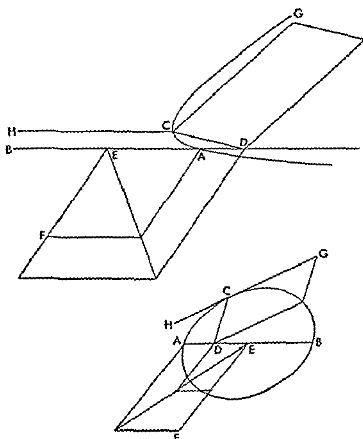
$DC : CH$ :: the upright : the transverse,

but

$DC : CH$ :: sq $DC$ : rect $DC, CH$

and

the upright the transverse sq $DC$ rect $BD$ $DA$ (I 21),




therefore

$$\text{rect } BD\ DA = \text{rect } DC\ CH$$

And since

$$DC \quad CG \text{ comp } AE \quad EF \text{ the upright the transverse}$$

or

$$DC \quad CG \text{ comp } AE \quad EF\ DC \quad CH,$$

and further

$$DC \quad CG \text{ comp } DC \quad CH\ CH \quad CG$$

therefore

$$\text{ratio comp } AE \quad EF\ DC \quad CH = \text{ratio comp } DC \quad CH\ CH \quad CG$$

Let the common ratio $DC \quad CH$ be taken away therefore

$$AE \quad EF \quad CH \quad CG$$

But

$$HC \quad CG \quad \text{rect } HC\ CD \quad \text{rect } GC\ CD,$$

and

$$AE \quad EF \quad \text{sq } AE \quad \text{rect } AE\ EF$$

therefore

$$\text{rect } HC,CD \quad \text{rect } GC\ CD \quad \text{sq } AE \quad \text{rect } AE\ EF$$

And it has been shown that

$$\text{rect } HC,CD = \text{rect } BD\ DA,$$

therefore

$$\text{rect } BD,DA \quad \text{rect } GC\ CD \quad \text{sq } AE \quad \text{rect } AE\ EF$$

Alternately

$$\text{rect } BD,DA \quad \text{sq } AE \quad \text{rect } GC\ CD \quad \text{rect } AE\ EF$$

And

$$\text{rect } GC,CD \quad \text{rect } AE\ EF \quad \text{pllg } DG \quad \text{pllg } FA$$

for they are equiangular and have to one another the ratio compounded of their sides $GC$ $AE$ and $CD$ $EF$ (Eucl vi 23) and therefore

$$\text{rect } BD\ DA \quad \text{sq } EA \quad \text{pllg } DG \quad \text{pllg } FA$$

Moreover in the case of the hyperbola we are to say *componendo*

$$\text{rect } BD\ DA + \text{sq } AE \quad \text{sq } AE \quad \text{pllg } GD + \text{pllg } AF \quad \text{pllg } AF,$$

or

$$\text{sq } DE \quad \text{sq } EA \quad \text{pllg } GD + \text{pllg } AF \quad \text{pllg } AF \text{ (Eucl ii 6)}$$

And as the square on $DE$ is to the square on $EA$ so is the figure described on $ED$ similar and similarly situated to the parallelogram $AF$, to the parallelogram $AF$ (Eucl vi 20 porism) therefore, with the figure on $ED$ similar to the parallelogram $AF$,

$$\text{pllg } GD + \text{pllg } AF \quad \text{pllg } AF \quad \text{figure on } ED \quad \text{pllg } AF$$

Therefore

$$\text{figure on } ED = \text{pllg } GD + \text{pllg } AF$$

the figure on $ED$ being similar to the parallelogram $AF$

And in the case of the ellipse and of the circumference of a circle we shall say since then

$$\text{whole sq } AE \quad \text{whole pllg } AF$$

$$\text{rect } AD\ DB \text{ subtracted} \quad \text{pllg } DG \text{ subtracted},$$

also remainder is to remainder as whole to whole (Eucl v 19)

And

$$\text{sq } AE - \text{rect } BD\ DA = \text{sq } DE \text{ (Eucl ii 5)},$$

therefore

$$\text{sq } DE \quad \text{pllg } AF - \text{pllg } DG \quad \text{sq } AE \quad \text{pllg } AF$$

But

$$\text{sq } AE \quad \text{pllg } AF \quad \text{sq } DE \quad \text{figure on } DE \text{ (Eucl vi 20 porism)}$$

the figure on $DE$ being similar to the parallelogram $AF$ Therefore the figure on $DE$ being similar to the parallelogram $AF$

$$\text{sq } DE \quad \text{pllg } AF - DG \quad \text{sq } DE \quad \text{figure on } DE$$

Therefore the figure on $DE$ being similar to the parallelogram $AF$,

$$\text{figure on } DE = \text{pllg } AF - \text{pllg } DG$$

Therefore

$$\text{figure on } DE + \text{pllg } DG = \text{pllg } AF$$

## Proposition 42

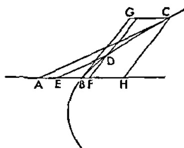
*If a straight line touching a parabola meets the diameter, and from the point of contact a straight line is dropped ordinatewise to the diameter and, some point being taken on the section two straight lines are dropped to the diameter, one of them parallel to the tangent and the other parallel to the straight line dropped from*

*the point of contact then the triangle resulting from them is equal to the parallelogram contained by the straight line dropped from the point of contact and by the straight line cut off by the parallel from the vertex of the section*

Let there be a parabola, whose diameter is the straight line $AB$ and let the straight line $AC$ be drawn tangent to the section, and let the straight line $CH$ be dropped ordinatewise and from some point at random let the straight line $DF$ be dropped ordinatewise and through the point $D$ let the straight line $DE$ be drawn parallel to the straight line $AC$, and through the point $C$ the straight line $CG$ parallel to the straight line $BF$ and through the point $B$ the straight line $BG$ parallel to the straight line $HC$

I say that

trgl $DEF$ = pllg $GF$




For since the straight line $AC$ touches the section and the straight line $CH$ has been dropped ordinatewise

$AB = BH$ (I 35),

therefore

$AH = 2BH$

Therefore

trgl $AHC$ = pllg $BC$ (Eucl I 41)

And since

sq $CH$  sq $DF$   $HB$  $BF$

because of the section (I 20) but

sq $CH$  sq $DF$   trgl $ACH$  trgl $EDF$ (Eucl VI 19)

and

$HB$  $BF$   pllg $GH$  pllg $GF$ (Eucl VI 1)

therefore

trgl $ACH$  trgl $EDF$   pllg $HG$  pllg $FG$

Therefore alternately

trgl $AHC$  pllg $BC$   trgl $EDF$  pllg $GF$

But

trgl $ACH$ = pllg $GH$

therefore

trgl $EDF$ = pllg $GF$

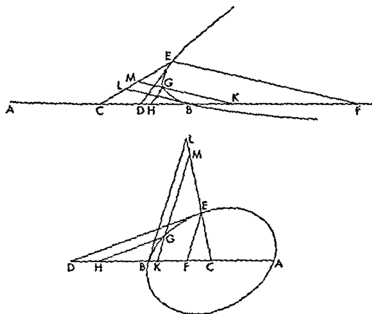
## Proposition 43

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter and from the point of contact a straight line is dropped ordinatewise to the diameter and a parallel to it is drawn through the vertex meeting the straight line drawn through the point of contact and the center, and some point being taken on the section two straight lines are drawn to the diameter one of which is parallel to the tangent and the other parallel to the straight line dropped from the point of contact then the triangle resulting from them in the case of the hyperbola will be less than the triangle the straight line through the center and the point of contact cuts off, by the triangle on the radius similar to the triangle cut off and in the case of the ellipse and the circumference of the circle together with the triangle cut off from the center, will be equal to the triangle on the radius similar to the triangle cut off*

Let there be an hyperbola or ellipse or circumference of a circle whose diam

eter is the straight line $AB$, and center the point $C$, and let the straight line $DE$ be drawn tangent to the section, and let the straight line $CE$ be joined and let the straight line $EF$ be dropped ordinatewise, and let some point $G$ be taken on the section and let the straight line $GH$ be drawn parallel to the tangent, and let the straight line $GK$ be dropped ordinatewise, and through $B$ let the straight line $BL$ be erected ordinatewise

I say that triangle $KMC$ differs from triangle $CLB$ by triangle $GKH$




For since the straight line $ED$ touches and the straight line $EF$ has been dropped hence

$$EF \quad FD \text{ comp } CF \quad FE \quad \text{the upright} \quad \text{the transverse (I 39)}$$

But

$$EF \quad FD \quad GK \quad KH$$

and

$$CF \quad FE \quad CB \quad BL \text{ (Eucl VI 4)}$$

therefore

$$GK \quad KH \text{ comp } BC \quad BL \quad \text{the upright} \quad \text{the transverse}$$

And through those things shown in the forty first theorem (I 41), triangle $CKM$ differs from triangle $BCL$ by triangle $GHK$ for the same things have also been shown in the case of the parallelograms their doubles

## Proposition 44

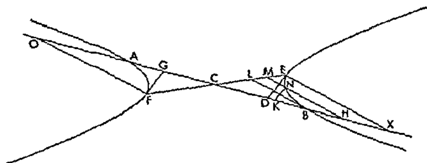
*If a straight line touching one of the opposite sections meets the diameter, and from the point of contact some straight line is dropped ordinatewise to the diameter and a parallel to it is drawn through the vertex of the other section meeting the straight line drawn through the point of contact and the center and some point being taken at random on the section let straight lines be dropped to the diameter, one of which*

*is parallel to the tangent and the other parallel to the straight line dropped ordinatewise from the point of contact then the triangle resulting from them will be less than the triangle the dropped straight line cuts off from the center of the section, by the triangle on the radius similar to the triangle cut off*

Let there be the opposite sections $AF$ and $BE$, and let their diameter be the straight line $AB$ and center the point $C$, and from some point $F$ of those on the section $FA$ let the straight line $FG$ be drawn tangent to the section, and the straight line $FO$ ordinatewise, and let the straight line $CF$ be joined and produced, as $CE$ (I 29), and through $B$ let the straight line $BL$ be drawn parallel to the straight line $FO$ and let some point $N$ be taken on the section $BE$, and from $N$ let the straight line $NH$ be dropped ordinatewise, and let the straight line $NK$ be drawn parallel to the straight line $FG$

I say that

$$\text{trgl } HKN + \text{trgl } CBL = \text{trgl } CMH$$




For through $E$ let the straight line $ED$ be drawn tangent to the section $BE$, and let the straight line $EX$ be drawn ordinatewise Since then $FA$ and $BE$ are opposite sections whose diameter is $AB$ and the straight line through whose center is $FCE$ and $FG$ and $ED$ are tangents to the section hence $DE$ is parallel to $FG$ [1] And the straight line $NK$ is parallel to $FG$, therefore $NK$ is also parallel to $ED$, and the straight line $MH$ to $BL$ Since then $BE$ is an hyperbola, whose diameter is the straight line $AB$, and whose center is $C$ and the straight line $DE$ is tangent to the section, and $EX$ drawn ordinatewise and $BL$ is parallel to $EX$ and $N$ has been taken on the section as the point from which $NH$ has

[1] Eutocius commenting says For since $AF$ is an hyperbola and $BG$ a tangent and $FO$ an ordinate rect $OC, CG$ = sq $CA$
(I 37) likewise then also rect $XC, CD$ = sq $CB$
Therefore rect $OC, CG$ : sq $AC$ = rect $XC, CD$ : sq $BC$
and alternately rect $OC, CG$ : rect $XC, CD$ = sq $AC$ : sq $CB$
But sq $AC$ = sq $CB$
therefore also rect $OC, CG$ = rect $XC, CD$
And $OC = CX$ (I 14 30)
and therefore $GC = CD$
and also $FC = CE$ (I 30)
therefore $FC : FC = CG : CD$
And they contain equal angles at the point $C$ for they are vertical And so also
$FG = FD$
and angle $CFG$ = angle $CFD$
And they are alternate therefore the straight line $FG$ is parallel to the straight line $ED$

been dropped ordinatewise and $KN$ has been drawn parallel to $DE$, therefore

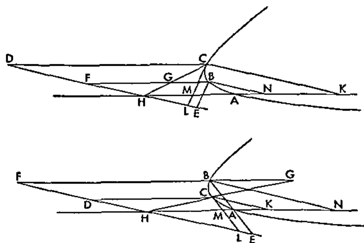
$$\text{trgl } NHK + \text{trgl } BCL = \text{trgl } HMC,$$

for this has been shown in the forty third theorem (I 43)

PROPOSITION 45

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter, and from the point of contact some straight line is dropped to the same diameter parallel to the other diameter, and through the point of contact and the center a straight line is produced, and some point being taken at random on the section two straight lines are drawn to the second diameter one of which is parallel to the tangent and the other parallel to the dropped straight line, then the triangle resulting from them is greater in the case of the hyperbola, than the triangle the dropped straight line cuts off from the center, by the triangle whose base is the tangent and vertex is the center of the section, and in the case of the ellipse and circle, together with the triangle cut off will be equal to the triangle whose base is the tangent and whose vertex is the center of the section*

Let there be an hyperbola or ellipse or circumference of a circle $ABC$, whose




diameter is the straight line $AH$ and second diameter $HD$, and center $H$ and let the straight line $CML$ touch it at $C$, and let the straight line $CD$ be drawn parallel to $AH$ and let the straight line $HC$ be joined and produced and let some point $B$ be taken at random on the section, and from $B$ let the straight lines $BE$ and $BF$ be drawn from $B$ parallel to the straight lines $LC$ and $CD$

I say that in the case of the hyperbola

$$\text{trgl } BEF = \text{trgl } GHF + \text{trgl } LCH,$$

and in the case of the ellipse and circle

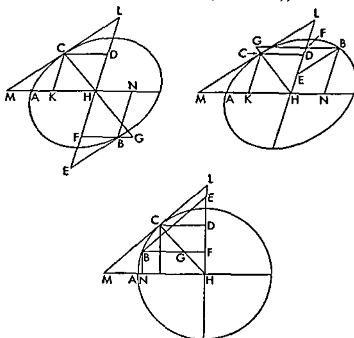
$$\text{trgl } BEF + \text{trgl } FGH = \text{trgl } CLH$$

For let the straight lines $CK$ and $BN$ be drawn parallel to $DH$ Since then the straight line $CM$ is tangent and the straight line $CK$ has been dropped ordinatewise hence

$$CK : KH \text{ comp } MK : KC \text{ the upright : the transverse (I 39)}$$

and

$$MK : KC = CD : DL \text{ (Eucl VI 4)},$$




therefore

$$CK : KH = \text{comp } CD : DL = \text{the upright : the transverse}$$

And triangle $CDL$ is the figure on $KH$ and triangle $CKH$, that is triangle $CDH$ is the figure on $CK$ that is on $DH$ therefore, in the case of the hyperbola,

$$\text{trgl } CDL = \text{trgl } CKH + \text{trgl on } AH \text{ similar to trgl } CDL,$$

and in the case of the ellipse and the circle

$$\text{trgl } CDH + \text{trgl } CDL = \text{trgl on } AH \text{ similar to trgl } CDL,$$

for this was also shown in the case of their doubles in the forty first theorem (I 41)

Since then triangle $CDL$ differs either from triangle $CKH$ or from triangle $CDH$ by the triangle on $AH$ similar to triangle $CDL$, and it also differs by triangle $CHL$ therefore

$$\text{trgl } CHL = \text{trgl on } AH \text{ similar to trgl } CDL$$

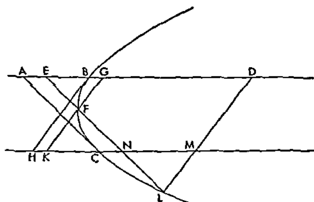
Since then triangle $BFE$ is similar to triangle $CDL$, and triangle $GFH$ to triangle $CDH$ therefore they have the same ratio [1] And triangle $BFE$ is described on $NH$ between the ordinate and the center and triangle $GFH$ on the ordinate $BN$, that is on $FH$ and by things already shown (I 41) triangle $BFE$ differs from triangle $GHF$ by the triangle on $AH$ similar to $CDL$, and so also by triangle $CHL$

[1] That is (Eucl VI 4) $BF : FE = CD : DL$
and $GF : FH = CD : DH = CK : KH$
Therefore these first ratios can be substituted in the central proportion of the theorem
$CK : KH = \text{comp } CD : DL = \text{the upright : the transverse}$
and so satisfy I 41

PROPOSITION 46

*If a straight line touching a parabola meets the diameter, the straight line drawn through the point of contact parallel to the diameter in the direction of the section bisects the straight lines drawn in the section parallel to the tangent*

Let there be a parabola whose diameter is the straight line $ABD$, and let the




straight line $AC$ touch the section (I 24) and through $C$ let the straight line $HCM$ be drawn parallel to the straight line $AD$ (I 26), and let some point $L$ be taken at random on the section, and let the straight line $LNFE$ (I 18,22) be drawn parallel to $AC$

I say that

$$LN = NF$$

Let the straight lines $BH$ $KFG$, and $LMD$ be drawn ordinatewise Since then by the things already shown in the forty-second theorem (I 42)

$$\text{trgl } ELD = \text{pllg } BM,$$

and

$$\text{trgl } EFG = \text{pllg } BK$$

therefore the remainders

$$\text{pllg } GM = \text{quadr } LFGD$$

Let the common pentagon $MDGFN$ be subtracted therefore the remainders

$$\text{trgl } KFN = \text{trgl } LMN$$

And $KF$ is parallel to $LM$ therefore

$$FN = LN \text{ (Eucl VI 22 lemma)}$$

PROPOSITION 47

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter and through the point of contact and the center a straight line is drawn in the direction of the section it bisects the straight lines drawn in the section parallel to the tangent*

Let there be an hyperbola or ellipse or circumference of a circle whose [diam]eter is the straight line $AB$ and center $C$ and let the straight line $DE$ [be] tangent to the section and let the straight line $CE$ be joined

let a point $N$ be taken at random on the section, and through $N$ let the straight line $HNOG$ be drawn parallel

I say that

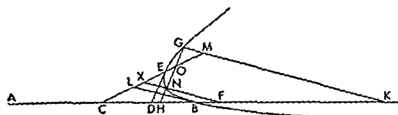
$$NO = OG$$

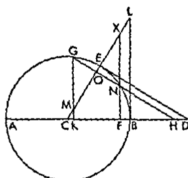
For let the straight lines $XNF$, $BL$ and $GMK$ be dropped ordinatewise Therefore by things already shown in the forty third theorem (I 43)

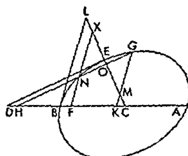
$$\text{trgl } HNF = \text{quadr } LBFX,$$

and

$$\text{trgl } GHK = \text{quadr } LBKM$$




Therefore the remainders

$$\text{quadr } NGKF = \text{quadr } MKFX,$$

Let the common pentagon $ONFKM$ be subtracted therefore the remainders

$$\text{trgl } OMG = \text{trgl } NXO$$

And the straight line $MG$ is parallel to the straight line $NX$, therefore

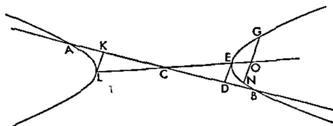
$$NO = OG \text{ (Eucl VI 22, lemma)}$$

## Proposition 48

*If a straight line touching one of the opposite sections meets the diameter, and through the point of contact and the center a straight line produced cuts the other section then whatever line is drawn in the other section parallel to the tangent, will be bisected by the straight line produced*

Let there be opposite sections whose diameter is the straight line $AB$ and center C and let the straight line $KL$ touch the section $A$ and let the straight line $LC$ be joined and produced (I 29), and let some point N be taken on the

section $B$, and through $N$ let the straight line $NG$ be drawn parallel to the straight line $LK$




I say that

$$NO = OG$$

For let the straight line $ED$ be drawn through $E$ tangent to the section therefore $ED$ is parallel to $LK$ (I 44 note) And so also to $NG$ Since then $BNG$ is an hyperbola whose center is $C$ and tangent $DE$ and since $CE$ has been joined and a point $N$ has been taken on the section and through it $NG$ has been drawn parallel to $DE$, by a theorem already shown (I 47) for the hyperbola

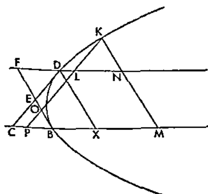
$$NO = OG$$

## Proposition 49

*If a straight line touching a parabola meets the diameter and through the point of contact a parallel to the diameter is drawn, and from the vertex a straight line is drawn parallel to an ordinate and it is contrived that as the segment of the tangent between the erected straight line and the point of contact is to the segment of the parallel between the point of contact and the erected straight line so is some straight line to the double of the tangent, then whatever straight line is drawn [parallel to the tangent] from the section to the straight line drawn through the point of contact parallel to the diameter will equal in square the rectangle contained by the straight line found and by the straight line cut off by it from the point of contact*

Let there be a parabola whose diameter is the straight line $MBC$ and $CD$ its tangent and through $D$ let the straight line $FDN$ be drawn parallel to the straight line $BC$, and let the straight line $FB$ be erected ordinatewise (I 17), and let it be contrived that

$ED$ $DF$ some straight line
$G$ $2CD$






and let some point $K$ be taken on the section and let the straight line $KLP$ be drawn through $K$ parallel to $CD$

I say that

$$\text{sq } KL = \text{rect } G\ DL$$

that is that with the straight line $DL$ as diameter the straight line $G$ is the upright side

For let the straight lines $DX$ and $KNM$ be dropped ordinatewise And since the straight line

$CD$ touches the section, and the straight line $DX$ has been dropped ordinatewise, then

$$CB = BX \quad (\text{I } 35)$$

But

$$BX = FD$$

and therefore

$$CB = FD$$

And so also

$$\text{trgl } ECB = \text{trgl } EFD$$

Let the common figure $DEBMN$ be added, therefore

$$\text{quadr } DCMN = \text{pllg } FM$$
$$= \text{trgl } KPM \quad (\text{I } 42)$$

Let the common quadrilateral $LPMN$ be subtracted therefore the remainders

$$\text{trgl } KLN = \text{pllg } LC$$

And

$$\text{angle } DLP = \text{angle } KLN,$$

therefore

$$\text{rect } KL\ LN = 2 \text{ rect } LD, DC$$ [1]

And since

$$ED \quad DF \quad G \quad 2CD$$

and

$$ED \quad DF \quad KL \quad LN,$$

therefore also

$$G \quad 2CD \quad KL \quad LN$$

But

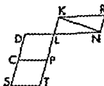
$$KL \quad LN \quad \text{sq } KL \quad \text{rect } KL\ LN,$$

and

$$G \quad 2CD \quad \text{rect } G\ DL \quad 2\text{rect } LD\ DC,$$

therefore

[1] Eutocius commenting says For let the triangle $KLN$ and the parallelogram $DLPC$ be set out And since $\text{trgl } KLN = \text{pllg } DP$ let the straight line $NR$ be drawn through $N$ parallel to $LK$ and through $K$ $KR$ parallel to $LN$ therefore $LR$ is a parallelogram and




$$\text{pllg } LR = 2 \text{ trgl } KLN$$

and so also $\text{pllg } LR = 2 \text{ pllg } DP$

Then let the straight lines $DC$ and $LP$ be produced to $S$ and $T$ and let $CS$ be made equal to $DC$ and $PT$ to $LP$ and let $ST$ be joined therefore

$$\text{pllg } DT = 2 \text{ pllg } DP$$

and so $\text{pllg } LR = \text{pllg } LS$

But it is also equiangular with it because of the angles at $L$ being vertical but in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional therefore $KL \quad LT$ or $DS \quad DL \quad LN$

and $\text{rect } KL\ LN = \text{rect } LD\ DS$

And since $DS = ?DC$

hence $\text{rect. } KL\ LN = 2 \text{ rect } LD\ DC$

And if $DC$ is parallel to $LP$ and $CP$ is not parallel to $LD$ it is clear $DCPL$ is a trapezoid and so I say that $\text{rect. } KL\ LN = \text{rect } DL\ CD + LR$

For if $LP$ is filled out as we have said before and the straight lines $DC$ and $LP$ are produced and $CS$ is made equal to $LP$ and $PT$ to $DC$ and the straight line $ST$ is joined then

$$\text{pllg } DT = 2DP$$

and the same demonstration will fit And this will be useful in what follows (I 50)

sq KL rect KL,LN rect G DL 2rect CD DL

And alternately but

rect KL LN = 2rect CD DL,

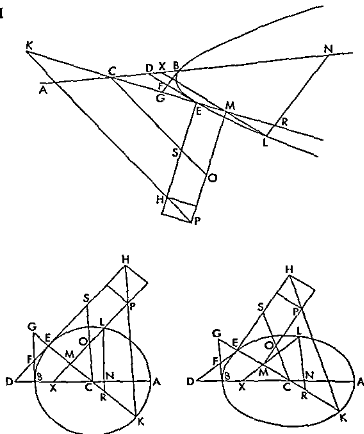
therefore also

sq KL = rect G,DL

## Proposition 50

*If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and a straight line is produced through the point of contact and the center and from the vertex a straight line erected parallel to an ordinate meets the straight line drawn through the point of contact and the center, and if it is construed that as the segment of the tangent between the point of contact and the straight line erected is to the segment of the straight line, drawn through the point of contact and the center, between the point of contact and the straight line erected, so some straight line is to the double of the tangent, then whatever straight line is drawn from the section to the straight line drawn through the point of contact and the center parallel to the tangent will equal in square a rectangular area applied to the straight line found having as breadth the straight line cut off by it from the*

Cases I

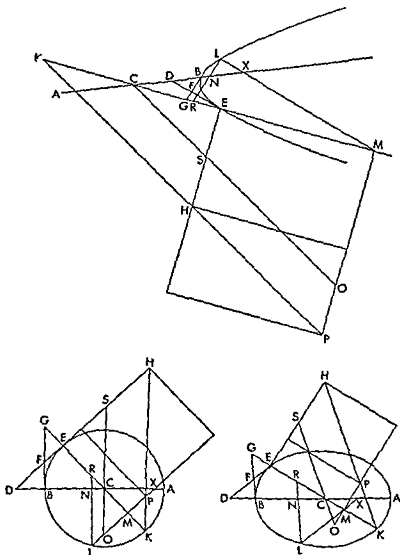


*point of contact, and exceeding, in the case of the hyperbola, by a figure similar to the rectangle contained by the double of the straight line between the center and the point of contact and by the straight line found but in the case of the ellipse and circle, defective by it*

Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line $AB$, and center $C$, and let the straight line $DE$ be a tangent, and let the straight line $CE$ be joined and produced both ways, and let the straight line $CK$ be made equal to the straight line $EC$, and through $B$ let the straight line $BFG$ be erected ordinatewise and through $E$ let the straight line $EH$ be drawn perpendicular to $\Gamma C$ and let it be that

$$\Gamma F \quad EG \qquad EH \quad 2FD,$$

CASES II



and let the straight line $HK$ be joined and produced and let some point $L$ be taken on the section and through it let the straight line $LMX$ be drawn parallel to $ED$, and the straight line $LRN$ parallel to $BG$, and the straight line $MP$ parallel to $EH$

I say that

$$\text{sq } LM = \text{rect } EM \; MP$$

For let the straight line $CSO$ be drawn through $C$ parallel to $KP$ And since

$$EC = CK,$$

and

*Cases I*

$$EC \quad KC \quad ES \quad SH$$

therefore also

$$ES = SH$$

*Cases II*

And since

$$FE \quad EG \quad HE \quad 2ED,$$

and

$$2ES = EH,$$

therefore also

$$FE \quad EG \quad SE \quad ED$$

And

$$FE \quad EG \quad LM \quad MR$$

therefore

$$LM \quad MR \quad SE \quad ED$$

And since it was shown (I 43) that, in the case of the hyperbola,

$$\text{trgl } RNC = \text{trgl } LNX + \text{trgl } GBC,$$
$$= \text{trgl } LNY + \text{trgl } CDE\;[1]$$

and, in the case of the ellipse and circle

$$\text{trgl } RXC + \text{trgl } LNY = \text{trgl } GBC$$
$$= \text{trgl } CDE$$

therefore in the case of the hyperbola with the common triangle $ECD$ and the common quadrilateral $NRMY$ subtracted and in the case of the ellipse and circle with the common triangle $MXC$ subtracted [2]

[1] That

$$\text{trgl. } GBC = \text{trgl } CDE$$

is proved by Apollonius in the course of another proof of I 43 reported by Eutocius It is also proved in III 1 without the help of intervening propositions

[2] The position of point $L$ furnishes different cases which at times as in the present theorem require a change in the course of the proof The figures marked Cases I are drawn to fit the proof as set down but we have added figures marked Cases II as an example of the possible differences

For the hyperbola of Case II instead of the subtraction in the theorem above we have

$$\text{trgl } RNC = \text{trgl } LNY + \text{trgl. } CDE$$
$$\text{quadr } MCNL = \text{quadr } MCNL$$

Subtracting the first equals from the second identity we have

$$\text{trgl. } LMR = \text{quadr } MEDY$$

The rest of the proof is the same

For the ellipse and circle of Case II we have as in the theorem above

$$\text{trgl } RXC + \text{trgl } LNY = \text{trgl } CDE$$

and subtracting the common triangle $CMY$

$$\text{trgl } LMR = \text{trgl } CDE - \text{trgl } CMY$$

therefore $$\text{rect } LM \; MR = \text{rect } EM \; LD + MY$$

For let $CM$ be made equal to $CM$ and $CX$ to $CX$ Then

$$\text{trgl } LMR = \text{quadr } MEDX$$

And $MX$ is parallel to $DE$, and

$$\text{angle } LMR = \text{angle } EMX,$$

therefore

rect $LM, MR$ = rect $EM, ED + MX$ (I 49 note, para 2)

And since

$$MC : CE = MX : ED$$

and

$$MC : CE = MO : ES,$$

therefore

$$MO : ES = MX : ED$$

And *componendo*

$$MO+ES : ES = MX+ED : ED,$$

alternately

$$MO+ES : MX+ED = ES : ED$$

But

$$MO+ES : MX+ED = \text{rect } MO+ES, EM : \text{rect } MX+ED, EM,$$

and

$$ES : ED = LM : MR = FE : EG \text{ (Eucl VI 4)}$$

or

$$ES : ED = \text{sq } LM : \text{rect } LM, MR$$

therefore

rect $MO+ES, ME$ : rect $MX+ED, EM$ = sq $LM$ : rect $LM, MR$

And alternately

rect $MO+ES, ME$ : sq $LM$ = rect $MX+ED, EM$ : rect $LM, MR$

But

rect $LM, MR$ = rect $ME, MX+ED$ (above),

therefore

$$\text{sq } LM = \text{rect } EM, MO+ES$$

And

$$SE = SH$$

and

$$SH = OP,$$

therefore

$$\text{sq } LM = \text{rect } EM, MP$$

## Proposition 51

*If a straight line touching either of the opposite sections meets the diameter and through the point of contact and the center some straight line is produced to the other section and from the vertex a straight line is erected parallel to an ordinate and meets the straight line drawn through the point of contact and the center and if it is contrived that as the segment of the tangent between the straight line erected*

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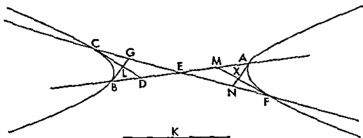
trgl $CDF$ − trgl $CMX$ = quadr $MFDX$
and $MX$ is parallel to $FD$ and
angle $FMX$ = angle $RMI$




These cases will come up again in Book III and in general it is convenient to think of quadrilateral $MFDX$ as standing for the difference of the two triangles when one pair of its sides cross each other

*and the point of contact is to the segment of the straight line, drawn through the point of contact and the center, between the point of contact and the straight line erected, so is some straight line to the double of the tangent, then whatever straight line in the other of the sections is drawn to the straight line through the point of contact and the center, parallel to the tangent, will equal in square the rectangle applied to the straight line found having as breadth the straight line cut off by it from the point of contact and exceeding by a figure similar to the rectangle contained by the straight line between the opposite sections and the straight line found*

Let there be opposite sections whose diameter is the straight line $AB$ and




center $E$, and let the straight line $CD$ be drawn tangent to the section $B$ and the straight line $CE$ joined and produced (I 29) and let the straight line $BLG$ be drawn ordinatewise (I 17) and let it be contrived that

$$LC \quad CG \quad \text{some straight line } K \quad 2CD$$

Now it is evident that the straight lines in the section $BC$, parallel to $CD$ and drawn to $EC$ produced are equal in square to the areas applied to $K$ having as breadths the straight line cut off by them from the point of contact and exceeding by a figure similar to the rectangle $CF$ $K$, for

$$FC=2CE$$

I say then that in section $FA$ the same thing will come about

For let the straight line $MF$ be drawn through $F$ tangent to the section $AF$, and let the straight line $AXN$ be erected ordinatewise And since $BC$ and $AF$ are opposite sections and $CD$ and $MF$ are tangents to them therefore $CD$ is equal and parallel to $MF$ (I 44 note) But also

$$CE=EF$$

therefore also

$$ED=EM$$

And since

$$LC \quad CG \quad K \quad 2CD \text{ or } 2MF$$

therefore also

$$XF \quad FN \quad K \quad 2MF$$

Since then $AF$ is an hyperbola whose diameter is $AB$ and tangent $MF$, and $AN$ has been drawn ordinatewise and

$$XF \quad FN \quad K \quad 2FM,$$

hence any lines drawn from the section to $EF$ produced parallel to $FM$ will equal in square the rectangle contained by the straight line $K$ and the line cut off by them from $F$ exceeding by a figure similar to the rectangle $CF$, $K$ (I 50)

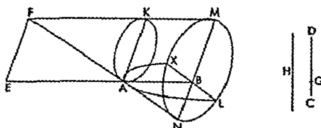
And with these things shown it is at once evident that in the parabola each of the straight lines drawn off parallel to the original diameter is a diameter

(I 46), but in the hyperbola and ellipse and opposite sections each of the straight lines drawn through the center is a diameter (I 47–48), and that in the parabola the straight lines dropped to each of the diameters parallel to the tangents will equal in square the rectangles applied to it (I 49), but in the hyperbola and opposite sections they will equal in square the areas applied to it and exceeding by the same figure (I 50–51), but in the ellipse the areas applied to it and defective by the same figure (I 50) and that all the things which have been already proved about the sections as following when the principal diameters are used,[1] will also, those very same things follow when the other diameters are taken

PROPOSITION 52 (PROBLEM)

*Given a straight line in a plane bounded at one point, to find in the plane the section of a cone called parabola, whose diameter is the given straight line, and whose vertex is the end of the straight line and where whatever straight line is dropped from the section to the diameter at a given angle will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line*

Let there be the straight line $AB$ given in position and bounded at the point $A$, and another straight line $CD$ given in magnitude and first let the given




angle be a right angle it is required then to find a parabola in the plane of reference whose diameter is the straight line $AB$ and whose vertex is the point $A$ and whose upright side is the straight line $CD$, and where the straight lines dropped ordinatewise will be dropped at a right angle that is so that $AB$ is the axis (First Def I 7)

Let $AB$ be produced to $E$ and let $CG$ be taken as the fourth part of $CD$, and let

$$EA > CG$$

and let

$$CD \quad H \quad H \quad EA$$

Therefore

$$CD \quad EA \quad \text{sq } H \quad \text{sq } EA$$

and

$$CD < 4EA$$

therefore also

$$\text{sq } H < 4 \text{ sq } EA$$

[1]The principal diameter (ἡ διάμετρος ἀρχική) is that whose being is established in I 7 porism.

Therefore

$$H < 2EA,$$

and so the two straight lines $EA$ are greater than $H$ It is therefore possible for a triangle to be constructed from $H$ and two straight lines $EA$ Then let the triangle $EAF$ be constructed on $EA$ at right angles to the plane of reference so that

$$EA = AF,$$

and

$$H = FE$$

and let the straight line $AK$ be drawn parallel to $FE$ and $FK$ to $EA$ and let a cone be conceived whose vertex is the point $F$ and whose base is the circle about diameter $KA$, at right angles to the plane through $AFX$ Then the cone will be a right cone (First Def I 3) for

$$AF = FK$$

And let the cone be cut by a plane parallel to the circle $KA$, and let it make as a section the circle $MNX$ (I 4) at right angles clearly to the plane through $MFN$, and let the straight line $MN$ be the common section of the circle $MNX$ and of the triangle $MFN$, therefore it is the diameter of the circle And let the straight line $XL$ be the common section of the plane of reference and of the circle Since then circle $MNX$ is at right angles to triangle $MFN$, and the plane of reference also at right angles to triangle $MFN$, therefore the straight line $LX$, their common section, is at right angles to triangle $MFN$, that is to triangle $KFA$ (Eucl XI 19), and therefore it is perpendicular to all the straight lines touching it and in the triangle, and so it is perpendicular to both $MN$ and $AB$

Again since a cone, whose base is the circle $MNX$ and whose vertex is the point $F$, has been cut by a plane at right angles to the triangle $MFN$ and makes as a section circle $MNX$ and since it has also been cut by another plane, the plane of reference, cutting the base of the cone in a straight line $XL$ at right angles to $MN$ which is the common section of the circle $MNX$ and the triangle $MFN$, and the common section of the plane of reference and of the triangle $MFN$ the straight line $AB$, is parallel to the side of the cone $FKM$, therefore the resulting section of the cone in the plane of reference is a parabola and its diameter $AB$ (I 11), and the straight lines dropped ordinatewise from the section to $AB$ will be dropped at right angles for they are parallel to $XL$ which is perpendicular to $AB$ And since

$$CD \quad H \quad H \quad EA,$$

and

$$EA = AF = FK$$

and

$$H = EF = AK$$

therefore

$$CD \quad AK \quad AK \quad AF$$

And therefore

$$CD \quad AF \quad \text{sq } AK \quad \text{sq } AF \text{ or rect } AF, FK$$

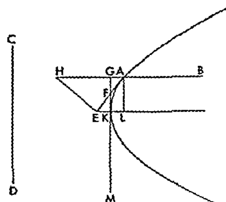
Therefore $CD$ is the upright side of the section, for this has been shown in the eleventh theorem (I 11)

PROPOSITION 53 (PROBLEM)

With the same things supposed let the given angle not be right, and let the angle $HAE$ be made equal to it and let

$AH$ = half $CD$,

and from $H$ let the straight line $HE$ be drawn perpendicular to $AE$, and through $E$ let the straight line $EL$ be drawn parallel to $BH$, and from $A$ let the straight line $AL$ be drawn perpendicular to $EL$, and let $EL$ be bisected at $K$, and from $K$ let the straight line $KM$ be drawn perpendicular to $EL$ and produced to $F$ and $G$ and let rect $LK,KM$ = sq $AL$ And given the two straight lines $LK$ and $KM$ $KL$ in position and bounded at $K$ and $KM$ in magnitude and let a parabola be described with a right angle whose diameter is the straight line $KL$, and whose vertex is the point $K$ and whose upright side is the straight line $KM$, as has been shown before (I 52), and it will pass through the point $A$ because




sq $AL$ = rect $LK,KM$ (I 11),

and the straight line $EA$ will touch the section because

$EK = KL$ (I 33)

And $HA$ is parallel to $EKL$, therefore $HAB$ is the diameter of the section and the straight lines dropped to it parallel to $AE$ will be bisected by $AB$ (I 46) And they will be dropped at angle $HAE$ And since

angle $AEH$ = angle $AGF$,

and angle at $A$ is common therefore triangle $AHE$ is similar to triangle $AGF$ Therefore

$HA$ $EA$ $FA$ $AG$,

therefore

$2AH$ $2AE$ $FA$ $AG$

But

$CD = 2AH$,

therefore

$FA$ $AG$ $CD$ $2AE$

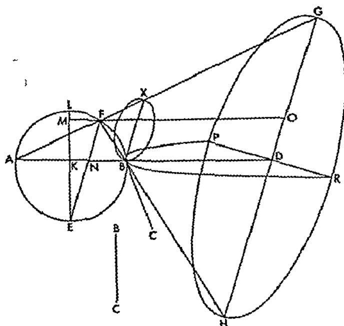
Then by things already shown in the forty ninth theorem (I 49) the straight line $CD$ is the upright side

## Proposition 54

*Given two bounded straight lines perpendicular to each other one of them being produced on the side of the right angle to find on the straight line produced the section of a cone called hyperbola in the same plane with the straight lines, so that the straight line produced is a diameter of the section and the point at the angle is the vertex, and where whatever straight line is dropped from the section to the diameter making an angle equal to the given angle will equal in square the rectangle applied to the other straight line having as breadth the straight line cut off by the dropped straight line beginning with the vertex and exceeding by a figure similar and similarly situated to that contained by the original straight lines*

Let there be the two bounded straight lines $AB$ and $BC$ perpendicular to

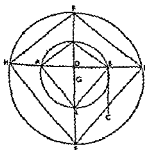
each other and let $AB$ be produced to $D$ it is required then to find in the plane through the lines $AB$, $BC$ an hyperbola whose diameter will be the straight




line $ABD$ and vertex $B$ and upright side the straight line $BC$ and where the straight lines dropped from the section to $BD$ at the given angle will equal in square the rectangles applied to $BC$ having as breadths the straight lines cut off by them from $B$ and exceeding by a figure similar and similarly situated to the rectangle $AB$ $BC$

First let the given angle be a right angle and on $AB$ let a plane be erected at right angles to the plane of reference and let the circle $AEBF$ be described in it about $AB$ so that the segment of the circle s diameter within the sector $AEB$ has to the segment of the diameter within the sector $AFB$ a ratio not greater than that of $AB$ to $BC$ [1] and let $AEB$ be bisected at $E$ and let the straight line

[1] Eutocius commenting, adds Let there be two straight lines $AB$ and $BC$ and let it be required to describe a circle on $AB$ so that its diameter is cut by $AB$ in such a way that the part of it on the side of $C$ has to the remainder a ratio not greater than that of $AB$ to $BC$




Now let it be supposed that they have the same ratio and let $AB$ be bisected at $D$ and through it let the straight line $EDF$ be drawn perpendicular to $AB$ and let it be contrived that $AB$ $BC$ $ED$ $DF$
and let $EF$ be bisected then it is clear that if

$$AB = BC$$

and

$$ED = DF$$

the point $D$ will be the midpoint of $EF$ and if

$$AB > BC$$

and

$$ED > DF$$

the midpoint will be below $D$ and if

$EK$ be drawn perpendicular from $E$ to the straight line $AB$ and let it be produced to $L$, therefore the straight line $EL$ is a diameter (Eucl III 1) If then

$$AB\ \ BC\ \ EK\ \ KL$$

we use point $L$, but if not, let it be contrived that

$$AB\ \ BC\ \ EK\ \ KM$$

with

$$KM < KL \text{ (Eucl v 8)},$$

and through $M$ let $MF$ be drawn parallel to $AB$, and let $A\Gamma$, $EF$, and $FB$ be joined, and through $B$ let $BX$ be drawn parallel to $FE$ Since then

$$\text{angle } AFE = \text{angle } EFB,$$

but

$$\text{angle } AFE = \text{angle } AXB,$$

and

$$\text{angle } KFB = \text{angle } XBF,$$

therefore also

$$\text{angle } XBF = \text{angle } FXB,$$

therefore also

$$FB = FX$$

Let a cone be conceived whose vertex is the point $F$ and whose base is the circle about diameter $BX$ at right angles to triangle $BFX$ Then the cone will be a right cone for

$$FB = FX$$

Then let the straight lines $BF$ $FX$ and $MF$ be produced, and let the cone be cut by a plane parallel to the circle $BX$ then the section will be a circle (I 4) Let it be the circle $GPR$ and so $GH$ will be the diameter of the circle (I 4 end) And let the straight line $PDR$ be the common section of circle $GH$ and of the plane of reference then $PDR$ will be perpendicular to both of the straight lines $GH$ and $DB$ for both of the circles $XB$ and $HG$ are perpendicular to triangle $FGH$ and the plane of reference is perpendicular to triangle $FGH$, and therefore their common section the straight line $PDR$ is perpendicular to triangle $FGH$ therefore it makes right angles also with all the straight lines touching it and in the same plane

---

$$AB < BC$$

it will be above $D$

And now let it be below as $G$ and with center $G$ and radius $GF$ let a circle be described, then it will have to pass either within or without the points $A$ and $B$ And if it should pass through the points $A$ and $B$ what was enjoined would be done but let it fall beyond the points $A$ and $B$ and let the straight line $AB$ produced both ways meet the circumference at $H$ and $K$ and let $FH$ $HF$ $EK$ and $KF$ be joined and let $MB$ be drawn through $B$ parallel to $FK$ and $BL$ parallel to $KF$ and let $MA$ and $AL$ be joined then these will also be parallel to $FH$ and $HE$ because $AD = DB$
and $DH = DK$
and $FDE$ is perpendicular to $HK$ And since the angle at $K$ is a right angle and $MB$ and $BL$ are parallel to $FK$ and $K\Gamma$ therefore the angle at $B$ is a right angle then for the same reasons also the angle at $A$ And so the circle described on $ML$ will pass through the points $A$ and $B$ (Eucl III 31) Let the circle $MALB$ be described And since $MB$ is parallel to $FK$,

$$FD\ \ DM\ \ KD\ \ DB$$

Then likewise also $KD\ \ DB\ \ FD\ \ DL$

And therefore $FD\ \ DM\ \ ED\ \ DL$

And alternately $ED\ \ DF\ \ AB\ \ BC\ \ LD\ \ DM$

And likewise if the circle described on $FE$ cuts $AB$ the same thing could be shown.

And since a cone whose base is circle $GH$ and vertex $F$, has been cut by a plane perpendicular to triangle $FGH$, and has also been cut by another plane the plane of reference, in the straight line $PDR$ perpendicular to the straight line $GDH$ and the common section of the plane of reference and of triangle $GFH$, that is the straight line $DB$, produced in the direction of $B$, meets the straight line $GF$ at $A$, therefore by things already shown before (I 12) the section $PBR$ will be an hyperbola whose vertex is the point $B$, and where the straight lines dropped ordinatewise to $BD$ will be dropped at a right angle, for they are parallel to straight line $PDR$ And since

$$AB \quad BC \quad EK \quad KM,$$

and

$$EK \quad KM \quad EN \quad NF \quad \text{rect } EN\,NF \quad \text{sq } NF,$$

therefore

$$AB \quad BC \quad \text{rect } EN\,NF \quad \text{sq } NF$$

And

$$\text{rect } EN\,NF = \text{rect } AN\,NB,$$

therefore

$$AB \quad CB \quad \text{rect } AN,NB \quad \text{sq } NF$$

But

$$\text{rect } AN,NB \quad \text{sq } NF \text{ comp } AN \quad NF\;BN \quad NF,$$

but

$$AN \quad NF \quad AD \quad DG \quad FO \quad OG,$$

and

$$BN \quad NF \quad FO \quad OH,$$

therefore

$$AB \quad BC \text{ comp } FO \quad OG, FO \quad OH,$$

that is

$$\text{sq } FO \quad \text{rect } OG\,OH$$

Therefore

$$AB \quad BC \quad \text{sq } FO \quad \text{rect } OG\,OH$$

And the straight line $FO$ is parallel to the straight line $AD$, therefore the straight line $AB$ is the transverse side and $BC$ the upright side, for these things have been shown in the twelfth theorem (I 12)

### Proposition 55 (Problem)

Then let the given angle not be a right angle and let there be the two given straight lines $AB$ and $AC$, and let the given angle be equal to angle $BAH$ then it is required to describe an hyperbola whose diameter will be the straight line $AB$, and upright side $AC$ and where the ordinates will be dropped at angle $HAB$

Let the straight line $AB$ be bisected at $D$ and let the semicircle $AFD$ be described on $AD$ and let some straight line $FG$ parallel to $AH$, be drawn to the semicircle making

$$\text{sq } FG \quad \text{rect } DG\,GA \quad AC \quad AB$$ [1]

[1] Eutocius, commenting gives this construction Let there be the semicircle $ABC$ on the diameter $AC$ and the given ratio $EF$ to $FG$ and let it be required to do what is proposed. Let $FH$ be made equal to $EF$ and let $HG$ be bisected at $K$ and let the straight line $CB$ be drawn in the semicircle at angle $ACB$ (the required angle) and from the center $L$ let the straight line $LS$ be drawn perpendicular to it and produced let it meet the circumference at

and let the straight line $FHD$ be joined and produced to $D$, and let

$$FD : DL = DL : DH,$$

and let $DK$ be made equal to $DL$ and let

$$\text{rect } LF, FM = \text{sq } AF$$

and let $KM$ be joined and through $L$ let $LN$ be drawn perpendicular to $KF$ and let it be produced towards $Y$ And with two given bounded straight lines $KL$ and $LN$ perpendicular to each other let an hyperbola be described whose transverse side is $KL$ and upright side $LN$ and where the straight lines dropped from the section to the diameter will be dropped at a right angle and will equal in square the rectangle applied to $LN$ having as breadths the straight lines cut off by them from $L$ and exceeding by a figure similar to rectangle $KL$, $LN$ (I 54) and the section will pass through $A$ for

$$\text{sq } AF = \text{rect } LF, FM \text{ (I 12)}$$

---

$N$ and through $K$ let $KM$ be drawn parallel to $CB$ therefore it will touch the circle And let it be contrived that $FH : HK = MX : XN$
and let $XO$ be made equal to $XK$ and let the straight lines $LX$ and $LO$ cutting the semicircle at $I$ and $P$ be joined and let the straight line $PRD$ be joined

Since then $XX = KO$
and $KL$ is common and perpendicular therefore

$$LO = LY$$

And also $LP = LR$

and therefore the remainders $PO = RX$

Therefore $PRD$ is parallel to $MO$ And

| | |
|---|---|
| | $FH : HK = MX : KX$ |
| and | $HK : HG = KX : XO$ |
| therefore *ex aequali* | $FH : HG = MX : XO$ |
| inversely | $HG : FH = XO : MX$ |
| *componendo* | $GF : FH = OM : MX$ |
| or | $CF : FE = PD : DR$ |
| And | $PD : DR = \text{rect } PD, DR : \text{sq } DP$ |
| but | $\text{rect } PD, DR = \text{rect } AD, DC$ (Eucl III 36) |
| therefore | $GF : FE = \text{rect } AD, DC : \text{sq } DR$ |
| Therefore inversely | $FE : GF = \text{sq } DK : \text{rect } AD, DC$ |

And $AH$ will touch it, for

$$\text{rect } FD, DH = \text{sq } DL \quad (\text{I } 37)$$

And so $AB$ is a diameter of the section (I 51) And since

$$CA \quad 2AD \text{ or } AB \quad \text{sq } \Gamma G \quad \text{rect } DG, GA,$$

but

$$CA \quad 2AD \text{ comp } CA \quad 2AH, 2AH \quad 2AD$$

or

$$CA \quad 2AD \text{ comp } CA \quad 2AH, AH \quad AD$$

and

$$AH \quad AD \quad FG \quad GD,$$

therefore

$$CA \quad AB \text{ comp } CA \quad 2AH, FG \quad GD$$

But also

$$\text{sq } FG \quad \text{rect } DG, GA \text{ comp } FG \quad GD, FG \quad GA,$$

therefore

$$\text{ratio comp } CA \quad 2AH, FG \quad GD = \text{ratio comp } FG \quad GA, FG \quad GD$$

Let the common ratio $FG \quad GD$ be taken away, therefore

$$CA \quad 2AH \quad FG \quad GA$$

But

$$FG \quad GA \quad OA \quad AX$$

therefore

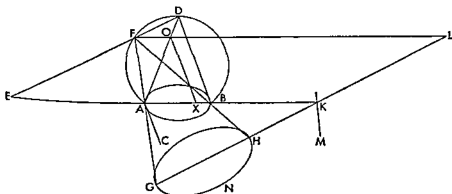
$$CA \quad 2AH \quad OA \quad AX$$

But whenever this is so the straight line $AC$ is a parameter for this has been shown in the fiftieth theorem (I 50)

### Proposition 56 (Problem)

*Given two bounded straight lines perpendicular to each other to find about one of them as diameter and in the same plane with the two straight lines the section of a cone called ellipse, whose vertex will be the point at the right angle and where the straight lines dropped ordinatewise from the section to the diameter at a given angle will equal in square the rectangles applied to the other straight line having as breadth the straight line cut off by them from the vertex of the section and defective by a figure similar and similarly situated to the rectangle contained by the given straight lines*

Let there be two given straight lines $AB$ and $AC$ perpendicular to each other,



of which the greater is the straight line $AB$, then it is required to describe in the plane of reference an ellipse whose diameter will be the straight line $AB$ and vertex $A$, and upright side $AC$, and where the ordinates will be dropped from the section to the diameter at a given angle and will equal in square the rectangles applied to $AC$ having as breadths the straight lines cut off by them from $A$ and defective by a figure similar and similarly situated to rectangle $BA, AC$

And first let the given angle be a right angle, and let a plane be erected from $AB$ at right angles to the plane of reference and in it on $AB$, let the sector of a circle $ADB$ be described and its midpoint be $D$ and let the straight lines $DA$ and $DB$ be joined and let the straight line $AX$ be made equal to $AC$, and through $X$ let the straight line $XO$ be drawn parallel to $DB$, and through $O$ let $OF$ be drawn parallel to $AB$, and let $DF$ be joined and let it meet $AB$ produced at $E$, then we will have

$$AB\ \ AC\quad AB\ \ AX\quad DA\ \ AO\quad DE\ \ EF$$

And let the straight lines $AF$ and $FB$ be joined and produced, and let some point $G$ be taken at random on $FA$, and through it let the straight line $GL$ be drawn parallel to $DE$ and let it meet $AB$ produced at $K$, then let $FO$ be produced and let it meet $GK$ at $L$ Since then

$$\text{arc } AD = \text{arc } DB,$$

$$\text{angle } ABD = \text{angle } DFB \text{ (Eucl III 27)}$$

And since

$$\text{angle } EFA = \text{angle } FDA + \text{angle } FAD,$$

but

$$\text{angle } FAD = \text{angle } FBD,$$

and

$$\text{angle } FDA = \text{angle } FBA,$$

therefore also

$$\text{angle } EFA = \text{angle } DBA = \text{angle } DFB$$

And also $DE$ is parallel to $LG$ therefore

$$\text{angle } EFA = \text{angle } FGH,$$

and

$$\text{angle } DFB = \text{angle } FHG$$

And so also

$$\text{angle } FGH = \text{angle } FHG,$$

and

$$FG = FH$$

Then let circle $GHN$ be described about $HG$ at right angles to triangle $HGF$, let a cone be conceived whose base is the circle $GHN$, and whose vertex is the point $F$, then the cone will be a right cone because

$$FG = FH$$

And since the circle $GHN$ is at right angles to plane $HGF$ and the plane of reference is also at right angles to the plane through $GH$ and $HF$ therefore their common section will be at right angles to the plane through $GH$ and $HF$ Then let their common section be the straight line $KM$ therefore the straight line $KM$ is perpendicular to both of the straight lines $AK$ and $KG$

And since a cone whose base is the circle $GHN$ and whose vertex is the point $F$ has been cut by a plane through the axis and makes as a section the triangle $GHF$, and has been cut also by another plane through $AK$ and $KM$,

which is the plane of reference in the straight line $KM$ which is perpendicular to $GK$ and the plane meets the sides of the cone $FG$ and $\Gamma H$, therefore the resulting section is an ellipse whose diameter is $AB$ and where the ordinates will be dropped at a right angle (I 13), for they are parallel to $KM$ And since

$$DE \quad EF \quad \text{rect } DE\ EF \text{ or rect } BE, EA \quad \text{sq } EF,$$

and

$$\text{rect } BE, EA \quad \text{sq } EF \text{ comp } BE \quad EF, AE \quad EF,$$

but

$$BE \quad EF \quad BK \quad KH,$$

and

$$AE \quad EF \quad AK \quad KG \quad FL \quad LG,$$

therefore

$$BA \quad AC \text{ comp } FL \quad LG\ FL \quad LH \text{ (see above)},$$

which is the same as

$$\text{sq } FL \quad \text{rect } GL, LH$$

therefore

$$BA \quad AC \quad \text{sq } FL \quad \text{rect } GL\ LH$$

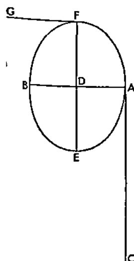
And whenever this is so, the straight line $AC$ is the upright side of the figure as has been shown in the thirteenth theorem (I 13)

## Proposition 57 (Problem)

With the same things supposed let the straight line $AB$ be less than $AC$, and let it be required to describe an ellipse about diameter $AB$ so that $AC$ is the upright

Let $AB$ be bisected at $D$ and from $D$ let the straight line $EDF$ be drawn perpendicular to $AB$, and let

$$\text{sq } FE = \text{rect } BA\ AC$$




so that

$$\Gamma D = DE,$$

and let $FG$ be drawn parallel to $AB$, and let it be contrived that

$$AC \quad AB \quad EF \quad FG,$$

therefore also

$$EF > FG$$

And since

$$\text{rect } CA, AB = \text{sq } EF,$$

hence

$$CA \quad AB \quad \text{sq } FE \quad \text{sq } AB \quad \text{sq } DF \quad \text{sq } DA$$

But

$$CA \quad AB \quad EF \quad FG$$

therefore

$$EF \quad FG \quad \text{sq } FD \quad \text{sq } DA$$

But

$$\text{sq } FD = \text{rect } FD\ DE,$$

therefore

$$EF \quad FG \quad \text{rect } ED\ DF \quad \text{sq } AD$$

Then with two bounded straight lines situated at right angles to each other and with $EF$ greater let an ellipse be described whose diameter is $EF$ and upright side $FG$ (I 56) then the section will pass through $A$ because

$$\text{rect } FD, DE \ \ \text{sq } DA \quad EF \ \ FG \ (\text{I } 21)$$

And

$$AD = DB$$

then it will also pass through $B$ Then an ellipse has been described about $AB$ And since

$$CA \ \ AB \quad \text{sq } FD \ \ \text{sq } DA,$$

and

$$\text{sq } DA = \text{rect } AD, DB,$$

therefore

$$CA \ \ AB \quad \text{sq } DF \ \ \text{rect } AD, DB$$

And so the straight line $AC$ is an upright side (I 21)

### Proposition 58 (Problem)

But then let the given angle not be a right angle and let the angle $BAD$ be equal to it and let the straight line $AB$ be bisected at $E$, and let the semicircle $AFE$ be described on $AE$, and in it let the straight line $FG$ be drawn parallel to $AD$ making

$$\text{sq } FG \ \ \text{rect } AG\,GE \quad CA \ \ AB$$[1]

and let the straight lines $AF$ and $EF$ be joined and produced, and let

$$DE \ \ EH \quad EH \ \ EF,$$

[1]Eutocius commenting gives this construction Let there be the semicircle $ABC$ and within it some straight line $AB$ (at the required angle to $AC$) and let two unequal straight lines $DE$ and $EF$ be laid down, and let $EF$ be produced to $G$ and let $FG$ be made equal to $DE$ and let the whole line $EG$ be bisected at $H$ and let the center of the circle $K$ be taken and from it let a perpendicular be drawn to $AB$ and let it meet the circumference at $L$ and through $L$ let $LM$ be drawn parallel to $AB$ and let $KA$ produced meet $LM$ at $M$ and let it be contrived that $HF \ \ FG \quad LM \ \ MN$
and let $LX = LN$
and let the straight lines $NK$ and $KX$ be joined and produced and let the circle finished out cut them at $P$ and $O$ and let the straight line $ORP$ be joined

| | | |
|---|---|---|
| Since then | $FH \ \ FG$ | $LM \ \ MN$ |
| *componendo* | $HG \ \ GF$ | $LN \ \ NM$ |
| inversely | $FG \ \ GH$ | $NM \ \ NL$ |
| and | $FG \ \ GE$ | $MN \ \ NX$ |
| *separando* | $FG \ \ FE$ | $NM \ \ MX$ |

And since $NL = LX$
and the straight line $LK$ is common and at right angles therefore also

$$KN = KX$$

And also $KO = KP$
therefore $NX$ is parallel to $OP$ Therefore triangle $KMN$ is similar to triangle $OKR$ and triangle $KMX$ to triangle $PRK$

| | | |
|---|---|---|
| Therefore | $KM \ \ KP$ | $MN \ \ RO$ |
| But also | $KM \ \ KR$ | $MX \ \ PR$ |
| and therefore | $NM \ \ RO$ | $MX \ \ PR$ |
| and alternately | $NM \ \ MX$ | $PO \ \ RP$ |

But $NM \ \ MX \quad GF \ \ FE \quad DE \ \ EF$
and $OR \ \ RP \quad \text{sq } OR \ \ \text{rect } OR\,RP$
and therefore

$$DE \ \ EF \quad \text{sq } OR \ \ \text{rect } OR\,RP$$

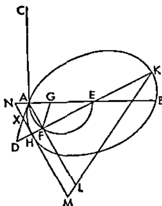
And rect $OR\,RP$ = rect $AR\,RC$ (Eucl III 35)
Therefore $DE \ \ EF \quad \text{sq } OR \ \ \text{rect } AR\,RC$

and let

$$EK = EH$$

and let it be contrived that

$$\text{rect } HF, FL = \text{sq } A\Gamma,$$




and let the straight line $KL$ be joined and from $H$ let the straight line $HMX$ be drawn perpendicular to $HF$ and so parallel to the straight line $AFL$ for the angle at $F$ is right And with the two given bounded straight lines $KH$, and $HM$ perpendicular to each other, let an ellipse be described whose transverse diameter is $KH$, and the upright side of whose figure is $HM$, and where the ordinates to $HK$ will be dropped at right angles (I 56–57) then the section will pass through $A$ because sq $FA$ = rect $HF, FL$ (I 13) And since

$$HE = EK,$$

and

$$AE = EB,$$

the section will also pass through $B$, and $E$ will be the center, and the straight line $AEB$ the diameter And the straight line $DA$ will touch the section because

$$\text{rect } DE\ EF = \text{sq } EH$$

And since

$$CA\ AB \quad \text{sq } FG \quad \text{rect } AG, GE$$

but

$$CA \quad AB \text{ comp } CA \quad 2AD\ 2AD \quad AB \text{ or } DA \quad AE,$$

and

$$\text{sq } FG = \text{rect } AG, GE \text{ comp } FG \quad GE\ FG \quad GA,$$

therefore

$$\text{ratio comp } CA \quad 2AD\ DA \quad AE = \text{ratio comp } FG \quad GE\ FG \quad GA$$

But

$$DA \quad AE \quad FG \quad GE$$

and the common ratio being taken away we will have

$$CA \quad 2AD \quad FG \quad GA$$

or

$$CA \quad 2AD \quad XA \quad AN$$

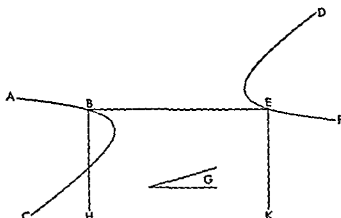
And whenever this is so the straight line $AC$ is the upright side of the figure (I 50)

## Proposition 59 (Problem)

*Given two bounded straight lines perpendicular to each other to find opposite sections whose diameter is one of the given straight lines and whose vertex is the ends of the straight line and where the straight lines dropped in each of the sections at a given angle will equal in square the rectangles applied to the other of the straight lines and exceeding by a figure similar to the rectangle contained by the given straight lines*

Let there be the two given bounded straight lines $BE$ and $BH$ perpendicular to each other and let the given angle be $G$ then it is required to describe

opposite sections about one of the straight lines $BE$ and $BH$, so that the ordinates are dropped at an angle $G$




And given the two straight lines $BE$ and $BH$ let an hyperbola be described whose transverse diameter will be the straight line $BE$ and the upright side of whose figure will be $HB$ and where the ordinates to $BE$ produced will be at an angle $G$, and let it be the line $ABC$ for we have already described how this must be done (I 55) Then let the straight line $EK$ be drawn through $E$ perpendicular to $BE$ and equal to $BH$, and let another hyperbola $DEF$ be likewise described whose diameter is $BE$ and the upright side of whose figure is $EK$ and where the ordinates from the section will be dropped at a same angle $G$ Then it is evident that $B$ and $E$ are opposite sections, and there is one diameter for them, and their uprights are equal

## Propositio 60 (Problem)

*Given two straight lines bisecting each other to describe about each of them opposite sections so that the straight lines are their conjugate diameters and the diameter of one pair of opposite sections is equal in square to the figure of the other pair and likewise the diameter of the second pair of opposite sections is equal in square to the figure of the first pair*

Let there be the two given straight lines $AC$ and $DE$ bisecting each other then it is required to describe opposite sections about each of them as a diameter so that the straight lines $AC$ and $DE$ are conjugates in them and $DE$ is equal in square to the figure about $AC$ and $AC$ is equal in square to the figure about $DE$

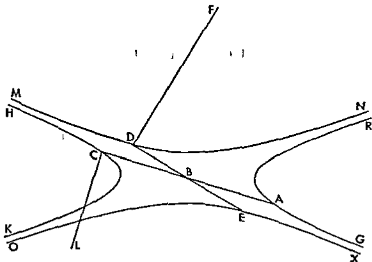
Let

$$\text{rect } AC\ CL = \text{sq } DE$$

and let $LC$ be perpendicular to $CA$ And given two straight lines $AC$ and $CL$ perpendicular to each other let the opposite sections $RAG$ and $HCK$ be described whose transverse diameter will be $CA$ and whose upright side will be $CL$ and where the ordinates from the sections to $CA$ will be dropped at the given angle (I 59) Then the straight line $DE$ will be a second diameter of the opposite sections (See Def I 11), for it is the mean proportion between the sides of the figure, and parallel to an ordinate it has been bisected at $B$

Then again let

$$\text{rect } DE, DF = \text{sq } AC,$$




and let $DF$ be perpendicular to $DE$ And given two straight lines $ED$ and $DF$ lying perpendicular to each other, let the opposite sections $MDN$ and $OEX$ be described whose transverse diameter will be $DE$ and the upright side of whose figure will be $DF$, and where the ordinates from the sections will be dropped to $DE$ at the given angle (I 59) then the straight line $AC$ will also be a second diameter of the sections $MDN$ and $XEO$ And so $AC$ bisects the parallels to $DE$ between the sections $RAG$ and $HCK$ and $DE$ the parallels to $AC$, and this it was required to do

And let such sections be called conjugate

# BOOK TWO

APOLLONIUS to EUDEMUS

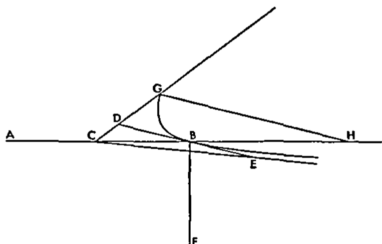
If you are well well and good and I too fare pretty well

I have sent you my son Apollonius bringing you the second book of the conics as arranged by us Go through it then carefully and acquaint those with it worthy of sharing in such things And Philonides the geometer, I introduced to you in Ephesus, if ever he happen about Pergamum, acquaint him with it too And take care of yourself, to be well Good bye

PROPOSITION 1

*If a straight line touch an hyperbola at its vertex and from it on both sides of the diameter a straight line is cut off equal to the straight line equal in square to the fourth of the figure, then the straight lines drawn from the center of the section to the ends thus taken on the tangent will not meet the section*

Let there be an hyperbola whose diameter is the straight line $AB$ and center $C$ and upright the straight line $BF$, and let the straight line $DE$ touch the




section at $B$ and let the squares on $BD$ and $BE$ each be equal to the fourth of the figure $AB$ $BF$ and the straight lines $CD$ and $CE$ be joined and produced

I say that they will not meet the section

For if possible, let $CD$ meet the section at $G$ and from $G$ let the straight line $GH$ be dropped ordinatewise therefore it is parallel to $DB$ (I 17) Since then

$$AB \quad BF \quad \text{sq } AB \quad \text{rect } AB\ BF,$$

but

sq $CB$ = fourth sq $AB$,

and

sq $BD$ = fourth rect $AB$ $BF$,

therefore

$AB$ $BF$ sq $CB$ sq $DB$ sq $CH$ sq $HG$

And also

$AB$ $BF$ rect $AH$ $HB$ sq $HG$ (I 21),

therefore

sq $CH$ sq $HG$ rect $AH, HB$ sq $HG$

Therefore

rect $AH$ $HB$ = sq $CH$

and this is absurd (Eucl II 6) Therefore the straight line $CD$ will not meet the section Then likewise we could show that neither does $CE$ therefore the straight lines $CD$ and $CE$ are asymptotes (ἀσύμπτωτοι)[1] to the section

PROPOSITION 2

*With the same things it is to be shown that a straight line cutting the angle contained by the straight lines DC and CE is not another asymptote*

For if possible let $CH$ be it and let the straight line $BH$ be drawn through $B$ parallel to $CD$ and let it meet $CH$ at $H$ and let $DG$ be made equal to $BH$ and

let $GH$ be joined and produced to the points $K$ $L$ and $M$ Since then $BH$ and $DG$ are equal and parallel $DB$ and $HG$ are also equal and parallel And since $AB$ is bisected at $C$ and a straight line $BL$ is added to it

rect $AL$ $LB$ + sq $CB$ = sq $CL$ (Eucl II 6)

Likewise then since $GM$ is parallel to $DE$ and

$DB = BE$

[1] The word ἀσύμπτωτος means literally not capable of meeting' and is used in a general way in Euclid to refer to any non secant lines or planes In Apollonius it is also used in this way as for instance in II 14 porism, where it refers to any straight lines not meeting the hyperbola The special case where in English the lines are spoken of as asymptotes is the one defined here Book II proposition 14 porism further declares their peculiar property and significance

therefore also

$$GL = LM$$

And since

$$GH = DB,$$

therefore

$$GK > DB$$

And also

$$KM > BE,$$

since also

$$LM > BE$$

therefore

rect $MK, KG >$ rect $DB$ $BE$
$> $ sq $DB$

Since then

$AB$ $BF$ sq $CB$ sq $BD$ (II 1),

but

$AB$ $BF$ rect $AL$ $LB$ sq $LK$ (I 21),

and

sq $CB$ sq $BD$ sq $CL$ sq $LG$,

therefore also

sq $CL$ sq $LG$ rect $AL$ $LB$ sq $LK$

Since then

whole sq $LC$ whole sq $LG$
part subtr rect $AL, LB$ part subtr sq $LK$,

therefore also

sq $LC$ sq $LG$ remainder sq $CB$ remainder rect $MK$ $KG$

that is

sq $CB$ rect $MK$ $KG$ sq $CB$ sq $DB$

Therefore

sq $DB =$ rect $MK$ $KG$,

and this is absurd for it has been shown to be greater than it Therefore the straight line $CH$ is not an asymptote to the section

## Proposition 3

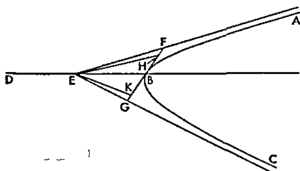
*If a straight line touches an hyperbola it will meet both of the asymptotes and it will be bisected at the point of contact, and the square on each of its segments will be equal to the fourth of the figure resulting on the diameter drawn through the point of contact*

Let there be the hyperbola $ABC$ and its center $E$, and asymptotes $FE$ and $EG$ and let some straight line $HK$ touch it at $B$

I say that the straight line $HK$ produced will meet the straight lines $FE$ and $EG$

For if possible, let it not meet them and let $EB$ be joined and produced and let $ED$ be made equal to $EB$ therefore the straight line $BD$ is a diameter Then let the squares on $HB$ and $BK$ each be made equal to the fourth of the figure on $BD$, and let $EH$ and $EK$ be joined Therefore they are asymptotes (II 1) and this is absurd (II 2) for $FE$ and $EG$ are supposed asymptotes Therefore $KH$ produced will meet the asymptotes $EF$ and $EG$ at $F$ and $G$

I say then also that the squares on $B\Gamma$ and $BG$ will each be equal to the fourth of the figure on $BD$

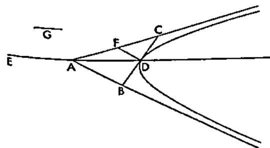



For let it not be, but if possible let the squares on $BH$ and $BK$ each be equal to the fourth of the figure Therefore $HE$ and $EK$ are asymptotes (II 1), and this is absurd (II 2) Therefore the squares on $FB$ and $BG$ will each be equal to the fourth of the figure on $BD$

### Proposition 4 (Problem)

*Given two straight lines containing an angle and a point within the angle to describe through the point the section of a cone called hyperbola so that the given straight lines are its asymptotes*

Let there be the two straight lines $AC$ and $AB$ containing a chance angle at $A$ and let some point $D$ be given and let it be required to describe through $D$ an hyperbola to the asymptotes $CA$ and $AB$




Let the straight line $AD$ be joined and produced to $E$ and let $AE$ be made equal to $DA$ and let the straight line $DF$ be drawn through $D$ parallel to $AB$, and let $FC$ be made equal to $AF$ and let $CD$ be joined and produced to $B$ and let it be contrived that

$$\text{rect } DE,G = \text{sq } CB$$

and with $AD$ extended let an hyperbola be described about it through $D$ so that the ordinates equal in square the areas applied to $G$ and exceeding by a figure similar to rectangle $DE$ $G$ Since then $DF$ is parallel to $BA$, and

$$CF = FA$$

therefore

$$CD = DB$$

and so

$$\text{sq } CB = 4 \text{ sq } CD$$

And

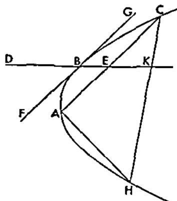
$$\text{sq } CB = \text{rect } DE,G$$

therefore the squares on $CD$ and $DB$ are each equal to the fourth part of the figure $DE$, $G$ Therefore the straight lines $AB$ and $AC$ are asymptotes to the hyperbola described

### Proposition 5

*If the diameter of a parabola or hyperbola bisects some straight line, the tangent to the section at the end of the diameter will be parallel to the bisected straight line*

Let there be the parabola or hyperbola $ABC$ whose diameter is the straight line $DBE$ and let the straight line $FBG$ touch the section and let some straight line $AEC$ be drawn in the section making $AE$ equal to $EC$




I say that $AC$ is parallel to $FG$

For if not let the straight line $CH$ be drawn through $C$ parallel to $FG$ and let $HA$ be joined Since then $ABC$ is a parabola or hyperbola whose diameter is $DE$, and tangent $FG$, and $CH$ is parallel to it therefore

$$CK = KH \quad (\text{I } 46,47)$$

But also

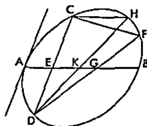
$$CE = EA$$

Therefore $AH$ is parallel to $KE$, and this is absurd, for produced it meets $BD$ (I 22)

### Proposition 6

*If the diameter of an ellipse or circumference of a circle bisects some straight line not through the center, the tangent to the section at the end of the diameter will be parallel to the bisected straight line*

Let there be an ellipse or circumference of a circle whose diameter is the straight line $AB$ and let $AB$ bisect $CD$ a straight line not through the center at the point $E$




I say that the tangent to the section at $A$ is parallel to $CD$

For let it not be but if possible, let $DF$ be parallel to the tangent at $A$ therefore

$$DG = FG$$

But also

$$DE = EC$$

therefore $CF$ is parallel to $GE$ and this is absurd For if $G$ is the center of the section $AB$ the straight line $CF$ will meet the straight line $AB$ (I 23), and if it is not suppose it to be $K$ and let $DK$ be joined and produced to $H$ and let $CH$ be joined Since then

$$DK = KH,$$

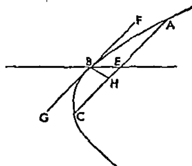
and also

$$DE = EC$$

therefore $CH$ is parallel to $AB$ But also $CF$ and this is absurd Therefore the tangent at $A$ is parallel to $CD$

PROPOSITION 7

*If a straight line touches a section of a cone or circumference of a circle and a parallel to it is drawn in the section and bisected the straight line joined from the point of contact to the midpoint will be a diameter of the section*




Let there be a section of a cone or circumference of a circle $ABC$, and $FG$ tangent to it, and $AC$ parallel to $FG$ and bisected at $E$, and let $BE$ be joined

I say that $BE$ is a diameter of the section

For let it not be, but if possible, let $BH$ be a diameter of the section Therefore

$$AH = HC \text{ (First Def I 4)},$$

and this is absurd, for

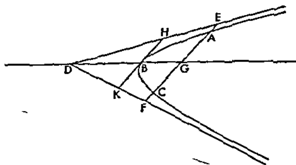
$$AE = EC$$

Therefore $BH$ will not be a diameter of the section Then likewise we could show that there is no other than $BE$

PROPOSITION 8

*If a straight line meets an hyperbola in two points produced both ways it will meet the asymptotes and the straight lines cut off on it by the section from the asymptotes will be equal*

Let there be the hyperbola $ABC$ and the asymptotes $ED$ and $DF$, and let some straight line $AC$ meet $ABC$




I say that produced both ways it will meet the asymptotes

Let $AC$ be bisected at $G$ and let $DG$ be joined Therefore it is a diameter of the section (II 7) therefore the tangent at $B$ is parallel to $AC$ (II 5 6) Then let $HBK$ be the tangent (I 32), then it will meet $ED$ and $DF$ (II 3) Since then $AC$ is parallel to $KH$ and $KH$ meets $DK$ and $DH$ therefore also $AC$ will meet $DE$ and $DF$

Let it meet them at $E$ and $F$ and

$$HB = BK \text{ (II 3)},$$

therefore also

$$FG = GE$$

And so also

$$CF = AE$$

PROPOSITION 9

*If a straight line meeting the asymptotes is bisected by the hyperbola, it will touch the section in one point only*

For let the straight line $CD$ meeting the asymptotes $CA, AD$ be bisected by the hyperbola at the point $E$

I say that it touches the hyperbola at no other point

For if possible, let it touch it at $B$ Therefore

$CE = BD$ (II 8),

and this is absurd for $CE$ is supposed equal to $ED$ Therefore it will not touch the section at another point

PROPOSITION 10

*If some straight line cutting the section meet both of the asymptotes the rectangle contained by the straight lines cut off between the asymptotes and the section is equal to the fourth of the figure resulting on the diameter bisecting the straight lines drawn parallel to the drawn straight line*

Let there be the hyperbola $ABC$ and let $DE, LF$ be its asymptotes, and let some straight line $DF$ be drawn cutting the section and the asymptotes and let $AC$ be bisected at $G$, and let $GE$ be joined, and let $EH$ be made equal to $BE$, and let $BM$ be drawn from $B$ perpendicular to $HEB$, therefore $BH$ is a diameter (II 7), and $BM$ the upright side

I say that

rect $DA$ $AF$ = fourth rect $HB, BM$,

then likewise also

rect $DC$ $CF$ = fourth rect $HB$ $BM$

For let $KL$ be drawn through $B$ tangent to the section therefore it is parallel to $DF$ (II 6) And since it has been shown

$HB$ $BM$ sq $EB$ sq $BK$ sq $FG$ sq $GD$ (II 1 3)

and

$HB$ $BM$ rect $HG$ $GB$ sq $GA$ (I 21),

therefore

$$\text{sq } EG \quad \text{sq } GD \quad \text{rect } HG,GB \quad \text{sq } GA$$

Since then

$$\text{whole sq } EG \quad \text{whole sq } GD$$

$$\text{part subtr rect } HG,GB \quad \text{part subtr sq } AG,$$

therefore also

$$\text{remainder sq } EB \quad \text{remainder rect } DA\ AF \quad \text{sq } EG \quad \text{sq } GD,$$

or

$$\text{remainder sq } EB \quad \text{remainder rect } DA,AF \quad \text{sq } EB \quad \text{sq } BK$$

Therefore

$$\text{rect } FA\ AD=\text{sq } BK$$

Then likewise it could be shown also that

$$\text{rect } DC,CF=\text{sq } BL,$$

therefore also

$$\text{rect } FA,AD=\text{rect } DC\ CF$$

## Proposition 11

*If some straight line cut each of the straight lines containing the angle adjacent to the angle containing the hyperbola, it will meet the section in one point only and the rectangle contained by the straight lines cut off between the containing straight lines and the section will be equal to the fourth part of the square on the diameter drawn parallel to the cutting straight line*

Let there be an hyperbola whose asymptotes are $CA,AD$ and let $DA$ be produced to $E$ and through some point $E$ let $EF$ be drawn cutting $EA$ and $AC$

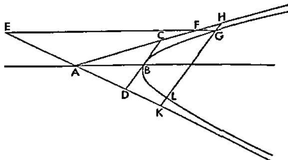
Now it is evident that it meets the section in one point only for the straight line drawn through $A$ parallel to $EF$ as $AB$ will cut angle $CAD$ and will meet the section (II 2) and be its diameter (I 50), therefore $EF$ will meet the section in one point only (I 26)

Let it meet it at $G$

I say then also that

$$\text{rect } EG\ GF=\text{sq } AB$$

For let the straight line $HGLK$ be drawn ordinatewise through $G$, therefore




the tangent through $B$ is parallel to $GH$ (II 5) Let it be $CD$ Since then

$$CB=BD \text{ (II 3)},$$

therefore

$$\text{sq } CB \text{ or rect } CB\ BD \quad \text{sq } BA \text{ comp } CB \quad BA\ DB \quad BA$$

But

$$CB \quad BA \qquad HG \quad G\Gamma,$$

and

$$DB \quad BA \qquad GK \quad GE$$

therefore

$$\text{sq } CB \quad \text{sq } B1 \text{ comp } HG \quad G\Gamma, KG \quad GE$$

But also

$$\text{rect } KG,GH \quad \text{rect } EG\, GF \text{ comp } HG \quad GF, KG \quad GE,$$

therefore

$$\text{rect } KG\, GH \quad \text{rect } EG\, GF \qquad \text{sq } CB \quad \text{sq } BA$$

Alternately

$$\text{rect } KG,GH \quad \text{sq } CB \qquad \text{rect } EG,G\Gamma \quad \text{sq } BA$$

But it was shown

$$\text{rect } KG,GH = \text{sq } CB \text{ (II 10)},$$

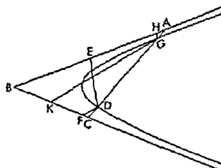
therefore also

$$\text{rect } EG\, GF = \text{sq } AB$$

PROPOSITION 12

*If two straight lines at chance angles are drawn to the asymptotes from some point of those on the ection and parallels are drawn to them from some point of those on the section then the rectangle contained by the parallels will be equal to that contained by those straight lines to which they were drawn parallel*

Let there be an hyperbola whose asymptotes are $AB$ and $BC$ and let some point $D$ be taken on the section, and from it let $DE$ and $DF$ be dropped to $AB$ and $BC$ and let some other point on the section $G$ be taken and through $G$ let $GH$ and $GK$ be drawn parallel to $ED$ and $DF$




I say that

$$\text{rect } ED\, DF = \text{rect } HG,GK$$

For let $DG$ be joined and produced to $A$ and $C$ Since then

$$\text{rect } AD\, DC = \text{rect } AG,GC \text{ (II 8)}$$

therefore

$$AG \quad AD \qquad DC \quad CG$$

But

$$AG \quad AD \qquad GH \quad ED,$$

and

$$DC \quad CG \qquad DF \quad GK$$

therefore

$$GH \quad DE \qquad DF \quad GK$$

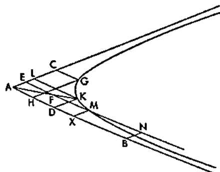
Therefore

$$\text{rect } ED\, D\Gamma = \text{rect } HG\, GK$$

PROPOSITION 13

*If in the place bounded by the asymptotes and the section some straight line is drawn parallel to one of the asymptotes, it will meet the section in one point only*

Let there be an hyperbola whose asymptotes are $CA$ and $AB$, and let some point $E$ be taken and through it let $EF$ be drawn parallel to $AB$




I say that it will meet the section

For if possible, let it not meet it and let some point $G$ on the section be taken and through $G$ let $GC$ and $GH$ be drawn parallel to $CA$ and $AB$ and let

$$\text{rect } CG, GH = \text{rect } AE\ EF$$

and let $AF$ be joined and produced, then it will meet the section (I 2) Let it meet it at $K$, and through $K$ parallel to $CA$ and $AB$ let $KL$ and $KD$ be drawn, therefore

$$\text{rect } CG, GH = \text{rect } LK\ KD \text{ (II 12)}$$

And it is supposed that also

$$\text{rect } CG, GH = \text{rect } AE\ EF$$

therefore

$$\text{rect } LK\ KD \text{ or rect } KL\ LA = \text{rect } AE, EF$$

and this is impossible for both

$$KL > EF$$

and

$$LA > AE$$

Therefore $EF$ will meet the section

Let it meet it at $M$

I say then that it will not meet it at any other point

For if possible let it also meet it at $N$ and through $M$ and $N$ let $MX$ and $NB$ be drawn parallel to $CA$ Therefore

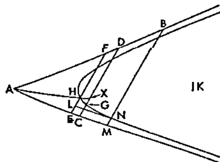
$$\text{rect } EM, MX = \text{rect } EN, NB \text{ (II 12)},$$

and this is impossible Therefore it will not meet the section in another point

## Proposition 14

*The asymptotes and the section if produced indefinitely draw nearer to each other and they reach a distance less than any given distance*

Let there be an hyperbola whose asymptotes are $AB$ and $AC$ and a given distance $K$




I say that $AB$ and $AC$ and the section if produced, draw nearer to each other and will reach a distance less than $K$

For let $EHF$ and $CGD$ be drawn parallel to the tangent and let $AH$ be joined and produced to $X$ Since then

$$\text{rect } CG, GD = \text{rect } FH\ HE \text{ (II 10)},$$

therefore

$$DG \quad FH \quad HE \quad CG$$

But

$$DG > FH \text{ (I 8, 26)},$$

therefore also

$$HE > CG$$

Then likewise we could show that the succeeding straight lines are less

Then let the distance $EL$ be taken less than $K$ and through $L$ let $LN$ be drawn parallel to $AC$ therefore it will meet the section (II 13) Let it meet it at $N$, and through $N$ let $MNB$ be drawn parallel to $EF$ Therefore

$$MN = EL$$

and so

$$MN < K$$

PORISM

Then from this it is evident that the straight lines $AB$ and $AC$ are nearer than all the asymptotes to the section and the angle contained by $BA, AC$ is clearly less than that contained by other asymptotes to the section

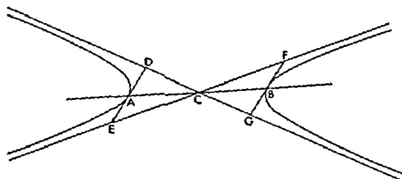
PROPOSITION 15

*The asymptotes of opposite sections are common*

Let there be opposite sections whose diameter is $AB$ and center $C$

I say that the asymptotes of the sections $A$ and $B$ are common

Let the straight lines $DAE$ and $FBG$ be drawn tangent to the sections through the points $A$ and $B$ they are therefore parallel (I 44, note) Then let




each of the straight lines $DA$ $AE$ $FB$ and $BG$ be cut off equal in square to the fourth of the figure applied to $AB$ therefore

$$DA = AE = FB = BG$$

Then let $CD$ $CE$ $CF$ and $CG$ be joined Then it is evident that $DC$ is in a straight with $CG$ and $CE$ with $CF$ because of the parallels Since then it is an hyperbola whose diameter is $AB$ and tangent $DE$ and $DA$ and $AE$ are each equal in square to the fourth of the figure applied to $AB$ therefore $DC$ and $CE$ are asymptotes (II 1) For the same reasons then $FC$ and $CG$ are also asymptotes to section $B$ Therefore the asymptotes of opposite sections are common

PROPOSITION 16

*If in opposite sections some straight line is drawn cutting each of the straight lines containing the angle adjacent to the angles containing the sections, it will meet*

*each of the opposite sections in one point only, and the straight lines cut off on it by the sections from the asymptotes will be equal*

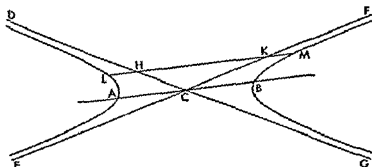
For let there be the opposite sections $A$ and $B$ whose center is $C$ and asymptotes $DCG$ and $ECF$, and let some straight line $HK$ be drawn through cutting each of the straight lines $DC$ and $CF$

I say that produced it will meet each of the sections in one point only

For since $DC$ and $CE$ are asymptotes of section $A$, and some straight line $HK$ has been drawn across cutting both of the straight lines containing the adjacent angle $DCF$, therefore $HK$ produced will meet the section (II 11) Then likewise also $B$

Let it meet them at $L$ and $M$

Let the straight line $ACB$ be drawn through $C$ parallel to $LM$, therefore




$$\text{rect } KL\ LH = \text{sq } AC \text{ (II 11)}$$

and

$$\text{rect } HM\ MK = \text{sq } CB \text{ (II 11)}$$

And so also

$$\text{rect } KL\ LH = \text{rect } HM, MK,$$

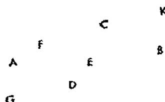
and

$$LH = KM$$

Proposition 17

*The asymptotes of conjugate opposite sections are common*

Let there be conjugate opposite sections whose conjugate diameters are $AB$ and $CD$, and whose center is $E$



I say that their asymptotes are common

For let the straight lines $FAG$, $GDH$, $HBK$, and $KCF$ be drawn through the points $A$, $B$, $C$, and $D$ touching the sections, therefore $FGHK$ is a parallelogram (I 44, note) Then let $FEH$ and $KEG$ be joined therefore they are straight lines (II 15) and diagonals of the parallelogram, and they are all bisected at the point $E$ And since the figure on $AB$ is equal to the square on $CD$ (I 60), and

$$CE = ED,$$

therefore each of the squares on $FA$ $AG$, $KB$, and $BH$ is equal to a fourth of the figure on $AB$ Therefore the straight lines $FEH$ and $KEG$ are asymptotes of the sections $A$ and $B$ (II 1) Then likewise we could show that the same straight lines are also asymptotes of the sections $C$ and $D$ Therefore the asymptotes of conjugate opposite sections are common

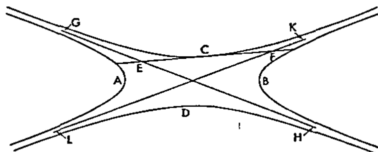
PROPOSITION 18

*If a straight line meeting one of the conjugate opposite sections when produced both ways, falls outside the section, it will meet both of the adjacent sections in one point only*

Let there be the conjugate opposite sections $A$, $B$, $C$, and $D$, and let some straight line $EF$ meet the section $C$ and produced both ways fall outside the section

I say that it will meet both of the sections $A$ and $B$ in one point only

For let $GH$ and $KL$ be asymptotes of the sections Therefore $EF$ meets both




$GH$ and $KL$ (II 3) Then it is evident that it will also meet the sections $A$ and $B$ in one point only (II 16)

PROPOSITION 19

*If some straight line is drawn touching some one of the conjugate opposite sections at random, it will meet the adjacent sections and will be bisected at the point of contact*

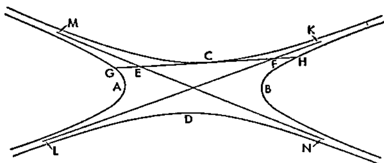
Let there be the conjugate opposite sections $A$, $B$, $C$ and $D$, and let some straight line $ECF$ touch it at $C$

I say that produced it will meet sections $A$ and $B$ and will be bisected at $C$

It is evident now that it will meet sections $A$ and $B$ (II 18), let it meet them at $G$ and $H$

I say that

$$CG = CH$$




For let the asymptotes of the sections $KL$ and $MN$ be drawn Therefore

$$EG = FH \text{ (II 16)},$$

and

$$CE = C\Gamma \text{ (II 3)},$$

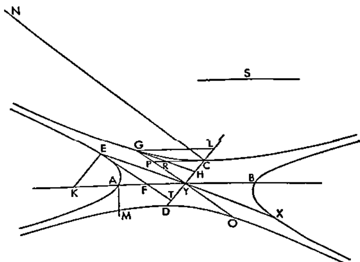
and

$$CG = CH$$

PROPOSITION 20

*If a straight line touches one of the conjugate opposite sections, and two straight lines are drawn through their center, one through the point of contact and one parallel to the tangent until it meet one of the adjacent sections then the straight line touching the section at the point of meeting will be parallel to the straight line drawn through the point of contact and the center, and those through the points of contact and the center will be conjugate diameters of the opposite sections*

Let there be conjugate opposite sections whose conjugate diameters are the



straight lines $AB$ and $CD$ and center $Y$, and let $EF$ be drawn touching the section $A$, and produced let it meet $CY$ at $T$, and let $EY$ be joined and produced to \ and through $Y$ let $YG$ be drawn parallel to $EF$, and through $G$ let $HG$ be drawn touching the section

I say that $HG$ is parallel to $YE$ and $GO$ and $E\backslash$ are conjugate diameters

For let the straight lines $KE$, $GL$ and $CRP$ be drawn ordinatewise, and let $AM$ and $CN$ be the parameters Since then

$$BA \quad AM \quad NC \quad CD \text{ (I 60)},$$

but

$$BA \quad AM \quad \text{rect } YK\ KF \quad \text{sq } KE \text{ (I 37)},$$

and

$$NC \quad CD \quad \text{sq } GL \quad \text{rect } YL\ LH \text{ (I 37)},$$

therefore also

$$\text{rect } YK\ KF \quad \text{sq } EK \quad \text{sq } GL \quad \text{rect } YL, LH$$

But

$$\text{rect } YK\ KF \quad \text{sq } EK \text{ comp } YK \quad KE\ FK \quad KE,$$

and

$$\text{sq } GL \quad \text{rect } YL\ LH \text{ comp } GL \quad LY\ GL \quad LH,$$

therefore

$$\text{ratio comp } YK \quad KE\ FK \quad KE = \text{ratio comp } GL \quad LY, GL \quad LH$$

and of these

$$FK \quad KE \quad GL \quad LY$$

for each of the straight lines $EK$ $KF$ and $FE$ is parallel to each of the straight lines $YL$ $LG$ and $GY$ respectively Therefore as remainder

$$YK \quad KE \quad GL \quad LH$$

Also the sides about the equal angles at $K$ and $L$ are proportional therefore *triangle $EKY$ is similar to triangle $GHL$* and will have equal the angles the corresponding sides subtend Therefore

$$\text{angle } EYK = \text{angle } LGH$$

But also

$$\text{angle } KYG = \text{angle } LGY$$

and therefore

$$\text{angle } EYG = \text{angle } HGY$$

Therefore $EY$ is parallel to $GH$

Then let it be contrived that

$$PG \quad GR \quad HG \quad S$$

*therefore $S$ is a half of the parameter of the ordinates to the diameter $GO$ in* sections $C$ and $D$ (I 51) And since $CD$ is the second diameter of the sections $A$ and $B$ and $ET$ meets it therefore

$$\text{rect } TY \quad EK = \text{sq } CY$$

for if we draw from $E$ a parallel to $KY$ the rectangle contained by $TY$ and the straight line cut off by the parallel will be equal to the square on $CY$ (I 38) And therefore

$$TY \quad EK \quad \text{sq } TY \quad \text{sq } YC \text{ (Eucl VI 20)}$$

But

$$TY \quad EK \quad TF \quad FE$$

or

$$TY \quad FK \quad \text{trgl } TYF \quad \text{trgl } EFY \text{ (Eucl VI 1)},$$

and

or

sq *TY* sq *CY* trgl *YTF* trgl *YCP* (Eucl vi 19)

Therefore

sq *TY* sq *CY* trgl *YTF* trgl *GHY* (iii 1)

Therefore

trgl *TYF* trgl *EFY* trgl *TFY* trgl *YGH*

Therefore

trgl *GHY* = trgl *YEF*

But they also have

angle *HGY* = angle *YEF*

for *EY* is parallel to *GH*, and *EF* to *GY* Therefore the sides about the equal angles are reciprocally proportional (Eucl vi 15) Therefore

*GH* *EY* *EF* *GY*

therefore

rect *HG GY* = rect *YE EF*

And since

*S HG* *RG GP*,

and

*RG GP* *YE EF*,

for they are parallel therefore also

*S HG* *YE EF*

But with *YG* taken as common height,

*S HG* rect *S YG* rect *HG GY*,

and

*YE EF* sq *YE* rect *YE EF*

And therefore

rect *S,YG* rect *HG GY* sq *YE* rect *YE FF*

Alternately

rect *S,GY* sq *EY* rect *HG GY* rect *FE,EY*

But

rect *HG,GY* = rect *YE EF* (above),

therefore also

rect *S GY* = sq *EY*

And rectangle *S GY* is a fourth of the figure on *GO* for

*GY* = half *GO*

and *S* is the parameter and

sq *EY* = fourth sq *EX*

for

*EY* = *YX*

Therefore the square on *EY* is equal to the figure on *GO* Then likewise we could show also that *GO* is equal in square to the figure on *EX* Therefore *EX* and *GO* are conjugate diameters of the opposite sections *A*, *B*, *C*, and *D*

## Proposition 21

*The same things being supposed it is to be shown that the point of meeting of the tangents is on one of the asymptotes*

Let there be conjugate opposite sections whose diameters are the straight lines *AB* and *CD* and let the straight lines *AE* and *EC* be drawn tangent

I say that the point *E* is on the asymptote

For since the square on $CY$ is equal to the fourth of the figure on $AB$ (I 60), and

sq $AE$ = sq $CY$ (II 17)

therefore also the square on $AE$ is equal to the fourth part of the figure on $AB$ Let $EY$ be joined therefore $EY$ is an asymptote (II 1) therefore the point $E$ is on the asymptote

PROPOSITION 22

*If in conjugate opposite sections a radius is drawn to any one of the sections, and a parallel is drawn to it meeting one of the adjacent sections and meeting the asymptotes, then the rectangle contained by the segments produced between the section and the asymptotes on the straight line drawn is equal to the square on the radius*

Let there be the conjugate opposite sections $A$ $B$ $C$ and $D$ and let there be the asymptotes of the sections $YEF$ and $YGH$ and from the center $Y$ let some straight line $YCD$ be drawn across and let $HE$ be drawn parallel to it cutting both the adjacent section and the asymptotes

I say that

rect $EK, KH$ = sq $CY$

Let $KL$ be bisected at $M$ and let $MY$ be joined and produced therefore $AB$ is the diameter of the sections $A$ and $B$ (I 51, end) And since the tangent at $A$ is parallel to $EH$ (II 5) therefore $EH$ has been dropped ordinatewise to $AB$ (I 17) And the center is $Y$ therefore $AB$ and $CD$ are conjugate diameters (First Def I 6) Therefore the square on $CY$ is equal to the fourth of the figure on $AB$ (I 60) And the rectangle $HK$, $KE$ is equal to the fourth part of the figure on $AB$ (II 10) therefore also

rect $HK$, $KE$ = sq $CY$

## Proposition 23

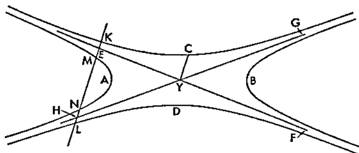
*If in conjugate opposite sections some radius is drawn to any one of the sections, and a parallel is drawn to it meeting the three adjacent sections, then the rectangle contained by the segments produced between the three sections on the straight line drawn is twice the square on the radius*

Let there be the conjugate opposite sections $A$, $B$, $C$, and $D$ and let the center of the section be $Y$, and from the point $Y$ let some straight line $CY$ be drawn to meet any one of the sections, and let $KL$ be drawn parallel to $CY$ cutting the three adjacent sections

I say that

$$\text{rect } KM, ML = 2 \text{ sq } CY$$

Let the asymptotes to the sections, $EF$ and $GH$, be drawn therefore




$$\text{sq } CY = \text{rect } HM\ ME \text{ (II 22)} = \text{rect } HK, KE \text{ (II 11)}$$

And

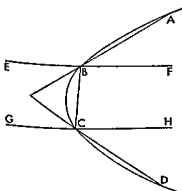
$$\text{rect } HM, ME + \text{rect } HK\ KE = \text{rect } LM\ MK$$

because of the straight lines on the ends being equal (II 8 16) Therefore also

$$\text{rect } LM\ MK = 2 \text{ sq } CY$$

## Proposition 24

*If two straight lines meet a parabola each at two points and if a point of meeting of neither one of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section*




Let there be the parabola $ABCD$ and let the two straight lines $AB$ and $CD$ meet $ABCD$ and let a point of meeting of neither of them be contained by the points of meeting of the other

I say that the straight lines produced will meet each other

Let the diameters of the section $EBF$ and $GCH$ be drawn through the points $B$ and $C$ therefore they are parallel (I 51, end) and each one cuts the section in one point only (I 26) Then let $BC$ be joined therefore

$$\text{angle } EBC + \text{angle } BCG = 2 \text{ rt angles}$$

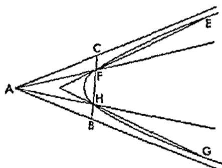
and $DC$ and $BA$ produced make angles less than two right angles Therefore they will meet each other outside the section (I 10, Eucl Post 5)

### Proposition 25

*If two straight lines meet an hyperbola each at two points and if a point of meeting of neither of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section, but within the angle containing the section*

Let there be an hyperbola, whose asymptotes are $AB$ and $AC$, and let the two straight lines $EF$ and $GH$ cut the section, and let a point of meeting of neither of them be contained by the points of meeting of the other

I say that the straight lines $EF$ and $GH$ produced will meet outside the section, but within the angle $CAB$



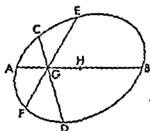

For let the straight lines $AF$ and $AH$ be joined and produced and let $FH$ be joined And since the straight lines $EF$ and $GH$ produced cut the angles $AFH$ and $AHF$, and the said angles are less than two right angles (Eucl I 17), the straight lines $EF$ and $GH$ produced will meet each other outside the section but within the angle $BAC$

Then we could likewise show it even if the straight lines $EF$ and $GH$ are tangents to the sections

### Proposition 26

*If in an ellipse or circumference of a circle two straight lines not through the center cut each other then they do not bisect each other*

For if possible, in the ellipse or circumference of a circle let the two straight lines $CD$ and $EF$ not through the center bisect each other at $G$ and let the point $H$ be the center of the section and let $GH$ be joined and produced to $A$ and $B$

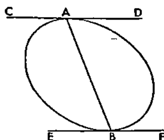


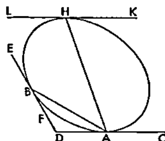

Since then the straight line $AB$ is a diameter bisecting $EF$, therefore the tangent at $A$ is parallel to $EF$ (II 6) We could then likewise show that it is also parallel to $CD$ And so also $EF$ is parallel to $CD$ And this is impossible Therefore $CD$ and $EF$ do not bisect each other

### Proposition 27

*If two straight lines touch an ellipse or circumference of a circle, and if the straight line joining the points of contact is through the center of the section the tangents will be parallel but if not they will meet on the same side of the center*

Let there be the ellipse or circumference of a circle $AB$, and let the straight lines $CAD$ and $EBF$ touch it, and $AB$ be joined, and first let it be through the center



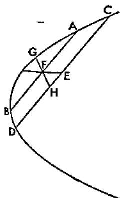

I say that $CD$ is parallel to $EF$

For since $AB$ is a diameter of the section, and $CD$ touches it at $A$, therefore $CD$ is parallel to the ordinates to $AB$ (I 17) Then for the same reasons $BF$ is also parallel to the same ordinates Therefore $CD$ is also parallel to $EF$

Then let $AB$ not be through the center, as in the second drawing, and let the diameter $AH$ be drawn, and let $KHL$ be drawn tangent through $H$ therefore $KL$ is parallel to $CD$ Therefore $EF$ produced will meet $CD$ on the same side of the center as $AB$

## Proposition 28

*If in a section of a cone or circumference of a circle some straight line bisects two parallel straight lines, then it will be a diameter of the section*




For let $AB$ and $CD$ two parallel straight lines in a conic section, be bisected at $E$ and $F$, and let $EF$ be joined and produced

I say that it is a diameter of the section

For if not let the straight line $GFH$ be so if possible Therefore the tangent at $G$ is parallel to $AB$ (II 5 6) And so the same straight line is parallel to $CD$ And $GH$ is a diameter therefore

$$CH = HD \text{ (First Def I 4)}$$

and this is impossible for it is supposed

$$CE = ED$$

Therefore $GH$ is not a diameter Then likewise we could show that there is no other except $EF$ Therefore $EF$ will be a diameter of the section

## Proposition 29

*If in a section of a cone or circumference of a circle two tangents meet the straight line drawn from their point of meeting to the midpoint of the straight line joining the points of contact is a diameter of the section*

Let there be a section of a cone or circumference of a circle to which let the straight lines $AB$ and $AC$ meeting at $A$ be drawn tangent and let $BC$ be joined and bisected at $D$, and let $AD$ be joined

I say that it is a diameter of the section

For if possible let $DE$ be a diameter, and let $EC$ be joined, then it will cut the section (I 35 36) Let it cut it at $F$, and through $F$ let $FKG$ be drawn parallel to $CDB$ Since then

$$CD = DB$$

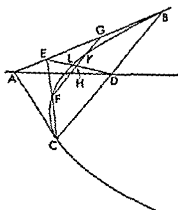
also

$$FH = HG$$

And since the tangent at $L$ is parallel to $BC$ (II 5, 6), and $FG$ is also parallel to $BC$, therefore also $FG$ is parallel to the tangent at $L$ Therefore

$$FH = HK \text{ (I 46, 47)}$$

and this is impossible Therefore $DF$ is not a diameter Then likewise we could show that there is no other except $AD$




## Proposition 30

*If two straight lines tangent to a section of a cone or to a circumference of a circle meet the diameter drawn from the point of meeting will bisect the straight line joining the points of contact*

Let there be the section of a cone or circumference of a circle $BC$, and let two tangents $BA$ and $AC$ be drawn to it meeting at $A$ and let $BC$ be joined and let $AD$ be drawn through $A$ as a diameter of the section

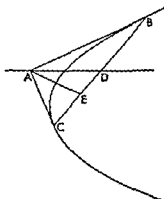
I say that

$$DB = DC$$

For let it not be but if possible let

$$BE = EC$$

and let $AE$ be joined therefore $AF$ is a diameter of the section (II 29) But $AD$ is also a diameter and this is absurd For if the section is an ellipse the point $A$ at which the diameters meet each other, will be a center outside the section and this is impossible and if the section is a parabola, the diameters meet each other (I 51 end) and if it is an hyperbola and the straight lines $BA$ and $AC$ meet the section without containing one another's points of meeting then the center is within the angle containing the hyperbola (II 25), but it is also on it, for it has been supposed a center since $DA$ and $AE$ are diameters (I 51 end) and this is absurd Therefore $BE$ is not equal to $EC$




## Proposition 31

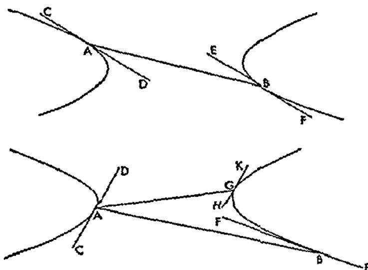
*If two straight lines touch each of the opposite sections then if the straight line joining the points of contact falls through the center the tangents will be parallel, but if not, they will meet on the same side as the center*

Let there be the opposite sections $A$ and $B$, and let the straight lines $CAD$ and $EBF$ be tangent to them at $A$ and $B$ and let the straight line joined from

$A$ to $B$ fall first through the center of the sections

I say that $CD$ is parallel to $EF$

For since they are opposite sections of which $AB$ is a diameter and $CD$ touches one of them at $A$, therefore the straight line drawn through $B$ parallel

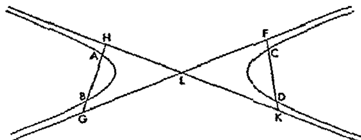



to $CD$ touches the section (I 44 note) But $EF$ also touches it therefore $CD$ is parallel to $EF$

Then let the straight line from $A$ to $B$ not be through the center of the sections and let $AG$ be drawn as a diameter of the sections and let $HK$ be drawn tangent to the section therefore $HK$ is parallel to $CD$ and since the straight lines $EF$ and $HK$ touch an hyperbola therefore they will meet (II 25 end) And $HK$ is parallel to $CD$ therefore also the straight lines $CD$ and $EF$ produced will meet And it is evident they are on the same side as the center

## Proposition 32

*If straight lines meet each of the opposite sections in one point when touching or in two points when cutting, and when produced the straight lines meet then their point of meeting will be in the angle adjacent to the angle containing the section*




Let there be opposite sections and the straight lines $AB$ and $CD$ ,

touching the opposite sections in one point or cutting them in two points, and let them meet when produced

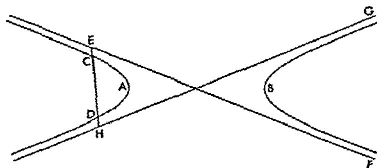
I say that their point of meeting will be in the angle adjacent to the angle containing the section

Let $FG$ and $HK$ be asymptotes to the sections, therefore $AB$ produced will meet the asymptotes (II 8) Let it meet them at $H$ and $G$ And since $FK$ and $HG$ are supposed as meeting it is evident that either they will meet in the place under angle $HLF$ or in that under angle $KLG$ And likewise also, if they touch (II 3)

PROPOSITION 33

*If a straight line meeting one of the opposite sections, when produced both ways, falls outside the section, it will not meet the other section, but will fall through the three places of which one is that contained by the angle containing the section, and two are those contained by the angle adjacent to the angle containing the section*

Let there be the opposite sections $A$ and $B$, and let some straight line $CD$




cut $A$, and when produced both ways, let it fall outside the section

I say that the straight line $CD$ does not meet the section $B$

For let $EF$ and $GH$ be drawn as asymptotes to the sections therefore $CD$ produced will meet the asymptotes (II 8) And it only meets them in the points $E$ and $H$ And so it will not meet the section $B$

And it is evident that it will fall through the three places For if some straight line meets both of the opposite sections it will meet neither of the opposite sections in two points For if it meets it in two points by what has just been proved it will not meet the other section

PROPOSITION 34

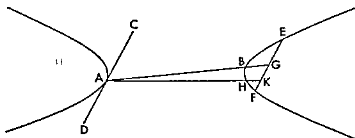
*If some straight line touch one of the opposite sections and a parallel to it be drawn in the other section then the straight line drawn from the point of contact to the midpoint of the parallel will be a diameter of the opposite sections*

Let there be the opposite sections $A$ and $B$, and let some straight line $CD$ touch one of them $A$ at $A$, and let $EF$ be drawn parallel to $CD$ in the other section, and let it be bisected at $G$ and let $AG$ be joined

I say that $AG$ is a diameter of the opposite sections

For if possible, let $AHK$ be Therefore the tangent at $H$ is parallel to $CD$

(II.31) But $CD$ is also parallel to $EF$, and therefore the tangent at $H$ is parallel to $EF$ Therefore




$$EK=KF \text{ (I 47)},$$

and this is impossible, for

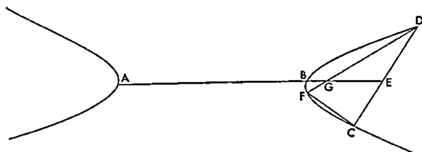
$$EG=GF$$

Therefore $AH$ is not a diameter of the opposite sections
Therefore $AB$ is

## Proposition 35

*If a diameter in one of the opposite sections bisects some straight line the straight line touching the other section at the end of the diameter will be parallel to the bisected straight line*

Let there be the opposite sections $A$ and $B$ and let their diameter $AB$ bisect the straight line $CD$ in section $B$ at $E$




I say that the tangent to the section at $A$ is parallel to $CD$

For if possible let $DF$ be parallel to the tangent to the section at $A$, therefore

$$DG=GF \text{ (I 48)}$$

But also

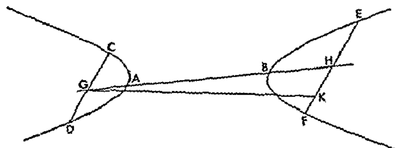
$$DE=EC$$

Therefore $CF$ is parallel to $EG$, and this is impossible, for produced it meets it (I 22) Therefore $DF$ is not parallel to the tangent to the section at $A$ nor is any other straight line except $CD$

## Proposition 36

*If parallel straight lines are drawn one in each of the opposite sections then the straight line joining their midpoints will be a diameter of the opposite sections*

Let there be the opposite sections $A$ and $B$, and let the straight lines $CD$ and $EF$ be drawn one in each of them, and let them be parallel, and let them both




be bisected at points $G$ and $H$ and let $GH$ be joined

I say that $GH$ is a diameter of the opposite sections

For if not, let $GK$ be one Therefore the tangent to $A$ is parallel to $CD$ (II 5), and so also to $EF$ Therefore

$$EK = KF \text{ (I 48)}$$

and this is impossible since also

$$EH = HF$$

Therefore $GK$ is not a diameter of the opposite sections Therefore $GH$ is

PROPOSITION 37

*If a straight line not through the center cuts the opposite sections then the straight line joined from its midpoint to the center is a so-called upright diameter of the opposite sections and the straight line drawn from the center parallel to the bisected straight line is a transverse diameter conjugate to it*

Let there be the opposite sections $A$ and $B$, and let some straight line $CD$ not through the center cut the sections $A$ and $B$ and let it be bisected at $E$, and let $Y$ be the center of the sections, and let $YE$ be joined, and through $Y$ let $AB$ be drawn parallel to $CD$

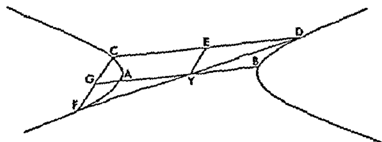
I say that the straight lines $AB$ and $EY$ are conjugate diameters of the sections

For let $DY$ be joined and produced to $F$, and let $CF$ be joined Therefore

$$DY = YF \text{ (I 30)}$$

But also

$$DE = EC,$$



therefore $EY$ is parallel to $FC$ Let $BA$ be produced to $G$ And since

$$DY = YF,$$

therefore also

$$EY = YG,$$

and so also

$$CG = FG$$

Therefore the tangent at $A$ is parallel to $CF$ (II 5), and so also to $EY$ Therefore $EY$ and $AB$ are conjugate diameters (I 16)

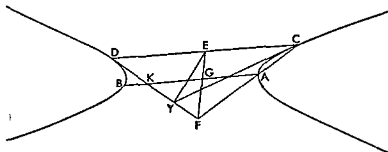
PROPOSITION 38

*If two straight lines meeting touch opposite sections the straight line joined from the point of meeting to the midpoint of the straight line joining the points of contact will be a so-called upright diameter of the opposite sections, and the straight line drawn through the center parallel to the straight line joining the points of contact is a transverse diameter conjugate to it*

Let there be the opposite sections $A$ and $B$, and $CY$ and $YD$ touching the sections and let $CD$ be joined and bisected at $E$, and let $EY$ be joined

I say that the diameter $EY$ is a so-called upright and the straight line drawn through the center parallel to $CD$ is a transverse diameter conjugate to it

For if possible, let $EF$ be a diameter and let $F$ be a point taken at random, therefore $DY$ will meet $EF$ Let it meet it at $F$, and let $CF$ be joined, therefore




$CF$ will hit the section (I 32) Let it hit it at $A$, and through $A$ let $AB$ be drawn parallel to $CD$ Since then $EF$ is a diameter, and bisects $CD$, it also bisects the parallels to it (First Def I 4) Therefore

$$AG = GB$$

And since

$$CE = ED$$

and is on triangle $CFD$ therefore also

$$AG = GK$$

And so also

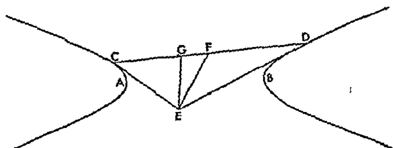
$$GK = GB$$

and this is impossible Therefore $EF$ will not be a diameter

PROPOSITION 39

*If two straight lines meeting touch opposite sections the straight line drawn through the center and the point of meeting of the tangents bisects the straight line joining the points of contact*

Let there be the opposite sections $A$ and $B$, and let two straight lines $CE$ and




$ED$ be drawn touching $A$ and $B$, and let $CD$ be joined, and let $EF$ be drawn as a diameter

I say that

$$CF = FD$$

For if not, let $CD$ be bisected at $G$, and let $GE$ be joined, therefore $GE$ is a diameter (II 38) But $EF$ is also, therefore $E$ is the center (I 31, end) Therefore the point of meeting of the tangents is at the center of the sections and this is absurd (II 32) Therefore $CF$ is not unequal to $FD$ Therefore equal

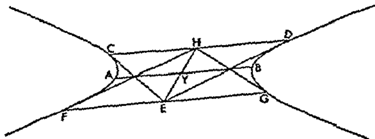
PROPOSITION 40

*If two straight lines touching opposite sections meet and through the point of meeting a straight line is drawn, parallel to the straight line joining the points of contact and meeting the sections then the straight lines drawn from the points of meeting to the midpoint of the straight line joining the points of contact touch the sections*

Let there be the opposite sections $A$ and $B$, and let two straight lines $CE$ and $ED$ be drawn touching $A$ and $B$, and let $CD$ be joined and through $E$ let $FEG$ be drawn parallel to $CD$, and let $CD$ be bisected at $H$, and let $FH$ and $HG$ be joined

I say that $FH$ and $HG$ touch the sections

Let $EH$ be joined, therefore $EH$ is an upright diameter and the straight line drawn through the center parallel to $CD$ a transverse diameter conjugate to it (II 38) And let the center $Y$ be taken and let $AYB$ be drawn parallel to $CD$ therefore $HE$ and $AB$ are conjugate diameters And $CH$ has been drawn




ordinatewise to the second diameter and $CE$ has been drawn touching the section and meeting the second diameter Therefore the rectangle $EY, YH$ is

equal to the square on the half of the second diameter (I 38); that is to the fourth part of the figure on $AB$ (Second Def I 10) And since $FE$ has been drawn ordinatewise and $FH$ joined, therefore $FH$ touches the section $A$ (I 38) Likewise then also $GH$ touches section $B$ Therefore $FH$ and $HG$ touch sections $A$ and $B$

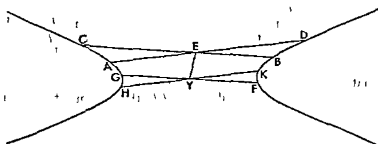
PROPOSITION 41

*If in opposite sections two straight lines not through the center cut each other, then they do not bisect each other*

Let there be the opposite sections $A$ and $B$, and in $A$ and $B$ let the two straight lines $CB$ and $AD$ not through the center cut each other at $E$

I say that they do not bisect each other

For if possible, let them bisect each other, and let $Y$ be the center of the sections and let $EY$ be joined, therefore $EY$ is a diameter (II 37) Let $YF$ be drawn through $Y$ parallel to $BC$, therefore $YF$ is a diameter and conjugate to

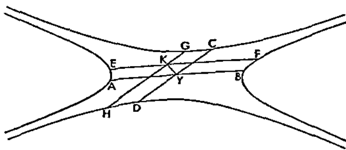



$EY$ (II 37) Therefore the tangent at $F$ is parallel to $EY$ (First Def I 6) Then for the same reasons with $HK$ drawn parallel to $AD$ the tangent at $H$ is parallel to $EY$ and so also the tangent at $F$ is parallel to the tangent at $H$, and this is absurd for it has been shown it also meets it (II 31) Therefore the straight lines $CB$ and $AD$ not being through the center do not bisect each other

PROPOSITION 42

*If in conjugate opposite sections two straight lines not through the center cut each other, they do not bisect each other*

Let there be the conjugate opposite sections $A$ $B$, $C$ and $D$, and in $A$, $B$, $C$




and $D$ let the two straight lines not through the center $EF$ and $GH$, cut each other at $K$

I say that they do not bisect each other

For if possible, let them bisect each other, and let the center of the sections be $Y$, and let $AB$ be drawn parallel to $EF$ and $CD$ to $HG$, and let $AY$ be joined, therefore $AY$ and $AB$ are conjugate diameters (II 37) Likewise $YK$ and $CD$ are also conjugate diameters And so also the tangent at $A$ is parallel to the tangent at $C$, and this is impossible, for it meets it, since the tangent at $C$ cuts the sections $A$ and $B$ (II 19), and the tangent at $A$ sections $C$ and $D$, it is evident also that their point of meeting is in the place under angle $AYC$ (II 21) Therefore the straight lines $EF$ and $GH$ not being through the center do not bisect each other

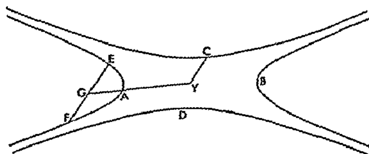
## Proposition 43

*If a straight line cuts one of the conjugate opposite sections in two points, and through the center one straight line is drawn to the midpoint of the cutting straight line and another straight line is drawn parallel to the cutting straight line, they will be conjugate diameters of the opposite sections*

Let there be the conjugate opposite sections $A$ $B$, $C$ and $D$, and let some straight line cut section $A$ at the two points $E$ and $F$, and let $FE$ be bisected at $G$, and let $Y$ be center, and let $YG$ be joined, and let $CY$ be drawn parallel to $EF$

I say that $AY$ and $YC$ are conjugate diameters

For since $AY$ is a diameter and bisects $EF$, the tangent at $A$ is parallel to




$EF$ (II 5) and so also to $CY$ Since then they are opposite sections and a tangent has been drawn to one of them $A$ at $A$ and from the center $Y$ one straight line $YA$ is joined to the point of contact and another $CY$ has been drawn parallel to the tangent therefore $YA$ and $CY$ are conjugate diameters for this has been shown before (II 20)

## Proposition 44 (Problem)

*Given a section of a cone to find a diameter*

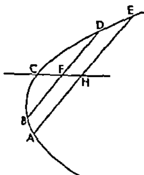
Let there be the given conic section on which are the points $A$ $B$, $C$ $D$ and $E$ Then it is required to find a diameter

Let it have been done and let it be $CH$ Then with $DF$ and $EH$ drawn ordinatewise and produced

$$DF = FB,$$

and

$$EH = HA \text{ (First Def I 4)}$$



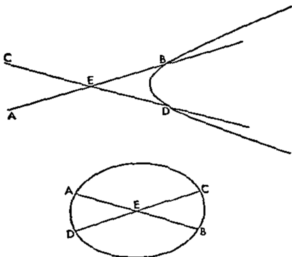

If then we fix the straight lines $BD$ and $EA$ in position to be parallel, the points $H$ and $F$ will be given And so $HFC$ will be given in position

Then it will be constructed (συντεθήσεται) thus let there be the given conic section on which are the points $A, B, C, D$ and $E$, and let the straight lines $BD$ and $AE$ be drawn parallel and be bisected at $F$ and $H$ And the straight line $FH$ joined will be a diameter of the section (First Def I 4) And in the same way we could also find an indefinite number of diameters

### Proposition 45 (Problem)

*Given an ellipse or hyperbola to find the center*

And this is evident, for if two diameters of the section, $AB$ and $CD$ are




drawn through (II 44) the point at which they cut each other will be the center of the section, as indicated below

### Proposition 46 (Problem)

*Given a section of a cone to find the axis*

Let the given section of a cone first be a parabola on which are the points $F$ $C$ and $E$ Then it is required to find its axis

For let $AB$ be drawn as a diameter of it (I 44) If then $AB$ is an axis what was enjoined would have been done but if not let it have been done and let $CD$ be the axis therefore the axis $CD$ is parallel to $AB$ (I 51, end) and bisects the straight lines drawn perpendicular to it (First Def I 7) And the perpendiculars to $CD$ are also perpendiculars to $AB$ and so $CD$ bisects the perpendic-

ulars to $AB$ If then I fix $EF$ the perpendicular to $AB$, it will be given in position, and therefore

$$ED = DF,$$

therefore the point $D$ is given Therefore through the given point $D$, $CD$ has been drawn parallel in position to $AB$, therefore $CD$ is given in position

Then it will be constructed thus let there be the given parabola on which are the points $F$, $E$ and $A$, and let $AB$, a diameter of it, be drawn (II 44), and let $BE$ be drawn perpendicular to it and let it be produced to $F$ If then

$$EB = BF$$

it is evident that $AB$ is the axis (First Def I 7), but if not, let $EF$ be bisected by $D$, and let $CD$ be drawn parallel to $AB$ Then it is evident that $CD$ is the axis of the section, for being parallel to a diameter that is being a diameter (I 51, end), it bisects $EF$ at right angles Therefore $CD$ has been found as the axis of the given parabola (First Def I 7)

And it is evident that the parabola has only one axis For if there is another, as $AB$, it will be parallel to $CD$ (I 51, end) And it cuts $EF$, and so it also bisects it (First Def I 4)

Therefore

$$BE = BF$$

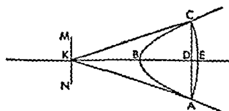
and this is absurd

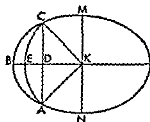
PROPOSITION 47 (PROBLEM)

*Given an hyperbola or ellipse to find the axis*

Let there be the hyperbola or ellipse $ABC$ then it is required to find its axis

Let it have been found and let it be $KD$, and $K$ the center of the section, therefore $KD$ bisects the ordinates to itself and at right angles (First Def I 7)




Let the perpendicular $CDA$ be drawn and let $KA$ and $KC$ be joined Since then

$$CD = DA,$$

therefore

$$CK = KA$$

If then we fix the given point $C$ $CK$ will be given And so the circle described with center $K$ and radius $KC$ will also pass through $A$ and will be given in position And the section $ABC$ is also given in position therefore the point $A$

is given But the point $C$ is also given therefore $CA$ is given in position Also

$$CD = DA$$

therefore the point $D$ is given. But also $K$ is given therefore $DK$ is given in position.

Then it will be constructed thus let there be the given hyperbola or ellipse $ABC$, and let $K$ be taken as its center and let a point $C$ be taken at random on the section and let the circle $CEA$ with center $K$ and radius $KC$ be described, and let $CA$ be joined and bisected at $D$, and let $KC$, $KD$ and $KA$ be joined and let $KD$ be drawn through to $B$

Since then

$$AD = DC$$

and $DK$ is common, therefore the two straight lines $CD$ and $DK$ are equal to the two straight lines $AD$ and $DK$ and

$$\text{base } KA = \text{base } KC$$

Therefore $KBD$ bisects $ADC$ at right angles Therefore $KD$ is an axis (First Def I 7)

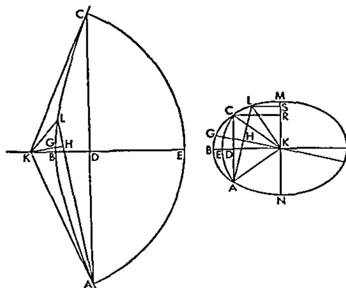
Let $MKN$ be drawn through $K$ parallel to $CA$ therefore $MN$ is the axis of the section conjugate to $BK$ (First Def I 8)

## Proposition 48 (Problem)

*Then with these things shown, let it be next in order to show that there are no other axes of the same sections*

For if possible let there also be another axis $KG$ Then in the same way as before with $AH$ drawn perpendicular

$$AH = HL \text{ (First Def I 4)},$$




and so also

$$AK = KL$$

But also

$$AK=KC$$

therefore

$$KL=KC,$$

and this is absurd

Now that the circle $AEC$ does not hit the section also in another point between the points $A$, $B$ and $C$, is evident in the case of the hyperbola, and in the case of the ellipse let the perpendiculars $CR$ and $LS$ be drawn Since then

$$KC=KL,$$

for they are radii, also

$$\text{sq } KC=\text{sq } KL$$

But

$$\text{sq } CR+\text{sq } RK=\text{sq } CK,$$

and

$$\text{sq } KS+\text{sq } SL=\text{sq } LK,$$

therefore

$$\text{sq } CR+\text{sq } RK=\text{sq } KS+\text{sq } SL$$

Therefore

$$\text{difference between sq } CR \text{ and sq } SL= \text{difference between sq } KS \text{ and sq } RK$$

Again since

$$\text{rect } MR,RN+\text{sq } RK=\text{sq } KM,$$

and also

$$\text{rect } MS\,SN+\text{sq } SK=\text{sq } KM \text{ (Eucl II 5)}$$

therefore

$$\text{rect } MR\,RN+\text{sq } RK=\text{rect } MS\,SN+\text{sq } SK$$

Therefore

$$\text{difference between sq } SK \text{ and sq } KR= \text{difference between rect } MR,RN \text{ and rect } MS,SN$$

And it was shown that

$$\text{difference between sq } SK \text{ and sq } KR= \text{difference between sq } CR \text{ and sq } SL$$

therefore

$$\text{difference between sq } CR \text{ and sq } SL= \text{difference between rect } MR\,RN \text{ and rect } MS,SN$$

And since $CR$ and $LS$ are ordinates

$$\text{sq } CR \quad \text{rect } MR,RN \quad \text{sq } SL \quad \text{rect } MS,SN \text{ (I 21)}$$

But the same difference was also shown for both therefore

$$\text{sq } CR=\text{rect } MR,RN$$

and

$$\text{sq } SL=\text{rect } MS\;SN \text{ (Eucl V 16, 17 9)}$$

Therefore the line $ICM$ is a circle, and this is absurd, for it is supposed an ellipse

## Proposition 49 (Problem)

*Given a section of a cone and a point not within the section to draw from the point a straight line touching the section in one point*

Let the given section of a cone first be a parabola whose axis is $BD$ Then it is

required to draw a straight line as prescribed from the given point which is not within the section

Then the given point is either on the line or on the axis or somewhere else outside

Now let it be on the line, and let it be $A$, and let it have been done, and let it be $AE$, and let $AD$ be drawn perpendicular, then it will be given in position And

$$BE=BD \text{ (I 35)}$$

and $BD$ is given, therefore $BE$ is also given And the point $B$ is given, therefore $E$ is also given But $A$ also therefore $AE$ is given in position

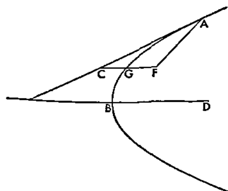
Then it will be constructed thus let $AD$ be drawn perpendicular from $A$, and let $BE$ be made equal to $BD$, and let $AE$ be joined Then it is evident that it touches the section (I 33)

Again let the given point $E$ be on the axis and let it have been done and let $AE$ be drawn tangent, and let $AD$ be drawn perpendicular, therefore

$$BE=BD \text{ (I 35)}$$

And $BE$ is given therefore also $BD$ is given And the point $B$ is given, therefore $D$ is also given And $DA$ is perpendicular, therefore $DA$ is given in position Therefore the point $A$ is given But also $E$, therefore $AE$ is given in position

Then it will be constructed thus let $BD$ be made equal to $BE$, and from $D$ let $DA$ be drawn perpendicular to $ED$, and let $AE$ be joined Then it is evident that $AE$ touches (I 33)




And it is evident also that even if the given point is the same as $B$ the straight line drawn from $B$ perpendicular touches the section (I 17)

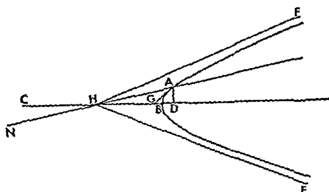
Then let $C$ be the given point and let it have been done and let $CA$ be it and through $C$ let $CF$ be drawn parallel to the axis that is to $BD$ therefore $CF$ is given in position And from $A$ let $AF$ be drawn ordinatewise to $CF$ then

$$CG=FG \text{ (I 35)}$$

And the point $G$ is given therefore $F$ is also given And $FA$ has been erected ordinatewise that is parallel to the tangent at $G$ (I 32) therefore $FA$ is given in position Therefore $A$ is also given but also $C$ Therefore $CA$ is given in position

It will be constructed thus let $CF$ be drawn through $C$ parallel to $BD$, and let $FG$ be made equal to $CG$, and let $FA$ be drawn parallel to the tangent at $G$ (above) and let $AC$ be joined It is evident then that this will do the problem (I 33)

Again let it be an hyperbola whose axis is $DBC$ and center $H$, and asymptotes $HE$ and $HF$. Then the given point will be given either on the section or




on the axis or within angle $EHF$ or in the adjacent place or on one of the asymptotes containing the section or in the place between the straight lines containing the angle vertical to angle $EHF$.

Let the point $A$ first be on the section and let it have been done, and let $AG$ be tangent and let $AD$ be drawn perpendicular and let $BC$ be the transverse side of the figure, then

$$CD \quad DB \quad CG \quad GB \text{ (I 36)}$$

And the ratio of $CD$ to $DB$ is given, for both the straight lines are given therefore also the ratio of $CG$ to $GB$ is given. And $BC$ is given, therefore point $G$ is given. But also $A$, therefore $AG$ is given in position.

It will be constructed thus: let $AD$ be drawn perpendicular from $A$, and let

$$CG \quad GB \quad CD \quad DB$$

and let $AG$ be joined. Then it is evident that $AG$ touches the section (I 34).

Then again let the given point $G$ be on the axis and let it have been done, and let $AG$ be drawn tangent, and let $AD$ be drawn perpendicular. Then for the same reasons

$$CG \quad GB \quad CD \quad DB \text{ (I 36)}$$

And $BC$ is given, therefore the point $D$ is given. And $DA$ is perpendicular, therefore $DA$ is given in position. And also the section is given in position, therefore the point $A$ is given. But also $G$, therefore $AG$ is given in position.

Then it will be constructed thus: let the other things be supposed the same, and let it be contrived that

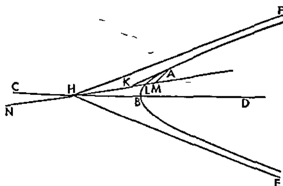
$$CG \quad GB \quad CD \quad DB$$

and let $DA$ be drawn perpendicular and let $AG$ be joined. Then it is evident that $AG$ does the problem (I 34) and that from $G$ another tangent to the section could be drawn on the other side.

With the same things supposed let the given point $K$ be in the place inside angle $EHF$, and let it be required to draw a tangent to the section from $K$. Let it have been done and it be $KA$, and let $KH$ be joined and produced, and let $HN$ be made equal to $LH$, therefore they are all given. Then also $LN$ will be given. Then let $AM$ be drawn ordinatewise to $MN$, then also

$$NK \quad KL \quad MN \quad ML$$

And the ratio of $NK$ to $KL$ is given, therefore also the ratio of $NM$ to $ML$ is given And the point $L$ is given, therefore also $M$ is given And $MA$ has been erected parallel to the tangent at $L$ therefore $MA$ is given in position And also the section $ALB$ is given in position, therefore the point $A$ is given But $K$ is also given, therefore $AK$ is given




Then it will be constructed thus let the other things be supposed the same and the given point $K$, and $KH$ be joined and produced, and let $HN$ be made equal to $HL$ and let it be contrived that

$$NK \; KL \quad NM \; ML$$

and let $MA$ be drawn parallel to the tangent at $L$ (above), and let $KA$ be joined, therefore $KA$ touches the section (I 34)

And it is evident that a tangent to the section could also be drawn to the other side

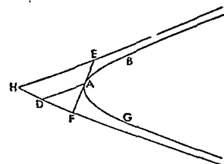
With the same things supposed let the given point $F$ be on one of the asymptotes containing the section, and let it be required to draw from $F$ a tangent to the section And let it have been done, and let it be $FAE$, and through $A$ let $AD$ be drawn parallel to $EH$ then

$$DH = DF$$

since also

$$FA = AE \text{ (II 3)}$$

And $FH$ is given, therefore also point $D$ is given And through the given point $D$ $DA$ has been drawn parallel in position to $EH$, therefore $DA$ is given in position And the section is also given in position therefore the point $A$ is given But $F$ is also given, therefore the straight line $FAE$ is given in position




Then it will be constructed thus let there be the section $AB$ and asymptotes $EH$ and $HF$, and the given point $F$ on one of the asymptotes containing the section and let $FH$ be bisected at $D$ and through $D$ let $DA$ be drawn parallel to $HE$ and let $FA$ be joined And since

$$FD = DH,$$

therefore also

$$FA = AE$$

And so by things shown before, the straight line $FAE$ touches the section (II 9)

With the same things supposed let the given point be in the place under the

angle adjacent to the straight lines containing the section, and let it be $K$, it is required then to draw a tangent to the section from $K$. And let it have been done and let it be $KA$ and let $KH$ be joined and produced, then it will be given in position. If then a given point $C$ is taken on the section, and through $C$, $CD$ is drawn parallel to $KH$, it will be given in position. And if $CD$ is bisected at $E$ and $HE$ is joined and produced, it will be in position, a diameter conjugate to $KH$ (First Def. I. 6). Then let $HG$ be made equal to $BH$ and through $A$ let $AL$ be drawn parallel to $BH$, then because $KL$ and $BG$ are conjugate diameters and $AK$ a tangent and $AL$ a straight line drawn parallel to $BG$, therefore rectangle $KH$, $HL$ is equal to the fourth part of the figure on $BG$ (I. 38). Therefore rectangle $KH$, $HL$ is given. And $KH$ is given, therefore $HL$ is also given. But it is also given in position, and the point $H$ is given, therefore $L$ is also given. And through $L$, $LA$ has been drawn parallel in position to $BG$, therefore $LA$ is given in position. And the section is also given in position, therefore the point $A$ is given. But also $K$; therefore $AK$ is given in position.

Then it will be constructed thus: let the other things be supposed the same and let the given point $K$ be in the aforesaid place, and let $KH$ be joined and produced, and let some point $C$ be taken, and let $CD$ be drawn parallel to $KH$, and let $CD$ be bisected by $E$ and let $EH$ be joined and produced, and let $HG$ be made equal to $BH$; therefore $GB$ is a transverse diameter conjugate to $KHL$ (First Def. I. 6). Then let rectangle $KH$, $HL$ be made equal to the fourth of the figure on $BG$, and through $L$ let $LA$ be drawn parallel to $BG$, and let $KA$ be joined; then it is clear that $KA$ touches the section by the converse of the theorem (I. 38).

And if it is given in the place between the straight lines $FH$ and $HP$, the problem is impossible. For the tangent will cut $CH$. And so it will meet both $FH$ and $HP$, and this is impossible by the things shown in the thirty-first theorem of the first book (I. 31) and in the third of this book (II. 3).

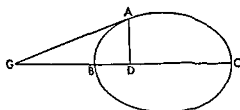
With the same things supposed let the section be an ellipse and the given point $A$ on the section, and let it be required to draw from $A$ a tangent to the section. Let it have been done and let it be $AG$, and let $AD$ be drawn from $A$ ordinatewise to the axis $BC$; then the point $D$ will be given, and

$$CD : DB = CG : GB \text{ (I. 36)}$$

And the ratio of $CD$ to $DB$ is given; therefore the ratio of $CG$ to $GB$ is also given. Therefore the point $G$ is given. But also $A$; therefore $AG$ is given in position.

Then it will be constructed thus let $AD$ be drawn perpendicular, and let

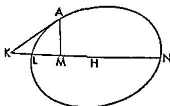
$CG$ $GB$ $CD$ $DE$,




and let $AG$ be joined Then it is evident that $AG$ touches, as also in the case of the hyperbola (I 34)

Then again let the given point be $K$, and let it be required to draw a tangent Let it have been done, and let it be $KA$, and let the straight line $KLH$ be joined to the center $H$ and produced to $N$, then it will be given in position And if $AM$ is drawn ordinatewise, then

$NK$ $KL$ $NM$ $ML$ (I 36)




And the ratio of $NK$ $KL$ is given, therefore the ratio of $MN$ to $LM$ is also given Therefore the point $M$ is given And $MA$ has been erected ordinatewise, for it is parallel to the tangent at $L$, therefore $MA$ is given in position Therefore the point $A$ is given But also $K$, therefore $KA$ is given in position

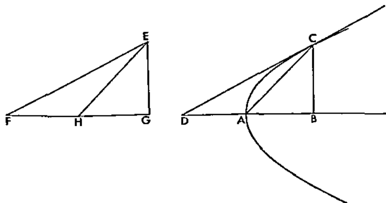
And the construction (συνθεσις) is the same as for the preceding

PROPOSITION 50 (PROBLEM)

*Given the section of a cone to draw a tangent which will make with the axis, on the same side as the section, an angle equal to a given acute angle*

Let the section of a cone first be a parabola whose axis is $AB$ then it is required to draw a tangent to the section which will make with the axis $AB$, on the same side as the section an angle equal to the given acute angle

Let it have been done and let it be $CD$ therefore angle $BDC$ is given Let $BC$




be drawn perpendicular then the angle at $B$ is also given Therefore the ratio of $DB$ to $BC$ is given But the ratio of $BD$ to $BA$ is given therefore also the ratio of $AB$ to $BC$ is given And the angle at $B$ is given therefore angle $BAC$

is also given And it is in position with respect to $BA$ and the given point $A$, therefore $CA$ is given in position And the section is also given in position, therefore the point $C$ is given And $CD$ touches, therefore $CD$ is given in position

Then the problem will be constructed thus let the given section of a cone first be a parabola whose axis is $AB$, and the given acute angle, angle $EFG$, and let some point $E$ be taken on $EF$, and let $EG$ be drawn perpendicular, and let $FG$ be bisected by $H$, and let $HE$ be joined, and let angle $BAC$ be constructed equal to angle $GHE$, and let $BC$ be drawn perpendicular, and let $AD$ be made equal to $BA$, and let $CD$ be joined Therefore $CD$ is tangent to the section (I 33)

I say then that

$$\text{angle } CDB = \text{angle } EFG$$

For since

$$FG \quad GH \quad DB \quad BA$$

and

$$HG \quad GE \quad AB \quad BC,$$

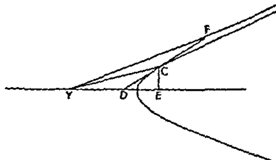
therefore *ex aequali*

$$FG \quad GE \quad DB \quad BC,$$

And the angles at $G$ and $B$ are right angles, therefore

$$\text{angle at } F = \text{angle at } D$$

Let the section be an hyperbola and let it have been done and let $CD$ be tangent and let the center of the section $Y$ be taken, and let $CY$ be joined and let $CE$ be perpendicular therefore the ratio of rectangle $YE$ $ED$ to the square on $CE$ is given for it is the same as the transverse to the upright (I 37) And the ratio of the square on $CE$ to the square on $ED$ is given, for each of the rectangles $CD$, $DE$ and $DE$ $EC$ is given Therefore the ratio of rectangle $YE$, $ED$ to the square on $ED$ is given and so also the ratio of $YE$ to $ED$ is given And the angle at $E$ is given, therefore the angle at $Y$ is also given Then some straight line $CY$ has been drawn across in position with respect to the straight line $YE$ and to the given point $Y$ at a given angle therefore $CY$ is given in position And the section is also given in position therefore the point $C$ is given And $CD$ has been drawn across as tangent therefore $CD$ is given in position




Let the asymptote to the section $YF$ be drawn therefore $CD$ produced will meet the asymptote (II 3) Let it meet it at $F$ Therefore

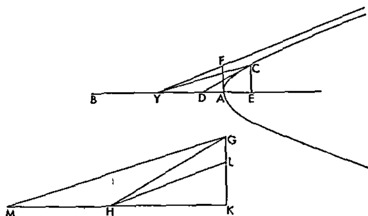
$$\text{angle } FDE > \text{angle } FYD$$

Therefore, for the construction the given acute angle will have to be greater than half the angle contained by the asymptotes

Then the problem will be constructed thus let there be the given hyperbola whose axis is $AB$, and asymptote $YF$, and the given acute angle $KHG$ greater than angle $AYF$, and let

$$\text{angle } KHL = \text{angle } AYF,$$

and let $AF$ be drawn from $A$ perpendicular to $AB$, and let some point $G$ be




taken on $GH$, and let $GK$ be drawn from it perpendicular to $HK$ Since then

$$\text{angle } FYA = \text{angle } LHK,$$

and also the angles at $A$ and $K$ are right, therefore

$$YA \quad AF \quad HK \quad KL$$

$$HK \quad KL > HK \quad KG$$

therefore also

$$YA \quad AF > HK \quad KG$$

And so also

$$\text{sq } YA \quad \text{sq } AF > \text{sq } HK \quad \text{sq } KG$$

But

$$\text{sq } YA \quad \text{sq } AF \quad \text{transverse} \quad \text{upright (II 1)}$$

therefore also

$$\text{transverse} \quad \text{upright} > \text{sq } HK \quad \text{sq } KG$$

If then we shall contrive that

$$\text{sq } YA \quad \text{sq } AF \quad \text{some other} \quad \text{sq } KG,$$

it will be greater than the square on $HK$ Let it be the rectangle $MK, KH$ and let $GM$ be joined Since then

$$\text{sq } MK > \text{rect } MK, KH,$$

therefore

$$\text{sq } MK \quad \text{sq } KG > \text{rect } MK \; KH \quad \text{sq } KG$$
$$> \text{sq } YA \quad \text{sq } AF$$

And if we shall contrive that

$$\text{sq } MK \quad \text{sq } KG \quad \text{sq } YA \quad \text{some other}$$

it will be to a magnitude less than the square on $AF$ and the straight line joined from $Y$ to the point taken will make similar triangles and therefore

$$\text{angle } FYA > \text{angle } CMK$$

Let angle $AYC$ be made equal to angle $GMK$ therefore $YC$ will cut the section (II 2) Let it cut it at $C$, and from $C$ let $CD$ be drawn tangent to the section (II 49) and $CE$ drawn perpendicular, therefore triangle $CYE$ is similar to triangle $GMK$ Therefore

$$\text{sq } YE \quad \text{sq } EC \quad \text{sq } MK \quad \text{sq } KG$$

But also

$$\text{transverse} \quad \text{upright} \quad \text{rect } YE, ED \quad \text{sq } EC \text{ (I 37)},$$

and

$$\text{transverse} \quad \text{upright} \quad \text{rect } MK, KH \quad \text{sq } KG$$

And inversely

$$\text{sq } CE \quad \text{rect } YE, ED \quad \text{sq } GK \quad \text{rect } MK, KH,$$

therefore *ex aequali*

$$\text{sq } YF \quad \text{rect } YE, FD \quad \text{sq } MK \quad \text{rect } MK, KH$$

And therefore

$$YE \quad ED \quad MK \quad KH$$

But also we had

$$CE \quad EY \quad GK \quad KM,$$

therefore *ex aequali*

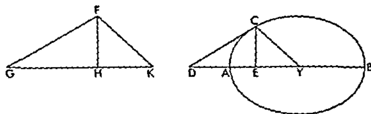
$$CE \quad ED \quad GK \quad KH$$

And the angles at $E$ and $K$ are right angles, therefore

$$\text{angle at } D = \text{angle } GHK$$

Let the section be an ellipse whose axis is $AB$ Then it is required to draw a tangent to the section which with the axis will contain, on the same side as the section an angle equal to the given acute angle

Let it have been done and let it be $CD$ Therefore angle $CDA$ is given Let $CE$ be drawn perpendicular, therefore the ratio of the square on $DE$ to the square on $EC$ is given Let $Y$ be the center of the section, and let $CY$ be joined Then the ratio of the square on $CE$ to the rectangle $DE$ $EY$ is given for it is the same as the ratio of the upright to the transverse (I 37) and therefore the ratio of the square on $DE$ to rectangle $DE, EY$ is given, and therefore the ratio of $DE$ to $EY$ is given And of $DE$ to $EC$ therefore also the ratio of $CE$ to $EY$ is given And the angle at $E$ is right, therefore the angle at $Y$ is given And it is given with respect to a straight line given in position and to a given point, therefore the point C is given And from the given point $C$ let $CD$ be drawn tangent therefore $CD$ is given in position




Then the problem will be constructed thus let there be the given acute angle $FGH$ and let some point $F$ be taken on $FG$, and let $FH$ be drawn perpendicular and let it be contrived that

$$\text{upright} \quad \text{transverse} \quad \text{sq } FH \quad \text{rect } GH\ HK,$$

and let $KF$ be joined and let $Y$ be the center of the section and let angle $AYC$ be constructed equal to angle $GKF$ and let $CD$ be drawn tangent to the section (II 49)

I say that $CD$ does the problem that is

$$\text{angle } CDE = \text{angle } FGH$$

For since

$$YE \quad EC \quad KH \quad FH,$$

therefore also

$$\text{sq } YE \quad \text{sq } EC \quad \text{sq } KH \quad \text{sq } FH$$

But also

$$\text{sq } EC \quad \text{rect } DE,EY \quad \text{sq } FH \quad \text{rect } KH,HG,$$

for each is the same ratio as that of the upright to the transverse (I 37, and above) And *ex aequali*, therefore

$$\text{sq } YE \quad \text{rect } DE\,EY \quad \text{sq } KH \quad \text{rect } KH,HG$$

And therefore

$$YE \quad ED \quad KH \quad HG$$

But also

$$YE \quad EC \quad KH \quad FH,$$

*ex aequali*, therefore

$$DE \quad EC \quad HG \quad FH$$

And the sides about the right angles are proportional therefore

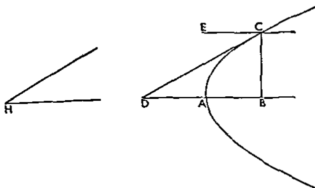
$$\text{angle } CDF = \text{angle } FGH$$

Therefore $CD$ does the problem

PROPOSITION 51 (PROBLEM)

*Given a section of a cone to draw a tangent which with the diameter drawn through the point of contact will contain an angle equal to a given acute angle*

Let the given section of a cone first be a parabola whose axis is $AB$ and the given angle $H$ then it is required to draw a tangent to the parabola which with




the diameter from the point of contact will contain an angle equal to the angle at $H$

Let it have been done and let $CD$ be drawn a tangent making with the diameter $EC$ drawn through the point of contact angle $ECD$ equal to angle $H$ and let $CD$ meet the axis at $D$ (I 24) Since then $AD$ is parallel to $EC$ (I 51, end)

$$\text{angle } ADC = \text{angle } ECD$$

But angle $ECD$ is given for it is equal to angle $H$, therefore angle $ADC$ is also given

Then it will be constructed thus let there be a parabola whose axis is $AB$,

and the given angle $H$ Let $CD$ be drawn a tangent to the section making with the axis the angle $ADC$ equal to angle $H$ (II 50), and through $C$ let $EC$ be drawn parallel to $AB$ Since then

$$\text{angle } H = \text{angle } ADC,$$

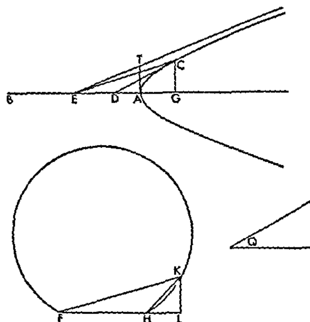
and

$$\text{angle } ADC = \text{angle } ECD,$$

therefore also

$$\text{angle } H = \text{angle } ECD$$

Let the section be an hyperbola whose axis is $AB$ and center $E$, and asymptote $ET$, and the given acute angle $Q$, and let $CD$ be tangent and let $CF$ be joined doing the problem, and let $CG$ be drawn perpendicular Therefore the ratio of the transverse to the upright is given, and so also the ratio of rectangle $EG, GD$ to the square on $CG$ (I 37) Then let some given straight line $FH$ be laid out, and on it let there be described a segment of a circle admitting an angle equal to angle $Q$ (Eucl III 33) therefore it will be greater than a semicircle (Eucl III 31) And from some point $K$ of those on the circumference let $KL$ be drawn perpendicular making




rect $FL$ $LH$ sq $LK$ transverse upright

and let $FK$ and $KH$ be joined Since then

$$\text{angle } FKH = \text{angle } LCD,$$

but also

rect $EG$ $GD$ sq $GC$ transverse upright,

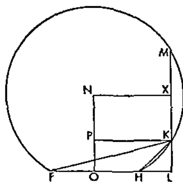
and

rect $FL$ $LH$ sq $LK$ transverse upright,

therefore triangle $KFL$ is similar to triangle $ECG$, and triangle $FHK$ to triangle $ECD$.[1] And so

$$\text{angle } HFK = \text{angle } CED$$

Then it will be constructed thus let there be the given hyperbola $AC$, and axis $AB$, and center $E$, and given acute angle $Q$ and let the given ratio of the transverse to the upright be the same as $YZ$ to $YW$, and let $WZ$ be bisected at $U$ and let a given straight line $FH$ be laid out, and on it let there be described a segment of a circle, greater than semicircle and admitting an angle equal to angle $Q$ (Eucl III 31, 33), and let it be $FKH$, and let the center of the circle $N$ be taken, and from $N$ let $NO$ be drawn perpendicular to $FH$ and let $NO$ be cut at $P$ in the ratio of $UW$ to $WY$ and through $P$ let $PK$ be drawn parallel to $FH$, and from $K$ let $KL$ be drawn perpendicular to $FH$ produced and let $FK$ and $KH$ be joined and let $LK$ be produced to $M$, and from $N$ let $NX$ be drawn perpendicular to it, therefore it is parallel to $FH$ And therefore




$$NP \quad PO \text{ or } UW \quad WY \quad XK \quad KL$$

And doubling the antecedents

$$ZW \quad WY \quad MK \quad KL,$$

*componendo*

$$ZY \quad YW \quad ML \quad LK$$

But

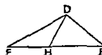
$$ML \quad LK \quad \text{rect } ML\ LK \quad \text{sq } LK$$

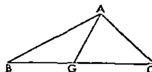
therefore

$$ZY \quad YW \quad \text{rect } ML, LK \quad \text{sq } LK \quad \text{rect } FL\ LH \quad \text{sq } LK \text{ (Eucl III 36)}$$

[1] Pappus in lemma IX to this book Let triangle $ABC$ be similar to triangle $DEF$ and triangle $AGB$ to $DEH$ the result is

$$\text{rect } BC\ CG \quad \text{sq } CA \quad \text{rect } EF\ FH \quad \text{sq } DF$$




For since because of similarity

whole angle $A$ = whole angle $D$

and angle $BAG$ = angle $EDH$

therefore remaining angle $GAC$ = remaining angle $HDF$

But also angle $C$ = angle $F$

therefore $GC \quad CA \quad HF \quad FD$

But also $BC \quad CA \quad EF \quad FD$

therefore also compounded ratio is the same with compounded. Therefore

$$\text{rect } BC\ CG \quad \text{sq } CA \quad \text{rect } EF\ FH \quad \text{sq } FD$$

But

$$ZI : IH :: \text{transverse} : \text{upright},$$

therefore also

$$\text{rect } FL,LH : \text{sq } LK :: \text{transverse} : \text{upright}$$

Then let $AT$ be drawn from $A$ perpendicular to $AB$ Since then

$$\text{sq } EA : \text{sq } AT :: \text{transverse} : \text{upright (II 1)},$$

and also

$$\text{transverse} : \text{upright} :: \text{rect } FL,LH : \text{sq } LK,$$

and

$$\text{sq } FL : \text{sq } LK > \text{rect } FL,LH : \text{sq } LK,$$

therefore also

$$\text{sq } FL : \text{sq } LK > \text{sq } EA : \text{sq } AT$$

And the angles at $A$ and $L$ are right angles, therefore

$$\text{angle } F < \text{angle } E$$

Then let angle $AEC$ be constructed equal to angle $LFK$, therefore $EC$ will meet the section (II 2) Let it meet it at $C$ Then let $CD$ be drawn tangent from $C$ (II 49), and let $CG$ be drawn perpendicular, then

$$\text{transverse} : \text{upright} :: \text{rect } EG,GD : \text{sq } CG \text{ (I 37)}$$

Therefore also

$$\text{rect } FL,LH : \text{sq } LK :: \text{rect } EG,GD : \text{sq } CG$$

therefore triangle $KFL$ is similar to triangle $ECG$ and triangle $KHL$ to triangle $CGD$, and triangle $KFH$ to triangle $CED$ And so

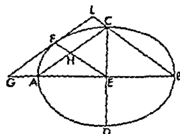
$$\text{angle } ECD = \text{angle } FKH = \text{angle } Q$$

And if the ratio of the transverse to the upright is equal to equal, $KI$ touches the circle $FKH$ (Eucl III 37), and the straight line joined from the center to $K$ will be parallel to $FH$ and itself will do the problem

PROPOSITION 52

*If a straight line touches an ellipse making an angle with the diameter drawn through the point of contact it is not less than the angle adjacent to the one contained by the straight lines deflected at the middle of the section*

Let there be an ellipse whose axes are $AB$ and $CD$ and center $E$ and let $AB$ be the major axis and let the straight line $GFL$ touch the section and let $AC$, $CB$ and $FE$ be joined and let $BC$ be produced to $L$




I say that angle $LFE$ is not less than angle $LCA$

For $FE$ is either parallel to $LB$ or not

Let it first be parallel and

$$AF = EB,$$

therefore also

$$AH = HC$$

And $FE$ is a diameter therefore the tangent at $F$ is parallel to $AC$ (II 6) But also $FE$ is parallel to $LB$ therefore $FHCL$ is a parallelogram and therefore

$$\text{angle } LFH = \text{angle } LCH$$

And since $AE$ and $EB$ are each greater than $EC$ angle $ACB$ is obtuse therefore angle $LCA$ is acute And so also angle $LFE$ And therefore angle $GFE$ is obtuse

Then let $FE$ not be parallel to $LB$, and let $FK$ be drawn perpendicular,

therefore $LBE$ is not equal to angle $FEA$ But

rt angle at $E$ = rt angle at $K$,

therefore it is not true that

$$\text{sq } BE \quad \text{sq } EC \quad \text{sq } EK \quad \text{sq } KF$$

But

sq $BE$ sq $EC$ rect $AE,EB$ sq $EC$ transverse upright (I 21)

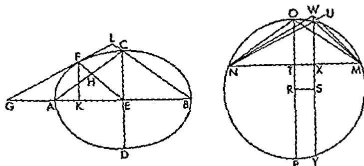
and

transverse upright rect $GK,KE$ sq $KF$ (I 37)

Therefore it is not true that

$$\text{rect } GK,KE \quad \text{sq } KF \quad \text{sq } KE \quad \text{sq } KF$$

Therefore $GK$ is not equal to $KE$ Let there be laid out a segment of a circle




$MUN$ admitting an angle equal to angle $ACB$ (Eucl III 33) and angle $ACB$ is obtuse therefore $MUN$ is a segment less than a semicircle (Eucl III 31) Then let it be contrived that

$$GK \quad KE \quad NX \quad XM$$

and from $X$ let $UXY$ be drawn at right angles and let $NU$ and $UM$ be joined, and let $MN$ be bisected at $T$ and let $OTP$ be drawn at right angles, therefore it is a diameter Let the center be $R$ and from it let $RS$ be drawn perpendicular, and $ON$ and $OM$ be joined Since then

$$\text{angle } MON = \text{angle } ACB$$

and $AB$ and $MN$ have been bisected the one at $E$ and the other at $T$ and the angles at $E$ and $T$ are right angles therefore triangles $OTN$ and $BEC$ are similar Therefore

$$\text{sq } TN \quad \text{sq } TO \quad \text{sq } BE \quad \text{sq } EC$$

And since

$$TR = SX$$

and

$$RO > SU$$

therefore

$$RO \quad TR > SU \quad SX$$

and *convertendo*

$$RO \quad OT < SU \quad UX$$

And doubling the antecedents therefore

$$PO \quad TO < YU \quad UX$$

And *separando*

$$PT \quad TO < YX \quad UX$$

But

$PT : TO = \text{sq } TN : \text{sq } TO = \text{sq } BE : \text{sq } EC = \text{transverse} : \text{upright}$ (I 21),

and

$\text{transverse} : \text{upright} = \text{rect } GK,KE : \text{sq } KF$ (I 37),

therefore

$$\text{rect } GK,KE : \text{sq } KF < YX : XU$$
$$< \text{rect } YX,XU : \text{sq } XU$$
$$< \text{rect } NX,XM : \text{sq } XU.$$

If then we contrive it that

$$\text{rect } GK,KE : \text{sq } KF = \text{rect } MX,XN : \text{some other},$$

it will be greater than the square on $XU$. Let it be to the square on $XW$. Since then

$$GK : KE = NX : XM,$$

and $KF$ and $YW$ are perpendicular and

$$\text{rect } GK,KF : \text{sq } KF = \text{rect } MX,XN : \text{sq } XW,$$

therefore

$$\text{angle } GFE = \text{angle } MWN.$$

Therefore

$$\text{angle } MUN \text{ or angle } ACB > \text{angle } GFE,$$

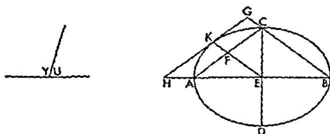
and the adjacent angle $LFH$ is greater than angle $LCH$.

Therefore angle $LFH$ is not less than angle $LCH$.

PROPOSITION 53 (PROBLEM)

*Given an ellipse, to draw a tangent which will make with the diameter drawn through the point of contact an angle equal to a given acute angle; then it is required that the given acute angle be not less than the angle adjacent to the angle contained by the straight lines deflected at the middle of the section.*

Let there be the given ellipse whose major axis is $AB$ and minor axis $CD$, and center $E$, and let $AC$ and $CB$ be joined; and let angle $U$ be the given angle




not less than angle $ACG$, and so also angle $ACB$ is not less than angle $Y$.

Therefore angle $U$ is either greater than or equal to angle $ACG$.

Let it first be equal, and through $E$ let $EK$ be drawn parallel to $BC$ and through $K$ let $KH$ be drawn tangent to the section (II 49). Since then

$$AE = EB$$

and

$$AE : EB = AF : FC,$$

therefore

$$AF = FC.$$

And $KE$ is a diameter therefore the tangent to the section at $K$, that is $HKG$, is parallel to $CA$ (II 6) And also $CK$ is parallel to $GB$, therefore $KFCG$ is a parallelogram, and therefore

$$\text{angle } GKF = \text{angle } GCF$$

And angle $GCF$ is equal to the given angle that is $U$, therefore also

$$\text{angle } GKE = \text{angle } U$$

Then let

$$\text{angle } U > \text{angle } ACG,$$

then inversely

$$\text{angle } Y < \text{angle } ACB$$

Let a circle be laid out and let a segment be taken from it and let it be $MNP$ admitting an angle equal to angle $Y$, and let $MP$ be bisected at $O$, and from $O$ let $NOR$ be drawn at right angles to $MP$, and let $NM$ and $NP$ be joined therefore

$$\text{angle } MNP < \text{angle } ACB$$

But

$$\text{angle } MNO = \text{half angle } MNP,$$

and

$$\text{angle } ACE = \text{half angle } ACB,$$

therefore

$$\text{angle } MNO < \text{angle } ACE$$

And the angles at $E$ and $O$ are right angles, therefore

$$AE : EC > OM : ON$$

And so also

$$\text{sq } AE : \text{sq } EC > \text{sq } MO : \text{sq } NO$$

But

$$\text{sq } AE = \text{rect } AE, EB,$$

and

$$\text{sq } MO = \text{rect } MO, OP = \text{rect } NO, OR \text{ (Eucl. III 35)}$$

therefore

$$\text{rect } AE, EB : \text{sq } EC \text{ or transverse : upright (I 21)} > PO : ON$$

Then let it be that

$$\text{transverse} \quad \text{upright} \quad QA' \quad A'Y',$$

and let $QY'$ be bisected at $Y'$ Since then

$$\text{transverse} \quad \text{upright} > RO \quad ON,$$

also

$$QA' \quad A'Y' > RO \quad ON$$

And *componendo*

$$QY' \quad Y'A' > RN \quad NO$$

Let the center of the circle be $W$, and so also

$$Y'Y' \quad Y'A' > WN \quad NO$$

And *separando*

$$A'Y' \quad A'Y' > WO \quad ON$$

Then let it be contrived that

$$A'Y' \quad A'X' \quad WO \text{ less than } ON$$

such as $IO$, and let $IX$ and $XT$ and $WZ$ be drawn parallel Therefore

$$A'Y' \quad A'X' \quad WO \quad OI \quad ZS \quad SX,$$

and *componendo*

$$Y'X' \quad XA' \quad ZX \quad XS$$

And doubling the antecedents

$$QX' \quad X'A' \quad TX \quad XS$$

And *separando*

$$QA' \quad A'X \text{ or transverse} \quad \text{upright} \quad TS \quad SX$$

Then let $MX$ and $XP$ be joined, and let angle $AEK$ be constructed on straight line $AE$ at point $E$ equal to angle $MPX$ and through $K$ let $KH$ be drawn touching the section (II 49), and let $KL$ be dropped ordinatewise Since then

$$\text{angle } MPX = \text{angle } AEK,$$

and

$$\text{rt angle at } S = \text{rt angle at } L$$

therefore triangle $XSP$ is equiangular with triangle $KEL$ And

$$\text{transverse} \quad \text{upright} \quad TS \quad SX \quad \text{rect } TS\,SX$$
$$\text{sq } SX \quad \text{rect } MX\,SP \quad \text{sq } SX$$

therefore triangle $KLE$ is similar to triangle $SXP$ and triangle $MXP$ to triangle $KHE$ and therefore

$$\text{angle } MXP = \text{angle } HKE$$

But

$$\text{angle } MXP = \text{angle } MNP = \text{angle } Y$$

therefore also

$$\text{angle } HKE = \text{angle } Y$$

And therefore

$$\text{adjacent angle } GKE = \text{adjacent angle } U$$

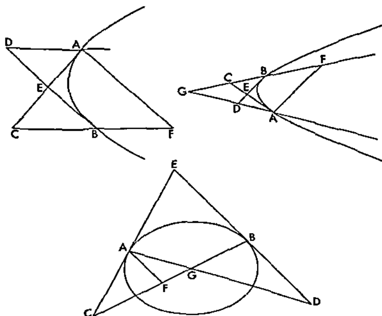
Therefore $GH$ has been drawn across tangent to the section and making with the diameter $KE$ drawn through the point of contact angle $GKE$ equal to the given angle $U$ and this it was required to do

# BOOK THREE

## Proposition 1

*If straight lines touching a section of a cone or circumference of a circle, meet and diameters are drawn through the points of contact meeting the tangents, the resulting vertically related triangles will be equal*

Let there be the section of a cone or circumference of a circle $AB$ and let $AC$ and $BD$, meeting at $E$, touch $AB$, and let the diameters of the section $CB$ and




$DA$ be drawn through $A$ and $B$ meeting the tangents at $C$ and $D$

I say that

$$\text{trgl } ADE = \text{trgl } EBC$$

For let $AF$ be drawn from $A$ parallel to $BD$ therefore it has been dropped ordinatewise (I 32) Then in the case of the parabola

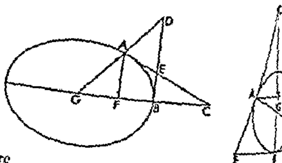
$$\text{pllg } ADBF = \text{trgl } ACF \text{ (I 42)}$$

and with the common area $AEBF$ subtracted

$$\text{trgl } ADE = \text{trgl } CBE$$

And in the case of the others let the diameters meet at center $G$ Since then

AF has been dropped ordinatewise and AC tangent ... rect FG, GC = sq BG ...




Therefore

therefore also FG GB BG GC,

But FG GC sq FG sq GB (Eucl ...

and sq FG sq GB trgl AGF trgl DGB ...

therefore also FG GC trgl AGF trgl AGC,

Therefore trgl AGF trgl AGC trgl AGF trgl ...

trgl AGC = trgl DGB

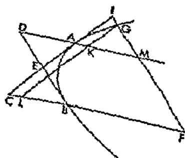
Let the common area DGBE be subtracted, therefore as ...

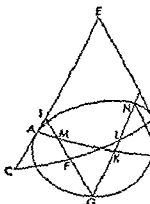
trgl AED = trgl CEB

PROPOSITION 2

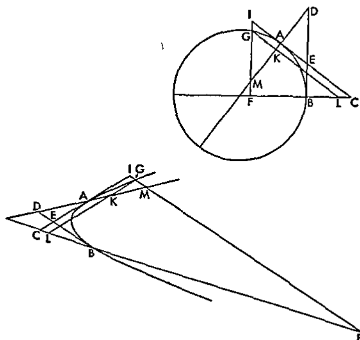
*With the same things supposed if some point is taken on the ... ence of a circle and through it parallels to the tangents are dr... diameters then the quadrilateral produced on one of the tangen... diameters will be equal to the triangle produced on the same tan... diameter*

For let there be a section of a cone or circumference of a ci...



*AEC* and *BED* be tangents, and *AD* and *BC* diameters, and let some point *G*




be taken on the section and *GKL* and *GMF* be drawn parallel to the tangents I say that

$$\text{trgl } AIM = \text{quadr } CLGI$$

For triangle *GKM* has been shown equal to quadrilateral *AL* (I 42, 43), let the common quadrilateral *IK* be added or subtracted, and

$$\text{trgl } AIM = \text{quadr } CG$$ [1]

## Proposition 3

*With the same things supposed, if two points are taken on the section or circumference of a circle and through them parallels to the tangents are drawn as far as the diameters, the quadrilaterals produced by the straight lines drawn and standing on the diameters as bases are equal to each other*

For let there be the section and tangents and diameters as said before and let two points at random *F* and *G* be taken on the section, and through *F* let the straight lines *FHKL* and *NFIM* be drawn parallel to the tangents and through *G* the straight lines *GXO* and *HPR*

I say that

$$\text{quadr } LG = \text{quadr } MH$$

[1]Eutocius commenting gives the proof for another and important case It must be remarked that if the point *G* is taken between *A* and *B* so that the parallels are for instance

and

$$\text{quadr } LN = \text{quadr } RN$$

For since it has already been shown that

$$\text{trgl } RPA = \text{quadr } CG \text{ (III. 2)},$$

and

$$\text{trgl } AMI = \text{quadr } CF \text{ (III. 2)},$$

and

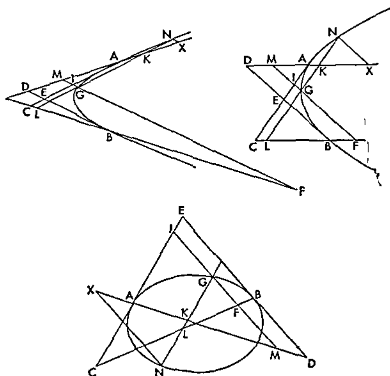
$$\text{trgl } RPA = \text{trgl } AMI + \text{quadr } PM,$$

therefore also

$$\text{quadr } CG = \text{quadr } C\Gamma + \text{quadr } PM,$$

---

$MIGI$ and $LGK$ one must draw $IK$ to the section at $N$ for instance and through $N$ draw




$KX$ parallel to $BD$ for by what was said in the forty ninth and fiftieth theorems of the first book (I 49 50) and in the notes to them

$$\text{trgl } KNX = \text{quadr } KC$$

But triangle $KNX$ is similar to triangle $KMG$ because $MG$ is parallel to $NX$ but it is also equal to it because $AC$ is a tangent and $CN$ is parallel to it and $MX$ is a diameter and

$$GK = KN$$

Since then $\text{trgl } KNX = \text{quadr } KC = \text{trgl } KMG$

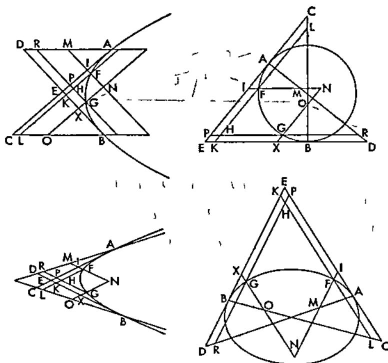
with the common quadrilateral $AC$ subtracted as remainders

$$\text{trgl } AIM = \text{quadr } CG$$

It will be noticed that, just as in the second note to I 50 the quadrilateral $CG$ is to be considered as the difference between the triangles $CIF$ and $CFI$

and so

$$\text{quadr } CG = \text{quadr } CH + \text{quadr } RF$$




be taker
I sa
t the common quadrilateral $CH$ be subtracted therefore as remainders
Fo
$$\text{quadr } LG = \text{quadr } HM$$
the and therefore as wholes

$$\text{quadr } LN = \text{quadr } RN$$

### Proposition 4

*If two straight lines touching opposite sections meet each other, and diameters are drawn through the points of contact meeting the tangents then the triangles at the tangents will be equal*

Let there be the opposite sections $A$ and $B$ and let the tangents to them $AC$ and $BC$ meet at $C$ and let $D$ be the center of the sections and let $AB$ and $CD$ be joined and $CD$ produced to $E$ and let $DA$ and $BD$ also be joined and produced to $F$ and $G$

I say that

$$\text{trgl } AGD = \text{trgl } BDF,$$

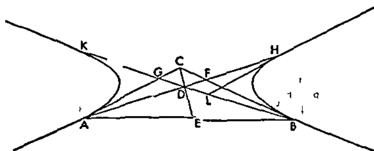
and

$$\text{trgl } ACF = \text{trgl } BCC$$

For let $HL$ be drawn through $H$ tangent to the section, therefore it is parallel to $AG$ (I 44 note) And since

$$AD = DH \text{ (I 30)},$$

trgl $AGD$ = trgl $DHL$ (Eucl vi 19)




But

trgl $DHL$ = trgl $BDF$ (iii 1),

therefore also

trgl $AGD$ = trgl $BDF$

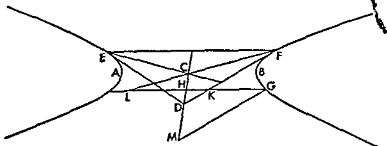
And so also

trgl $ACF$ = trgl $BCG$

PROPOSITION 5

*If two straight lines touching opposite sections meet, and some p[…] either of the sections, and from it two straight lines are drawn, t[…] the tangent, the other parallel to the line joining the points of […] triangle produced by them on the diameter drawn through the […] differs from the triangle cut off at the point of meeting of the tang[…] angle cut off on the tangent and the diameter drawn through the po[…]*

Let there be the opposite sections $A$ and $B$ whose center is $C$, […] gents $ED$ and $DF$ meet at $D$, and let $EF$ and $CD$ be joined and let […]

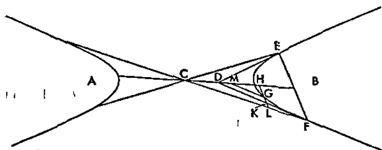



duced, and let $FC$ and $EC$ be joined and produced and let some point $G$ be taken on the section, and through it let $HGKL$ be drawn parallel to $EF$, and $GM$ parallel to $DF$

I say that triangle $GHM$ differs from triangle $KHD$ by triangle $KLF$

For since $CD$ has been shown to be a diameter of the opposite sections (ii

39, 38), and $EF$ to be an ordinate to it (II 38, First Def I 5), and $GH$ has been




drawn parallel to $EF$, and $MG$ parallel to $DF$, therefore triangle $GHM$ differs from triangle $CLH$ by triangle $CDF$ (I 45, or I 44, according to the case) And so triangle $GHM$ differs from triangle $KHD$ by triangle $KLF$

And it is evident that

$$\text{trgl } KLF = \text{quadr } MGKD$$

## Proposition 6

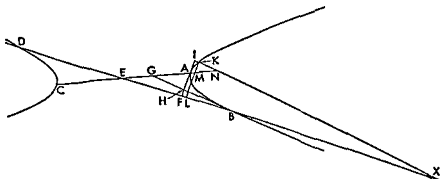
...ame things supposed if some point is taken on one of the opposite sec-... from it parallels to the tangents are drawn meeting the tangents and the ... then the quadrilateral produced by them on one of the tangents and on ... diameters will be equal to the triangle produced on the same tangent and ...iameter

...re be opposite sections of which $AEC$ and $BED$ are diameters and let ...$G$ touch the section $AB$ meeting each other at $H$ and let some point ...en on the section and from it let $KML$ and $KNX$ be drawn parallel ...angents

...r that

$$\text{quadr } KF = \text{trgl } AIN$$

...y since $AB$ and $CD$ are opposite sections, and $AF$, meeting $BD$, touches




...ction $AB$ and $KL$ has been drawn parallel to $AF$ therefore

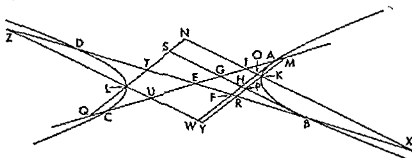
trgl $AIN$ = quadr $KF$ (III 2)[1]

### Proposition 7

*With the same things supposed, if points are taken on each of the sections, and from them parallels to the tangents are drawn meeting the tangents and the diameters, then the quadrilaterals produced by the straight lines drawn and standing on the diameters as bases will be equal to each other*

For let the aforesaid things be supposed and let points $K$ and $L$ be taken on both sections and through them let $MKPRY$ and $NSTLQ$ be drawn parallel to $AF$ and $NIOKX$ and $YWULZ$ parallel to $BG$

I say that what was said in the enunciation will be so




For since

$$\text{trgl } AOI = \text{quadr } RO \text{ (III 2)},$$

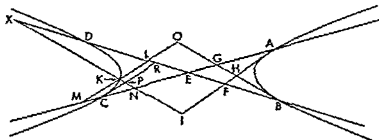
[1] Another and important case where the point $K$ falls between $C$ and $D$ is given by Eu in his commentary to this proposition It is as follows and let $CPR$ be drawn tangent to the section then it is evident that it is parallel to $AF$ and $ML$ (I 44 note) since it has been shown in the second theorem (III 2) in the figure of the hyperbola

$$\text{trgl } PNC = \text{quadr } LP \text{ (III 2 note)}$$

let the common quadrilateral $MP$ be added therefore

$$\text{trgl } MKN = \text{quadr } MLRC$$

Let there be added the common triangle $CRE$ which is equal to triangle $AEF$ by I 44 (and I 30) therefore




$$\text{whole trgl } MEL = \text{trgl } MKN + \text{trgl } AEF$$

With common triangle $KMN$ subtracted as remainders

$$\text{trgl } AEF = \text{quadr } KIFN$$

Let the common quadrilateral $FENI$ be added therefore in whole

$$\text{trgl } AIN = \text{quadr } KIFI$$

And likewise also

$$\text{trgl } BOL = \text{quadr } KNGO$$

let the quadrilateral $EO$ be added to both therefore

whole trgl $AEF$ = quadr $KE$

But also

trgl $BGE$ = quadr $LE$ (III 5, note),

and

trgl $AEF$ = trgl $BGE$ (III 1),

therefore

quadr $LE$ = quadr $IKRE$

Let the common quadrilateral $NE$ be added, therefore as wholes

whole quadr $TK$ = quadr $IL$,

and also

quadr $KU$ = quadr $RL$

PROPOSITION 8

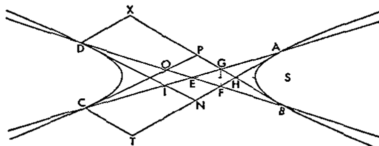
With the same things supposed, instead of $K$ and $L$ let there be taken the points $C$ and $D$ at which the diameters hit the sections and through them let the parallels to the tangents be drawn

I say that

quadr $DG$ = quadr $FC$

and

quadr $XI$ = quadr $OT$




For since it was shown

trgl $AGH$ = trgl $HBF$ (III 1)

and the straight line from $A$ to $B$ is parallel to the straight line from $G$ to $F$[1] therefore

$AE$ $EG$ $BE$ $EF$

and *convertendo*

$EA$ $AG$ $EB$ $BF$

And also

$CA$ $AE$ $DB$ $BE$,

[1]For the point $H$ falls within the angle $AEB$ (II 25) and the straight line drawn from $H$ to the midpoint of $AB$ that is $S$ is a diameter (II 29) and must therefore pass through $E$ (I 51 end) An analogous series of propositions is found for the opposite sections II. 32 38 39

Then, since trgl $GHA$ = trgl $FHB$

therefore trgl $GFB$ = trgl $GFA$

Their bases are the same therefore their heights are equal (Eucl. VI 1)

for each is double the other therefore *ex aequali*

$$CA \quad AG \quad DB \quad BF$$

And the triangles are similar because of the parallels, therefore

trgl $CTA$ trgl $AHG$ trgl $\lambda BD$ trgl $HBF$ (Eucl VI 19)

And alternately, but

trgl $AHG$ = trgl $HBF$ (III 1),

therefore

trgl $CTA$ = trgl $\lambda BD$

As parts of these it was shown

trgl $AHG$ = trgl $HBF$,

therefore also as remainders

quadr $DH$ = quadr $CH$

And so also

quadr $DG$ = quadr $CF$

And since $CO$ is parallel to $AF$

trgl $COE$ = trgl $AEF$

And likewise also

trgl $DEI$ = trgl $BEG$

But

trgl $BEG$ = trgl $AEF$ (III 1),

therefore also

trgl $COE$ = trgl $DEI$

And also

quadr $DG$ = quadr $CF$ (above)

Therefore as wholes,

quadr $XI$ = quadr $OT$

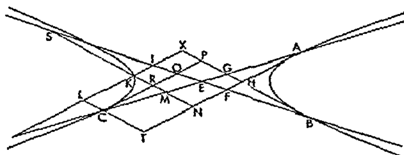
PROPOSITION 9

With the same things supposed if one of the points is between the diameters as $K$, and the other is the same with one of the points $C$ and $D$ for instance $C$, and the parallels are drawn I say that

trgl $CEO$ = quadr $KE$

and

quadr $LO$ = quadr $LM$




And this is evident For since it was shown

trgl $CEO$ = trgl $AEF$,

and

trgl $AEF$ = quadr $KE$ (III 5, note),

therefore also

trgl $CEO$ = quadr $KE$

And so also

trgl $CRM$ = quadr $KO$,

and

quadr $KC$ = quadr $LO$

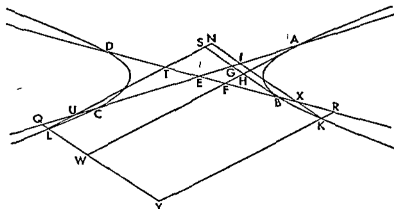
## Proposition 10

With the same things supposed, let $K$ and $L$ be taken not as points at which the diameters hit the sections

Then it is to be shown that

quadr $LTRY$ = quadr $QYKI$

For since the straight lines $AF$ and $BG$ touch, and $AE$ and $BE$ are diameters




through the points of contact and $LT$ and $KI$ are parallel to the tangents,

trgl $TUE$ = trgl $UQL$ + trgl $EFA$ (I 44)

And likewise also

trgl $XEI$ = trgl $XRK$ + trgl $BEG$

But

trgl $EFA$ = trgl $BEG$ (III 1)

therefore

trgl $TUE$ − trgl $UQL$ = trgl $XEI$ − trgl $XRK$

Therefore

trgl $TUE$ + trgl $XRK$ = trgl $XEI$ + trgl $UQL$

Let the common area $KXEULY$ be added therefore

quadr $LTRY$ = quadr $QYKI$

## Proposition 11

*With the same things supposed if some point is taken on either of the sections and from it parallels are drawn one parallel to the tangent and the other parallel to the straight line joining the points of contact then the triangle produced by them on the diameter drawn through the point of meeting of the tangents differs from the tri-*

for each is double the other, therefore *ex aequali*

$CA : AG :: DB : BF$

And the triangles are similar because of the parallels, therefore

trgl $CTA$ : trgl $AHG$ :: trgl $XBD$ : trgl $HBF$ (Eucl vi 19)

And alternately, but

trgl $AHG$ = trgl $HBF$ (iii 1),

therefore

trgl $CTA$ = trgl $XBD$

As parts of these it was shown

trgl $AHG$ = trgl $HBF$

therefore also as remainders

quadr $DH$ = quadr $CH$

And so also

quadr $DG$ = quadr $CF$

And since $CO$ is parallel to $AF$

trgl $COE$ = trgl $AEF$

And likewise also

trgl $DEI$ = trgl $BEG$

But

trgl $BEG$ = trgl $AEF$ (iii 1),

therefore also

trgl $COE$ = trgl $DEI$

And also

quadr $DG$ = quadr $CF$ (above)

Therefore, as wholes

quadr $XI$ = quadr $OT$

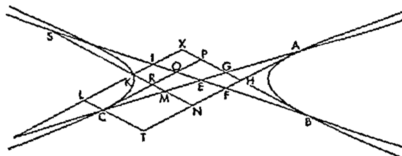
PROPOSITION 9

With the same things supposed, if one of the points is between the diameters, as $K$, and the other is the same with one of the points $C$ and $D$ for instance $C$, and the parallels are drawn, I say that

trgl $CEO$ = quadr $KE$

and

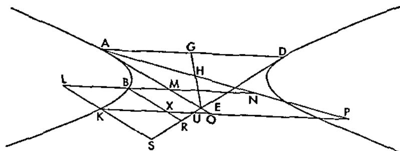
quadr $LO$ = quadr $LM$

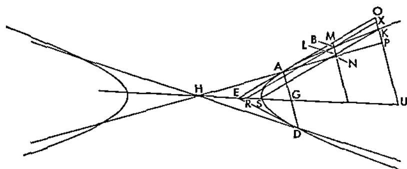



And this is evident For since it was shown

trgl $CEO$ = trgl $AEF$,

For let there be the same things as before and let the points $B$ and $K$ be taken at random on section $AB$ and through them let $LBMN$ and $KXOUP$




be drawn parallel to $AD$ and $BXR$ and $LKS$ parallel to $AE$ I say that

quadr $BP$ = quadr $KR$

For since it has been shown

trgl $AOP$ = quadr $KOES$ (III 11 end)

and

trgl $AMN$ = quadr $BMER$ (III 11 end),

therefore, as remainders either

quadr $KR$ − quadr $BO$ = quadr $MP$

or

quadr $KR$ + quadr $BO$ = quadr $MP$

And with the common quadrilateral $BO$ added or subtracted

quadr $BP$ = quadr $XS$

## Proposition 13

*If in conjugate opposite sections straight lines tangent to the adjacent sections meet and diameters are drawn through the points of contact then the triangles whose common vertex is the center of the opposite sections will be equal*

Let there be conjugate opposite sections on which there are the points $A$, $B$ $C$ and $D$ and let $BE$ and $AE$ meeting at $E$ touch the sections $A$ and $B$ and let $H$ be the center and let $AH$ and $BH$ be joined and produced to $D$ and $C$

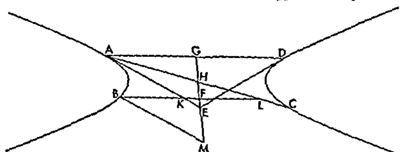
*angle cut off on the tangent and the diameter drawn through the point of contact by the triangle cut off at the point of meeting of the tangents*

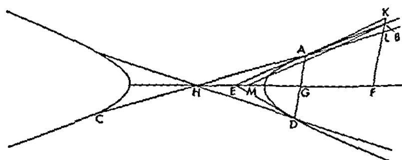
Let there be the opposite sections $AB$ and $CD$, and let the tangents $AE$ and $DE$ meet at $E$ and let the center be $H$, and let $AD$ and $EHG$ be joined, and let some point $B$ be taken at random on the section $AB$, and through it let $BFL$ be drawn parallel to $AG$, and $BM$ parallel to $AE$

I say that triangle $BFM$ differs from triangle $AKL$ by triangle $KEF$

For it is evident that $AD$ is bisected by $EH$ (II 39 and II 29), and that $EH$ is a diameter conjugate to the diameter drawn through $H$ parallel to $AD$ (II 38), and so $AG$ is an ordinate to $EG$ (First Def I 6)

Since then $GE$ is a diameter and $AE$ touches and $AG$ is an ordinate and with point $B$ taken on the section, $BF$ has been dropped to $EG$ parallel to $AG$,




and $BM$ parallel to $AE$, therefore it is clear that triangle $BMF$ differs from triangle $LHF$ by triangle $HAE$ (I 45, I 43) [1] And so also triangle $BMF$ differs from triangle $AKL$ by triangle $KFE$

And it has been proved at the same time that

quadr $BKEM$ = trgl $LKA$

PROPOSITION 12

*With the same things being so if on one of the sections two points are taken and parallels are drawn from each of them, likewise the quadrilaterals produced by them will be equal*

[1] That is in the first case

trgl $BMF$ = trgl $LHF$ + trgl $HAF$ (I 45)

In the second case only the more general statement differs holds true (I 43) It will be noticed these are different cases of I 43 and I 45 from those given in the text itself

*B* and through it let *XRS* be drawn parallel to *AG* and *XTO* parallel to *BE*

I say that triangle *OHT* differs from triangle *XST* by triangle *HBF*

For let *AU* be drawn from *A* parallel to *BF* Since then because of the same things as before, *LHM* is a diameter of the section *AL*, and *DHB* is a second diameter and conjugate to it (II 20), and *AG* is a tangent at *A*, and *AU* has been dropped parallel to *LM*, therefore

$$AU \quad UG \text{ comp } HU \quad UA,$$
$$\text{transverse side of figure on } LM \quad \text{upright (I 40)}$$

But

$$AU \quad UG \quad XT \quad TS,$$

and

$$HU \quad UA \quad HT \quad TO \quad HB \quad BF,$$

and

$$\text{transverse side of figure on } LM \quad \text{upright}$$
$$\text{upright side of figure on } BD \quad \text{transverse (I 60)}$$

Therefore

$$XT \quad TS \text{ comp } HB \quad BF, \text{ upright side of figure on } BD \quad \text{transverse}$$

or

$$XT \quad TS \text{ comp } HT \quad TO \text{ upright side of figure on } BD \quad \text{transverse}$$

And by things shown in the forty first theorem of the first book (I 41), triangle *THO* differs from triangle *XTS* by triangle *BFH*

And so also by triangle *AGH* (III 13)

### Proposition 15

*If straight lines touching one of the conjugate opposite sections meet and diameters are drawn through the points of contact and some point is taken on any one of the conjugate sections and from it parallels to the tangents are drawn as far as the diameters then the triangle produced by them at the section is greater than the triangle produced at the center by the triangle having the tangent as base and the center of the opposite sections as vertex* [1]

Let there be conjugate opposite sections *AB GS T* and *Y* whose center is *H* and let *ADE* and *BDC* touch the section *AB* and let the diameters *AHFW* and *BHT* be drawn through the points of contact *A* and *B* and let some point *S* be taken on the section *GS* and through it let *SFL* be drawn parallel to *BC* and *SU* parallel to *AE*

I say that

$$\text{trgl } SLU = \text{trgl } HLF + \text{trgl } HCB$$

For let *YHG* be drawn through *H* parallel to *BC* and *KIG* through *G* parallel to *AE* and *SO* parallel to *BT* then it is evident that *YG* is a diameter conjugate to *BT* (II 20) and that *SO* being parallel to *BT* has been dropped ordinatewise to *HGO* (First Def I 6) and that *SLHO* is a parallelogram

Since then *BC* touches and *BH* is through the point of contact and *AE* is another tangent let it be contrived that

$$DB \quad BE \quad MN \quad 2BC,$$

therefore *MN* is the so-called upright side of the figure on *BT* (I 50) Let *MN* be bisected at *P* therefore

[1] This proposition comes as a climax to a long series and shows that the conjugate opposite sections taken as a unit have the same property as the other conic sections The conjugate opposite sections seem to be a sort of fifth section.

$$DB \quad BE \quad MP \quad BC$$

Then let it be contrived that

$$\lambda G \quad TB \quad TB \quad R, \qquad (\alpha)$$

then $R$ also will be the so-called upright side of the figure applied to $\lambda G$ (I 16, 60)

Since then

$$DB \quad BE \quad MP \quad BC, \qquad (\beta)$$

but

$$DB \quad BE \quad \text{sq } DB \quad \text{rect } DB\,BE,$$

and

$$MP \quad BC \quad \text{rect } MP, BH \quad \text{rect } CB\,BH,$$

therefore

$$\text{sq } DB \quad \text{rect } DB\,BE \quad \text{rect } MP\,BH \quad \text{rect } CB, BH$$

And

$$\text{rect } MP\,BH = \text{sq } HG$$

because

$$\text{sq } \lambda G = \text{rect } TB\,MN \ (\text{I } 16) \qquad (\gamma)$$

and

$$\text{rect } MP\,BH = \text{fourth rect } TB\,MN$$

and

$$\text{sq } HG = \text{fourth sq } \lambda G,$$

therefore

$$\text{sq } DB \quad \text{rect } DB, BE \quad \text{sq } HG \quad \text{rect } CB\,BH$$

Alternately

$$\text{sq } DB \quad \text{sq } HG \quad \text{rect } DB\,BE \quad \text{rect } CB, BH$$

But

$$\text{sq } DB \quad \text{sq } HG \quad \text{trgl } DBE \quad \text{trgl } GHI,$$

for they are similar and

$$\text{rect } DB\,BE \quad \text{rect } CB\,BH \quad \text{trgl } DBE \quad \text{trgl } CBH$$

therefore

$$\text{trgl } DBE \quad \text{trgl } GHI \quad \text{trgl } DBE \quad \text{trgl } CBH$$

Therefore

$$\text{trgl } GHI = \text{trgl } CBH$$

Again since

$$HB : BC = \text{comp } HB : MP,\ MP : BC,$$

but

$$HB : MP = TB : MN = R : XG \text{ (above } \alpha \text{ and } \gamma),$$

and

$$MP : BC = DB : BE \text{ (above, } \beta)$$

therefore

$$HB : BC = \text{comp } DB : BE,\ R : XG$$

And since $BC$ is parallel to $SL$, and triangle $HCB$ is similar to triangle $HLF$, and

$$HB : BC = HL : LF,$$

therefore

$$HL : LF = \text{comp } R : XG,\ DB : BE$$

or

$$HL : LF = \text{comp } R : XG,\ HG : HI$$

Since then $GS$ is an hyperbola having $XG$ as a diameter and $R$ as an upright side and from some point $S$ $SO$ has been dropped ordinatewise and figure $HIG$ has been described on radius $HG$ and figure $HLF$ has been described on the ordinate $SO$ or its equal $HL$, and on $HO$ the straight line between the center and the ordinate or on $SL$ its equal the figure $SLU$ has been described similar to the figure $HIG$ described on the radius, and there are the compounded ratios as already given therefore

$$\text{trgl } SLU = \text{trgl } HLF + \text{trgl } HCB \text{ (I 41)}$$

## Proposition 16

*If two straight lines touching a section of a cone or circumference of a circle meet and from some point of those on the section a straight line is drawn parallel to one tangent and cutting the section and the other tangent then as the squares on the tangents are to each other so the area contained by the straight lines between the section and the tangent will be to the square cut off at the point of contact*

Let there be the section of a cone or circumference of a circle $AB$ and let the straight lines $AC$ and $CB$ meeting at $C$ touch it and let some point $D$ be taken on the section $AB$ and through it let $EDF$ be drawn parallel to $CB$

I say that

$$\text{sq } BC : \text{sq } AC = \text{rect } FE, ED : \text{sq } EA$$

For let the diameters $AGH$ and $KBL$ be drawn through $A$ and $B$, and $DMN$ through $D$ parallel to $AL$, it is at once evident that

$$DK = KF \text{ (I 46 47)}$$

and

$$\text{trgl } AEG = \text{quadr } LD \text{ (III 2)},$$

and

$$\text{trgl } BLC = \text{trgl } ACH \text{ (III 1)}$$

Since then

$$DK = KF$$

and $DE$ is added

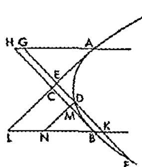
$$\text{rect } FE, ED + \text{sq } DK = \text{sq } KE$$

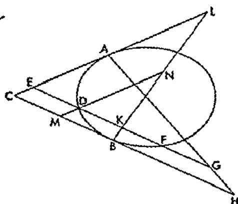
And since triangle $ELK$ is similar to triangle $DNK$

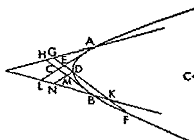
$$\text{sq } EK : \text{sq } KD = \text{trgl } EKL : \text{trgl } DNK$$

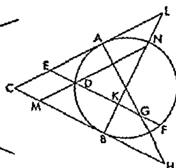
And alternately

whole sq *EK* whole trgl *ELK*
part subtracted sq *DK* part subtracted trgl *DNK*




Therefore also

remainder rect *FE ED* remainder quadr *DL* sq *EK* trgl *ELK*

But

sq *EK* trgl *ELK* sq *CB* trgl *BLC*

therefore also

rect *FE ED* quadr *LD* sq *CB* trgl *LCB*

But

quadr *LD* = trgl *AEG*

and

trgl *BLC* = trgl *ACH*,

therefore also

rect *FE ED* trgl *AEG* sq *CB* trgl *ACH*

Alternately

rect *FE ED* sq *CB* trgl *AEG* trgl *ACH*

But

trgl *AEG* trgl *ACH* sq *EA* sq *AC*

therefore also

rect *FE ED* sq *CB* sq *EA* sq *AC*

And alternately

## PROPOSITION 17

*If two straight lines touching a section of a cone or circumference of a circle meet, and two points are taken at random on the section, and from them in the section are drawn parallel to the tangents straight lines cutting each other and the line of the section, then as the squares on the tangents are to each other, so will the rectangles contained by the straight lines taken similarly*

Let there be the section of a cone or circumference of a circle $AB$, and tangents to $AB$ $AC$ and $CB$ meeting at $C$, and let points $D$ and $E$ be taken at random on the section and through them at $EFIK$ and $DFGH$ be drawn parallel to $AC$ and $CB$

I say that

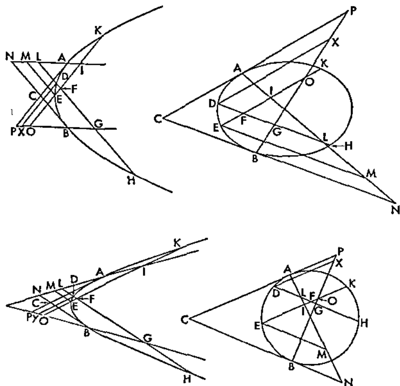
sq $CA$ sq $CB$ rect $KF$ $FE$ rect $HF$ $FD$

For let the diameters $ALMN$ and $BOXP$ be drawn through $A$ and $B$, and let the tangents and parallels be produced to the diameters and let $DX$ and $EM$ be drawn from $D$ and $E$ parallel to the tangents, then it is evident that

$KI=IE$, $HG=GD$ (I 46 47)

Since then $KE$ has been cut equally at $I$ and unequally at $F$,

rect $KF, FE$+sq $FI$=sq $EI$ (Eucl II 5)




And since the triangles are similar because of the parallels

whole sq $EI$ whole trgl $IME$

part subtracted sq $IF$ part subtracted trgl $FIL$

Therefore also

remainder rect $KF, FE$ remainder quadr $FM$ whole sq $EI$ whole trgl $IME$

But

sq $EI$ trgl $IME$ sq $CA$ trgl $CAN$

Therefore

rect $KF, FE$ quadr $FM$ sq $CA$ trgl $CAN$

But

trgl $CAN$ = trgl $CPB$ (III 1),

and

quadr $FM$ = quadr $F\Lambda$ (III 3)

therefore

rect $KF$ $FE$ quadr $F\Lambda$ sq $CA$ trgl $CPB$

Then likewise it could be shown that

rect $HF, FD$ quadr $FY$ sq $CB$ trgl $CPB$

Since then

rect $KF$ $FE$ quadr $FY$ sq $CA$ trgl $CPB$

and *inversely*

quadr $F\Lambda$ rect $HF$ $FD$ trgl $CPB$ sq $CB$

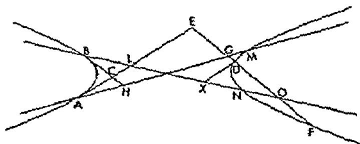
therefore *ex aequali*

sq $CA$ sq $CB$ rect $KF$ $FE$ rect $HF$ $FD$

## Proposition 18

*If two straight lines touching opposite sections meet and some point is taken on either one of the sections and from it some straight line is drawn parallel to one of the tangents cutting the section and the other tangent then as the squares on the tangents are to each other so will the rectangle contained by the straight lines between the section and the tangent be to the square on the straight line cut off at the point of contact*

Let there be the opposite sections $AB$ and $MN$ and tangents $ACL$ and $BCH$ and through the points of contact the diameters $AM$ and $BN$ and let some point $D$ be taken at random on the section $MN$ and through it let $EDF$ be drawn parallel to $BH$




I say that

sq $BC$ sq $CA$ rect $FE, FD$ sq $AF$

For let $DX$ be drawn through $D$ parallel to $AL$ Since then $AB$ is an hyper

bola and $BN$ its diameter and $BH$ a tangent and $DF$ parallel to $BH$, therefore

$$FO = OD \text{ (I 48)}$$

And $ED$ is added therefore

$$\text{rect } FE, ED + \text{sq } DO = \text{sq } EO \text{ (Eucl II 6)}$$

And since $EL$ is parallel to $D\Lambda$, triangle $EOL$ is similar to triangle $D\Lambda O$. Therefore

$$\text{whole sq } EO \quad \text{whole trgl } EOL$$
$$\text{part subtracted sq } DO \quad \text{part subtracted trgl } D\Lambda O,$$

therefore also

$$\text{remainder rect } DE, EF \quad \text{remainder quadr } DL \quad \text{sq } EO \quad \text{trgl } EOL$$

But

$$\text{sq } OE \quad \text{trgl } EOL \quad \text{sq } BC \quad \text{trgl } BCL$$

therefore also

$$\text{rect } FE, ED \quad \text{quadr } DL \quad \text{sq } BC \quad \text{trgl } BCL$$

And

$$\text{quadr } DL = \text{trgl } AEG \text{ (III 6 note)},$$

and

$$\text{trgl } BCL = \text{trgl } ACH \text{ (III 1)},$$

therefore

$$\text{rect } FE, ED \quad \text{trgl } AEG \quad \text{sq } BC \quad \text{trgl } ACH$$

But also

$$\text{trgl } AEG \quad \text{sq } EA \quad \text{trgl } ACH \quad \text{sq } AC,$$

therefore *ex aequali*

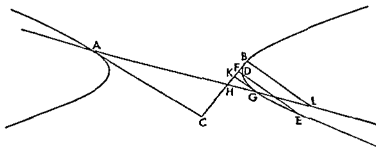
$$\text{sq } BC \quad \text{sq } AC \quad \text{rect } FE\,ED \quad \text{sq } EA\ [1]$$

[1] Eutocius gives an alternative proof of Apollonius demonstrating another and important case For let there be the opposite sections $A$ and $B$ and tangents to them $AC$ and $CB$ meeting at $C$ and let $D$ be taken on section $B$ and through it let $EDF$ be drawn parallel to $AC$ I say that

$$\text{sq } AC \quad \text{sq } CB \quad \text{rect } EF\,FD \quad \text{sq } FB$$

For let $AHG$ be drawn as a diameter through $A$ and through $B$ and $G$ $GK$ and $BL$ parallel to $EF$ Since then $BH$ touches the hyperbola at $B$ and $BL$ has been drawn ordinatewise

$$AL \quad LG \quad AH \quad HG \text{ (I 36)}$$




| | |
|---|---|
| But | $AL \quad LG \quad CB \quad BK$ |
| and | $AH \quad HG \quad AC \quad KG$ |
| therefore also | $CB \quad BK \quad AC \quad KG$ |
| And alternately | $AC \quad CB \quad KG \quad KB$ |
| and | $\text{sq } AC \quad \text{sq } CB \quad \text{sq } GK \quad \text{sq } KB$ |
| But it was shown | $\text{sq } GK \quad \text{sq } KB \quad \text{rect } EF\,FD \quad \text{sq } FB$ |
| therefore also | $\text{sq } AC \quad \text{sq } CB \quad \text{rect } EF\,FD \quad \text{sq } FB$ |

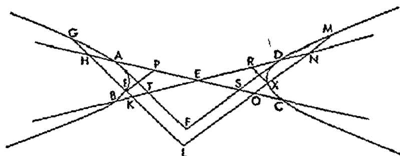
### Proposition 19

*If two straight lines touching opposite sections meet and parallels to the tangents are drawn cutting each other and the section then, as the squares on the tangents are to each other so will the rectangle contained by the straight lines between the section and the point of meeting of the straight lines be to the rectangle contained by the straight lines taken similarly*

Let there be opposite sections whose diameters are $AC$ and $BD$ and center in $E$, and let the tangents $AF$ and $FD$ meet at $F$, and let $GHIKL$ and $MNXOL$ be drawn from any points parallel to $AF$ and $FD$

I say that

sq $AF$  sq $FD$   rect $GL\ LI$   rect $ML, L\lambda$




Let $IP$ and $XR$ be drawn through $\lambda$ and $I$ parallel to $AF$ and $FD$

$BH$ touches the hyperbola at $B$ and $BL$ has been drawn ordinatewise,

$AL$  $LG$   $AH$  $HG$ (I, 36)

And since

sq $AF$  trgl $AFS$   sq $HL$  trgl $HLO$   sq $HI$  trgl $HIP$

therefore

remainder rect $GL\ LI$  remainder quadr $IPOL$   sq $AF$  trgl $AFS$

But

trgl $AFS$ = trgl $DTF$ (III 4)

and

quadr $IPOL$ = quadr $KR\lambda L$ (III 7)

therefore also

sq $AF$  trgl $DTF$   rect $GL\ LI$  quadr $KRXL$

But

trgl $DTF$  sq $FD$   quadr $KRXL$  rect $ML\ LX$ (likewise),

and therefore *ex aequali*

sq $AF$  sq $FD$   rect $GL\ LI$  rect $ML\ LX$

### Proposition 20

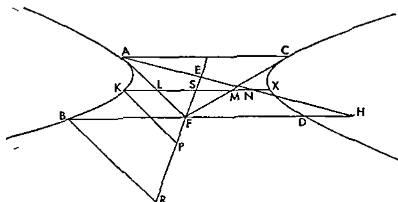
*If two straight lines touching opposite sections meet and through the point of meeting some straight line is drawn parallel to the straight line joining the points of contact and meeting each of the sections and some other straight line is drawn parallel to the same straight line and cutting the sections and the tangents then as the rectangle contained by the straight lines drawn from the point of meeting to cut the sections is to the square on the tangent so is the rectangle contained by the*

*straight lines between the sections and the tangent to the square on the straight line cut off at the point of contact*

Let there be the opposite sections $AB$ and $CD$ whose center is $E$ and tangents $AF$ and $CF$ and let $AC$ be joined, and let $EF$ and $AE$ be joined and produced and let $BFH$ be drawn through $F$ parallel to $AC$, and let the point $K$ be taken at random and through it let $KLSMNX$ be drawn parallel to $AC$

I say that

$$\text{rect } BF, FD \ \ \text{sq } FA \quad \text{rect } KL, LX \ \ \text{sq } AL$$




For let $KP$ and $BR$ be drawn from $K$ and $B$ parallel to $AF$ Since then

$$\text{sq } BF \ \ \text{trgl } BFR \quad \text{sq } KS \ \ \text{trgl } KSP \quad \text{sq } LS \ \ \text{trgl } LSF$$

and

$$\text{sq } KS \ \ \text{trgl } KSP$$
$$\text{remainder rect } KL \ LX \text{ (Eucl II 5)}$$
$$\text{remainder quadr } KLFP \text{ (Eucl V 19)}$$

and

$$\text{sq } BF = \text{rect } BF, FD \text{ (II 39, 38)}$$

and

$$\text{trgl } BRF = \text{trgl } AFH \text{ (III 11 and special case)}$$

and

$$\text{quadr } KLFP = \text{trgl } ALN \text{ (III 5)}$$

therefore

$$\text{rect } BF, FD \ \ \text{trgl } AFH \quad \text{rect } KL\ LX \ \ \text{trgl } ALN$$

And

$$\text{trgl } AFH \ \ \text{sq } AF \quad \text{trgl } ALN \ \ \text{sq } AL,$$

then

$$\text{rect } BF\ FD \ \ \text{sq } FA \quad \text{rect } KL, LX \ \ \text{sq } AL$$

## Proposition 21

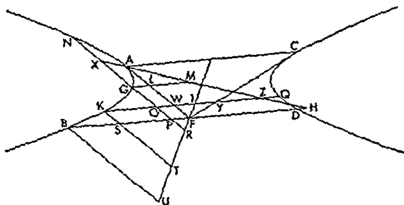
*With the same things supposed if two points are taken on the section and through them straight lines are drawn the one parallel to the tangent, the other parallel to the straight line joining the points of contact and cutting each other and the sections then as the rectangle contained by the straight lines drawn from the point*

*of meeting to cut the sections is to the square on the tangent, so will the rectangle contained by the straight lines between the sections and the point of meeting be to the rectangle contained by the straight lines between the section and the point of meeting*

For let there be the same things as before, and let points $G$ and $K$ be taken and through them let $NXGOPR$ and $KST$ be drawn parallel to $AF$, and $GLM$ and $KOWIYZQ$ parallel to $AC$

I say that

rect $BF, FD$ sq $FA$ rect $KO, OQ$ rect $NO, OG$




For since

sq $AF$ trgl $AFH$ sq $AL$ trgl $ALM$ sq $XO$ trgl $XOZ$

and

sq $XO$ trgl $XOZ$ sq $XG$ trgl $XGM$,

therefore whole sq $XO$ whole trgl $XOZ$

part subtracted sq $XG$ part subtracted trgl $XGM$,

therefore also

remainder rect $NO\ OG$ remainder quadr $GOZM$ sq $AF$ trgl $AFH$

But

trgl $AFH$ = trgl $BUF$ (III 11 end special case),

and

quadr $GOZM$ = quadr $KORT$ (III 12),

therefore

sq $AF$ trgl $BFU$ rect $NO\ OG$ quadr $KORT$

But it was shown (in the course of III 20) trgl $BUF$ sq $BF$ or rect $BF\ FD$ (II 30 38) quadr $KORT$ rect $KO\ OQ$

therefore *ex aequali*

sq $AF$ rect $BF\ FD$ rect $KO\ OG$ rect $KO\ OQ$

And inversely

rect $BF\ FD$ sq $FA$ rect $KO\ OQ$ rect $NO, OG$

PROPOSITION 22

*If two parallel straight lines touch opposite sections and any straight lines are drawn cutting each other and the sections one parallel to the tangent the other parallel to the straight line joining the points of contact, then as the transverse side*

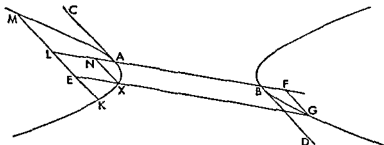
*of the figure on the straight line joining the points of contact is to the upright, so the rectangle contained by the straight lines between the sections and the point of meeting will be to the rectangle contained by the straight lines between the section and the point of meeting*

Let there be the opposite sections $A$ and $B$, and let $AC$ and $BD$ be parallel and tangent to them, and let $AB$ be joined Then let $EXG$ be drawn across parallel to $AB$ and $KELM$ parallel to $AC$

I say that

$AB$ upright side of the figure rect $GE$ $EX$ rect $KE,EM$

Let $XN$ and $GF$ be drawn through $G$ and $X$ parallel to $AC$




For since $AC$ and $BD$ are parallels tangent to the sections $AB$ is a diameter (II 31), and $KL$, $XN$ and $GF$ are ordinates to it (I 32), then (I 21)

$AB$ upright side

rect $BL,LA$ sq $LK$ rect $BN$ $NA$ sq $NX$ or sq $LK$

Therefore

whole rect $BL$ $LA$ whole sq $LK$

part subtracted rect $BN$ $NA$ part subtracted sq $LE$,

or

rect $BL,LA$ sq $LK$ rect $FA$ $AN$ sq $LE$,

for

$NA = BF$ (I 21),

therefore also

remainder rect $FL$ $LN$ remainder rect $KE,EM$ $AB$ upright

But

rect $FL$ $LN =$ rect $GE$ $EX$,

therefore

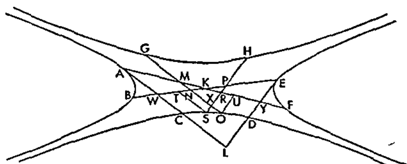
$AB$ the transverse side of figure upright

rect $GE$ $EX$ rect $KE,EM$

## Proposition 23

*If in conjugate opposite sections two straight lines touching contrary sections meet in any one section at random and any straight lines are drawn parallel to the tangents and cutting each other and the other opposite sections then as the squares on the tangents are to each other so the rectangle contained by the straight lines between the sections and the point of meeting will be to the rectangle contained by the straight lines similarly taken*

Let there be the conjugate opposite sections $AB$ $CD$, $EF$ and $GH$, and their center $K$, and let $AWCL$ and $EYDL$, tangents to the sections $AB$ and $EF$




meet at $L$ and let $AK$ and $EK$ be joined and produced to $B$ and $F$, and let $GMNXO$ be drawn from $G$ parallel to $AL$ and $HPRXS$ from $H$ parallel to $EL$

I say that

$$\text{sq } EL \ \ \text{sq } LA \quad \text{rect } HX, XS \ \ \text{rect } GX\ XO$$

For let $ST$ be drawn through $S$ parallel to $AL$, and $OU$ from $O$ parallel to $EL$ Since then $BE$ is a diameter of the conjugate opposite sections $AB$ $CD$, $EF$ and $GH$ and $EL$ touches the section and $HS$ has been drawn parallel to it,

$$HP = PS \text{ (II 20, First Def I 5),}$$

and for the same reasons

$$GM = MO$$

And since

$$\text{sq } EL \ \ \text{trgl } EWL \quad \text{sq } PS \ \ \text{trgl } PTS \quad \text{sq } PX \ \ \text{trgl } PNX,$$

also

$$\text{remainder rect } HX, XS \ \ \text{remainder quadr } TNXS \quad \text{sq } EL \ \ \text{trgl } WLE$$

But

$$\text{trgl } EWL = \text{trgl } ALY \text{ (III 4),}$$

and

$$\text{quadr } TNXS = \text{quadr } XRUO \text{ (III 15)}$$ [1]

therefore

$$\text{sq } EL \ \ \text{trgl } ALY \quad \text{rect } HX\ XS \ \ \text{quadr } XRUO$$

But

$$\text{trgl } AYL \ \ \text{sq } AL \quad \text{quadr } XRUO \ \ \text{rect } GX\ XO \text{ (same way),}$$

therefore *ex aequali*

$$\text{sq } EL \ \ \text{sq } AL \quad \text{rect } HX, XS \ \ \text{rect } GX, XO$$

## Proposition 24

*If in conjugate opposite sections two straight lines are drawn from the center through to the sections and one of them is taken as the transverse diameter and the other as the upright diameter and any straight lines are drawn parallel to the two diameters and meeting each other and the sections and the point of meeting of the straight lines is the place between the four sections then the rectangle contained by*

[1] This is the case of III 15 where the tangents are one to each of the opposite sections Compare with the two cases of III 12 and III 18

For trgl $TSK$ − trgl $KPR$ = trgl $ANK$ (III 15)

and trgl $MOU$ − trgl $MNK$ = trgl $ANK$ (III 15)

*the segments of the parallel to the transverse diameter together with the rectangle to which the rectangle contained by the segment of the parallel to the upright diameter has the ratio which the square on the upright diameter has to the square on the transverse will be equal to twice the square on the half of the transverse*

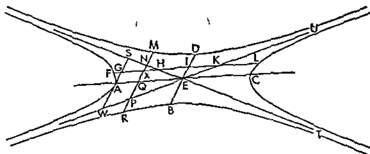
Let there be the conjugate opposite sections $A$, $B$, $C$ and $D$ whose center is $E$, and from $E$ let the transverse diameter $AEC$ and the upright diameter $DEB$ be drawn through and let $FGHIKL$ and $MNXOPR$ be drawn parallel to $AC$ and $DB$ and meeting each other at $X$, and first let $X$ be within the angle $SEW$ or the angle $UET$

I say that the rectangle $FX$, $XL$ together with the rectangle to which the rectangle $RX$, $XM$ has the ratio which the square on $DB$ has to the square on $AC$, is equal to twice the square on $AE$

For let the asymptotes of the sections $SET$ and $UEW$ be drawn and through $A$, $SGAW$ tangent to the section

Since then

rect $SA, AW$ = sq $DE$ (I 60, II 1),




therefore

rect $SA$ $AW$ sq $EA$ sq $DE$ sq $EA$

And

rect $SA, AW$ sq $AE$ comp $SA$ $AE$ $WA$ $AE$

But

$SA$ $AE$ $NX$ $XH$

and

$WA$ $AE$ $PX$ $XK$

therefore

sq $DE$ sq $AE$ comp $NX$ $XH, PX$ $XK$

But

rect $PX$ $XN$ rect $KX$ $XH$ comp $NX$ $XH$ $PX$ $XK$,

therefore

sq $DE$ sq $AE$ rect $PX, XN$ rect $KX$ $XH$

Therefore also

sq $DE$ sq $AE$ sq $DE$+rect $PX, XN$ sq $AE$+rect $KX$ $XH$

And

sq $DE$=rect $PM$ $MN$ (II 11)=rect $RN$ $NM$ (II 16)

and

sq $AE$=rect $KF$ $FH$ (II 11)=rect $LH, HF$ (II 16),

therefore

$$\text{sq } DE \quad \text{sq } AE \quad \text{rect } PX, XN + \text{rect } RN, NM$$
$$\text{rect } KX, XH + \text{rect } LH, HF$$

And

$$\text{rect } PX\ XN + \text{rect } RN, NM = \text{rect } RX, XM,^{1}$$

therefore

$$\text{sq } DE \quad \text{sq } AE \quad \text{rect } RX, XM \quad \text{rect } KX, XH + \text{rect } KF, FH$$

Then it must be shown that

$$\text{rect } FX\ XL + \text{rect } KX\ XH + \text{rect } KF, FH = 2 \text{ sq } AE$$

Let the common square $AE$ that is rectangle $KF, FH$, be subtracted, therefore it remains to be shown that

$$\text{rect } FX\ XL + \text{rect } KX, XH = \text{sq } AE$$

And this is so, for

$$\text{rect } FX\ XL + \text{rect } KX\ XH = \text{rect } LH\ HF, \dagger$$
$$\text{rect } FX, XL + \text{rect } KX\ XH = \text{rect } KF, FH \text{ (II 16)}$$
$$= \text{sq } AE \text{ (II 11)}$$

Then let the straight lines $FL$ and $MR$ meet on one of the asymptotes at $H$ Then

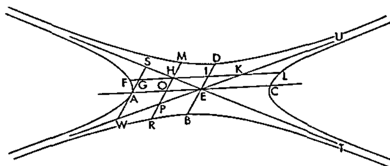
$$\text{rect } FH\ HL = \text{sq } AE,$$

and

$$\text{rect } MH\ HR = \text{sq } DE \text{ (II 11 16)},$$

therefore

$$\text{sq } DE \quad \text{sq } AE \quad \text{rect } MH\ ER \quad \text{rect } FH, HL$$




And so we want twice rectangle $FH\ HL$ to equal twice the square on $AE$ And it does

[1]For $RP = NM$ (II 8)
and $RO = OM$ (II 3)
therefore $PO = ON$
But $\text{rect } PX\ XN + \text{sq } OX = \text{sq } ON$ (Eucl II 5)
and for the same reasons
$\text{rect } RN\ NM + \text{sq } ON = \text{sq } OM$
and $\text{rect } RX\ XM + \text{sq } OX = \text{sq } OM$
Hence $\text{rect } RN\ NM + \text{sq } ON = \text{rect } RX\ XM + \text{sq } OX$
and adding equals to equals,
$\text{rect. } RN\ NM + \text{sq } ON + \text{rect } PX\ XN + \text{sq } OX = \text{rect } RX\ XM + \text{sq } OX + \text{sq } ON$
Subtracting the common squares
$\text{rect } RN\ NM + \text{rect } PX\ XN = \text{rect } RX\ XM$

†By the same manner of proof as in the note above but using also Euclid II 6 because of the different position of the point X

And let the point $Y$ be within the angle $SEK$ or angle $WET$ Then likewise by the composition of ratios

$$\text{sq } DE \quad \text{sq } AE \qquad \text{rect } PY, XN \quad \text{rect } KY\ XH$$

And

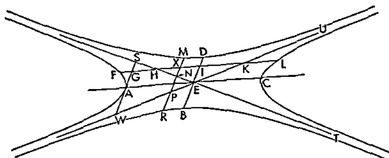
$$\text{sq } DE = \text{rect } PM, RN = \text{rect } RN\ NM,$$

and

$$\text{sq } AL = \text{rect } FH\ HL,$$

therefore

$$\text{rect } RN, NM \quad \text{rect } FH, HL$$




$$\text{part subtracted rect } PY\ XN \quad \text{part subtracted rect } KX\ YH$$

Therefore also

$$\text{rect } RN\ NM \quad \text{rect } FH\ HL$$

$$\text{remainder rect } RX\ XM \quad \text{remainder (sq } AE - \text{rect } KX\ XH)$$

Therefore it must be shown that

$$\text{rect } FX\ YL + (\text{sq } AE - \text{rect } KX\ YH) = 2 \text{ sq } AE$$

Let the common square on $AE$ that is rectangle $FH$ $HL$ be subtracted therefore it remains to be shown that

$$\text{rect } KX\ YH + (\text{sq } AE - \text{rect } KY\ YH) = \text{sq } AE$$

And this is so for

$$\text{rect } KY\ XH + \text{sq } AE - \text{rect } KX\ XH = \text{sq } AE$$

## Proposition 25

With the same things supposed let the point of meeting of the parallels to $AC$ and $BD$ be within one of the sections $D$ and $B$ as set out below at $X$

I say that the rectangle contained by the segments of the parallel to the transverse that is rectangle $OX$ $XN$, will be greater than the rectangle to which the rectangle contained by the segments of the parallel to the upright diameter that is rectangle $RX$ $XM$ has the ratio which the square on the upright diameter has to the square on the transverse by twice the square on the half of the transverse

For for the same reasons

$$\text{sq } DE \quad \text{sq } AE \qquad \text{rect } PX\ XH \quad \text{rect } SY\ YL$$

and

$$\text{sq } DE = \text{rect } PM\ MH$$

and

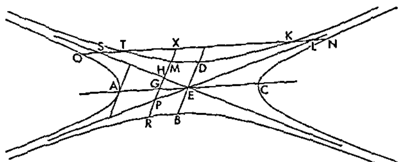
$$\text{sq } AE = \text{rect } LO\ OS \text{ (II 11)}$$

therefore also

sq $DE$  sq $AE$  rect $PM, MH$  rect $LO, OS$

And since

whole rect $PX, XH$  whole rect $LX, XS$




part subtracted rect $PM\ MH$  part subtracted rect $LO\ OS$
or rect $ST, TL$ (II 22),

therefore also

remainder rect $RX, SM$  remainder rect $TX, XK$
(first note to III 24  II 8)  sq $DE$  sq $AE$

Therefore it must be shown that

rect $OX\ XN$ = rect $TX\ XK$ + 2 sq $AE$

Let the common rectangle $TX, XK$ be subtracted, therefore it must be shown that

rect $OT, TN$ (first note to III 24) = 2 sq $AE$

And it is (II 23)

PROPOSITION 26

And if the point of meeting of the parallels at $X$ is within one of the sections $A$ and $C$ as set out below then the rectangle contained by the segments of the parallel to the transverse that is rectangle $LX, XF$ will be less than the rectangle to which the rectangle contained by the segments of the other parallel that is rectangle $RX\ XG$ has the ratio which the square on the upright diameter has to the square on the transverse by twice the square on half of the transverse

For, since for the same reasons as before

sq $DE$  sq $AE$  rect $WX\ XS$  rect $KX\ XH$

therefore also

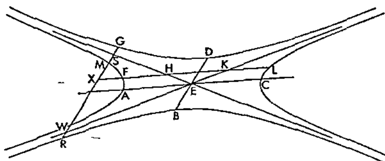
whole rect $RX\ XG$[1]  whole rect $KX\ XH$ + sq $AE$
sq upright diameter  sq transverse

Therefore it must be shown that

rect $LX, XF$ + 2 sq $AL$ = rect $KX\ XH$ + sq $AE$

[1]For by II 11  rect $WG\ GS$ = sq $DE$
and  $JW$ = $GS$ (II 16)
Therefore by the first note to III 24 and II 16
rect $WX\ XS$ + sq $DE$ = rect $WX\ XS$ + rect $WG\ GS$ = rect $RX\ XG$

Let the common square on $AE$ be subtracted, therefore it remains to be shown that




$$\text{rect } LX\ XF + \text{sq } AE = \text{rect } KX, XH$$

or

$$\text{rect } LX, XF + \text{rect } LH\ HF = \text{rect } KX\ XH \quad (\text{II } 16\ 11)$$

And it is for

$$\text{rect } LH, HF + \text{rect } LX, XF = \text{rect } KX, XH$$ [1]

PROPOSITION 27

*If the conjugate diameters of an ellipse or circumference of a circle are drawn and one of them is called the upright diameter and the other the transverse and two straight lines meeting each other and the line of the section, are drawn parallel to them then the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the line of the section plus the figures described on the straight lines cut off on the straight line drawn parallel to the upright diameter between the point of meeting of the straight lines and the line of the section figures similar and similarly situated to the figure on the upright diameter will be equal to the square on the transverse diameter*

For let there be the ellipse or circumference of a circle $ABCD$, whose center is $E$ and let two of its conjugate diameters be drawn the upright $AFC$ and the transverse $BED$ and let $NGFH$ and $KFLM$ be drawn parallel to $AC$ and $BD$

I say that the squares on $NF$ and $FH$ plus the figures described on $KF$ and $FM$, similar and similarly situated to the figure on $AC$ will be equal to the square on $BD$

Let $NX$ be drawn from $N$ parallel to $AF$ therefore it has been dropped ordinatewise to $BD$ And let $BP$ be the upright side Now since

$$BP\ AC \quad AC\ BD \ (\text{I } 15),$$

therefore also

$$BP\ BD \quad \text{sq } AC\ \text{sq } BD$$

And

$$\text{sq } BD = \text{figure on } AC,$$

therefore

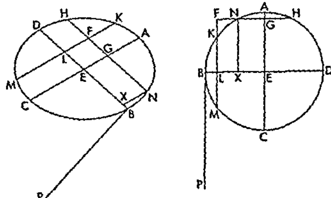
$$BP\ BD \quad \text{sq } AC\ \ \text{figure on } AC$$

[1] This is another case of the first note to III 21

And

sq $AC$ figure on $AC$

sq $NX$ figure on $NX$ similar to the figure on $AC$ (Eucl VI 22),




therefore also

$BP$ $BD$ sq $NX$ figure on $NX$ similar to the figure on $AC$

And also

$BP$ $BD$ sq $NX$ rect $BX, XD$ (I 21)

therefore

figure on $NX$ or $FL$ similar to the figure on $AC$ = rect $BX$ $XD$

Then likewise we could show that

figure on $KL$ similar to the figure on $AC$ = rect $BL$ $LD$

And since the straight line $NH$ has been cut equally at $G$ and unequally at $F$,

sq $HF$ + sq $FN$ = 2[sq $HG$ + sq $GF$] = 2[sq $NG$ + sq $GF$] (Eucl VI 9)

Then for the same reasons also

sq $MF$ + sq $FK$ = 2[sq $KL$ + sq $LF$]

and the figure on $MF$ and $FK$ similar to the figure on $AC$ are double the similar figures on $KL$ and $LF$

And

figure on $KF$ + figure on $FL$ = rect $BX$ $XD$ + rect $BL$ $LD$ (above),

and

sq $NG$ + sq $GF$ = sq $XE$ + sq $EL$,

therefore

sq $NF$ + sq $FH$ + figures on $KF$ and $FM$ similar to the figure on $AC$ =
2[rect $BX$ $XD$ + rect $BL$ $LD$ + sq $XE$ + sq $EL$]

And since the straight line $BD$ has been cut equally at $E$ and unequally at $X$,

rect $BX$ $XD$ + sq $XE$ = sq $BE$ (Eucl II 5)

And likewise also

rect $BL$ $LD$ + sq $LE$ = sq $BE$,

and so

rect $BX$ $XD$ + rect $BL$ $LD$ + sq $XE$ + sq $LE$ = 2 sq $BE$

Therefore the squares on $NF$ and $FH$ together with figures on $KF$ and $FM$ similar to the figure on $CA$ are double the square on $BL$ But also

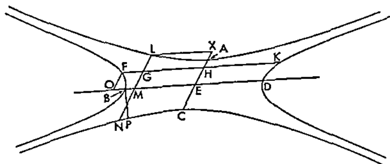
sq $BD$ = 2 sq $BE$,

therefore the squares on $NF$ and $FH$ plus the figures on $KF$ and $FM$ similar to the figure on $AC$ are equal to the square on $BD$

PROPOSITION 28

*If in conjugate opposite sections conjugate diameters are drawn, and one of them is called the upright, and the other the transverse and two straight lines are drawn parallel to them and meeting each other and the sections then the squares on the straight lines cut off on the straight line drawn parallel to the upright between the point of meeting of the straight lines and the sections have to the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the sections the ratio which the square on the upright diameter has to the square on the transverse diameter*

Let there be the conjugate opposite sections *A*, *B*, *C*, and *D* and let *AEC* be the upright diameter and *BED* the transverse and let *FGHK* and *LGMN*




be drawn parallel to them and cutting each other and the sections

I say that

sq *LG*+sq *GN*  sq *FG*+sq *GK*  sq *AC*  sq *BD*

For let *LY* and *FO* be drawn ordinatewise from *F* and *L*, therefore they are parallel to *AC* and *BD* And from *B* let the upright side for *BD* *BP* be drawn, then it is evident that

*PB*  *BD*  sq *AC*  sq *BD* (I 15)  sq *AE*  sq *EB*

sq *FO*  rect *BO OD* (I 21)  rect *CX*, *XA*  sq *LY* (I 60 21)

Therefore as one of the antecedents is to one of the consequents so are all of the antecedents to all of the consequents (Eucl v 12) therefore

sq *AC*  sq *BD*  rect *CX*, *XA*+sq *AE*+sq *OF*  rect *DO*, *OB*+sq *BE*, +sq *LX*

or

sq *AC*  sq *BD*  rect *CX*, *YA*+sq *AE*+sq *EH*  rect *DO*,*OB*+sq *BE*+sq *ME*

But

rect *CX*,*XA*+sq *AE*=sq *YE*,

and

rect *DO OB*+sq *BE*=sq *OE* (Eucl II 6),

therefore

sq *AC*  sq *BD*  sq *YE*+sq *EH*  sq *OE*+sq *EM*  sq *LM*+sq *MG*  sq *FH*+sq *HG*

And as has been shown

sq *NG*+sq *GL*=2[sq *LM*+sq *MG*]

and

$$\text{sq } FG+\text{sq } GK=2(\text{sq } FH+\text{sq } HG) \text{ (Eucl II 9)},$$

therefore also

$$\text{sq } AC \quad \text{sq } BD \quad \text{sq } NG+\text{sq } GL \quad \text{sq } FG+\text{sq } GK$$

## Proposition 29

*With the same things supposed, if the parallel to the upright diameter cuts the asymptotes then the squares on the straight lines cut off on the straight line drawn parallel to the upright between the point of meeting of the straight lines and the asymptotes plus the half of the square on the upright diameter has to the squares on the straight lines cut off on the straight line drawn parallel to the transverse between the point of meeting of the straight lines and the sections the ratio which the square on the upright diameter has to the square on the transverse*

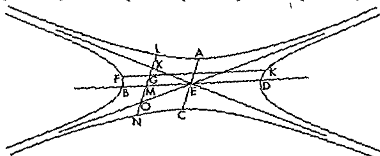
For let there be the same things as before and let $NL$ cut the asymptotes at $X$ and $O$

It is to be shown that

$$\text{sq } XG+\text{sq } GO+\text{half sq } AC \quad \text{sq } FG+\text{sq } GK \quad \text{sq } AC \quad \text{sq } BD$$

or

$$\text{sq } XG+\text{sq } GO+2 \text{ sq } AE \quad \text{sq } FG+\text{sq } GK \quad \text{sq } AC \quad \text{sq } BD$$




For since

$$LX=ON \text{ (II 16)},$$

$$\text{sq } LG+\text{sq } GN+2 \text{ rect } NX, XL=\text{sq } XG+\text{sq } GO,[1]$$

therefore

$$\text{sq } XG+\text{sq } GO+2 \text{ sq } AE=\text{sq } LG+\text{sq } GN$$

And

$$\text{sq } LG+\text{sq } GN \quad \text{sq } FG+\text{sq } GK \quad \text{sq } AC \quad \text{sq } BD \text{ (III 28)},$$

therefore also

$$\text{sq } XG+\text{sq } GO+2 \text{ sq } AE \quad \text{sq } FG+\text{sq } GK \quad \text{sq } AC \quad \text{sq } BD$$

[1] For

$$OM=MX$$

Therefore as in a lemma of Pappus since

$$2 \text{ rect } NX, XL+2 \text{ sq } MX=2 \text{ sq } ML \text{ (Eucl II 5)}$$

adding the common square on $GM$

$$2 \text{ rect. } NX, XL+2 \text{ sq } MX+2 \text{ sq } GM=2 \text{ sq } ML+2 \text{ sq } GM$$

And

$$2 \text{ sq } ML+2 \text{ sq } GM=\text{sq } NG+\text{sq } LG$$

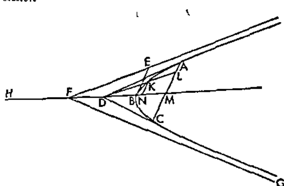
and

$$2 \text{ sq } MX+2 \text{ sq } GM=\text{sq } OG+\text{sq } GX \text{ (Eucl II 9)}$$

Therefore as above

PROPOSITION 30

*If two straight lines touching an hyperbola meet and through the points of contact a straight line is produced and through the point of meeting a straight line is drawn parallel to some one of the asymptotes and cutting both the section and the straight line joining the points of contact then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section*[1]




Let there be the hyperbola $ABC$, and let $AD$ and $DC$ be tangents and $EF$ and $FG$ asymptotes and let $AC$ be joined and through $D$ parallel to $FE$ let $DKL$ be drawn

I say that

$$DK = KL$$

For let $FDBM$ be joined and produced both ways and let $FH$ be made equal to $BF$ and through the points $B$ and $K$ let $BE$ and $KN$ be drawn parallel to $AC$ therefore they have been dropped ordinatewise (II 30 5 7) And since triangle $BEF$ is similar to triangle $DNK$, therefore

$$\text{sq } DN \quad \text{sq } NK \qquad \text{sq } BF \quad \text{sq } BE \qquad (\alpha)$$

And

$$\text{sq } BF \quad \text{sq } BE \qquad HB \quad \text{upright (II 1)},$$

therefore also

$$\text{sq } DN \quad \text{sq } NK \qquad HB \quad \text{upright}$$

But

$$HB \quad \text{upright} \qquad \text{rect } HN\,NB \quad \text{sq } NK \text{ (I 21)},$$

therefore also

$$\text{sq } DN \quad \text{sq } NK \qquad \text{rect } MN,NB \quad \text{sq } NK \qquad (\beta)$$

Therefore

$$\text{rect } HN\,NB = \text{sq } DN$$

[1]The propositions from 30 to 34 inclusive are one special case and propositions 35 and 36 are another special case of proposition 37 The first group takes the line drawn through the intersection of the tangents as parallel to an asymptote The second group takes one of the tangents as an asymptote Proposition 34 lying between is special in both these ways F

In proposition 37 we have the line $CF$ divided by the section at $D$ and $F$ and at $E$ by the straight line joining the points of contact in such a way that

$$CF \quad CD \qquad FE \quad ED$$ E

This is the same form of the harmonic proportion as we found in I 34 and $DF$ is the harmonic mean between $CF$ and $FE$ D

If we argue by analogy from this proportion treating infinity as a definite magnitude and two such infinities as would occur here as equal and subject to the general laws of magnitudes we can immediately deduce the special cases of propositions 30 to 36 Thus in the case of the first group $CF$ and $FE$ both become infinite therefore $CD$ is equal to $ED$ C

And also

$$\text{rect } MF\ FD = \text{sq } FB \ (\text{I } 37),$$

because $AD$ touches and $AM$ has been dropped ordinatewise, and so also

$$\text{rect } HN, NB + \text{sq } FB = \text{rect } MF, FD + \text{sq } DN$$

But

$$\text{rect } HN, NB + \text{sq } FB = \text{sq } FN \ (\text{Eucl II } 6),$$

and therefore

$$\text{rect } MF\ FD + \text{sq } DN = \text{sq } FN$$

Therefore $DM$ has been bisected at $N$ with $DF$ added (Eucl II 6) And $KN$ and $LM$ are parallel, therefore

$$DK = KL$$

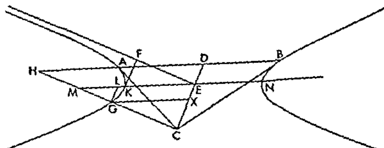
## Proposition 31

*If two straight lines touching opposite sections meet and a straight line is produced through the points of contact, and through the point of meeting a straight line is drawn parallel to the asymptote and cutting both the section and the straight line joining the points of contact then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section*

Let there be the opposite sections $A$ and $B$ and tangents $AC$ and $CB$, and let $AB$ be joined and produced, and let $FE$ be an asymptote and through $C$ let $CGH$ be drawn parallel to $FE$

I say that

$$CG = GH$$




Let $CE$ be joined and produced to $D$ and through $E$ and $G$ let $NEKM$ and $GX$ be drawn parallel to $AB$, and through $G$ and $K$ let $KF$ and $GL$ be drawn parallel to $CD$

Since triangle $KFE$ is similar to triangle $MLG$

$$\text{sq } KE \quad \text{sq } KF \quad \text{sq } ML \quad \text{sq } LG$$

And it has been shown

$$\text{sq } KE\ KE \quad \text{sq } KF \quad \text{rect } NL\ LK \quad \text{sq } LG \ (\alpha \text{ and } \beta \text{ of III } 30),$$

therefore

$$\text{rect } NL\ LK = \text{sq } ML$$

Let the square on $KF$ be added to each therefore

$$\text{rect } NL\ LK + \text{sq } KE = \text{sq } LE = \text{sq } GX = \text{sq } ML + \text{sq } KE$$

And

$$\text{sq } GX \quad \text{sq } ML + \text{sq } KE \quad \text{sq } XC \quad \text{sq } LG + \text{sq } KF \ (\text{Eucl VI } 4, \text{V } 12),$$

therefore

$$\text{sq } XC = \text{sq } LG + \text{sq } KF$$

And

$$\text{sq } LG = \text{sq } XE$$

and

$$\text{sq } KF = \text{sq on half of second diameter (II 1)},$$
$$= \text{rect } CE, ED \text{ (I 38)},$$

therefore

$$\text{sq } XC = \text{sq } XE + \text{rect } CE\ ED$$

Therefore the straight line $CD$ has been cut equally at $X$ and unequally at $E$ (Eucl II 5)
And $DH$ is parallel to $GX$, therefore

$$CG = GH$$

PROPOSITION 32

*If two straight lines touching an hyperbola meet, and a straight line is produced through the points of contact and a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact and a straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of the asymptotes, then the straight line cut off between this midpoint and the parallel will be bisected by the section*

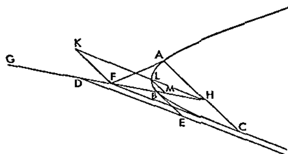
Let there be the hyperbola $ABC$, whose center is $D$, and asymptote $DE$, and let $AF$ and $FC$ touch and let $CA$ and $FD$ be joined and produced to $G$ and $H$, then it is evident that

$$AH = HC$$

Then let $FK$ be drawn through $F$ parallel to $AC$ and $HLK$ through $H$ parallel to $DE$

I say that

$$KL = HL$$




Let $LM$ and $BE$ be drawn through $B$ and $L$ parallel to $AC$ then as has been already shown (III 30 $\alpha$, $\beta$ and conclusion)

$$\text{sq } DB\ \ \text{sq } BE\ \ \text{sq } HM\ \ \text{sq } ML\ \ \text{rect } BM, MG\ \ \text{sq } ML$$

therefore

$$\text{rect } GM, MB = \text{sq } MH$$

And also

$$\text{rect } HD\ DF = \text{sq } DB$$

because $AF$ touches and $AH$ has been dropped ordinatewise (I 37) therefore

$$\text{rect } GM\ MB + \text{sq } DB = \text{rect } HD\ DF + \text{sq } MH = \text{sq } DM \text{ (Eucl II 6)}$$

Therefore $FH$ has been bisected at $M$ with $DF$ added And $KF$ and $LM$ are parallel, therefore

$$KL = LH$$

PROPOSITION 33

*If two straight lines touching opposite sections meet, and one straight line is produced through the points of contact and another straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of*

*contact and still another straight line is drawn through the midpoint of the straight line joining the points of contact parallel to one of the asymptotes and meeting the section and the parallel drawn through the point of meeting, then the straight line between the midpoint and the parallel will be bisected by the section*

Let there be the opposite sections $ABC$ and $DEF$, and tangents $AG$ and $DG$ and center $H$, and asymptote $KH$, and let $HG$ be joined and produced and also let $ALD$ be joined then it is evident that it is bisected at $L$ (II 30) Then let $BHE$ and $CGF$ be drawn through $G$ and $H$ parallel to $AD$, and $LMN$ through $L$ parallel to $HK$

I say that

$$LM = MN$$

For let $EK$ and $MX$ be dropped from $E$ and $M$ parallel to $GH$ and $MP$ through $M$ parallel to $AD$

Since then through things already shown (III 30 $\alpha$ and $\beta$)

$$\text{sq } HE : \text{sq } EK = \text{rect } BX, XE : \text{sq } XM$$

therefore

sq $HE$ : sq $EK$ = rect $BX, XE$+sq $HE$ : sq $KE$+sq $XM$ (Eucl v 12)

or

$$\text{sq } HE : \text{sq } EK = \text{sq } HX : \text{sq } KE+\text{sq } XM \text{ (Eucl II 6)}$$

But it has been shown (I 38, II 1)

$$\text{sq } EK = \text{rect } GH, HL$$

and

$$\text{sq } XM = \text{sq } HP$$

therefore

$$\text{sq } HE : \text{sq } EK = \text{sq } HX \text{ or sq } MP : \text{rect } GH, HL+\text{sq } HP$$

And

$$\text{sq } HE : \text{sq } EK = \text{sq } MP : \text{sq } PL \text{ (Eucl VI 4)}$$

therefore

$$\text{sq } MP : \text{sq } PL = \text{sq } MP : \text{rect } GH, HL+\text{sq } HP$$

Therefore

$$\text{sq } PL = \text{rect } GH, HL+\text{sq } HP$$

Therefore the straight line $LG$ has been cut equally at $P$ and unequally at $H$ (Eucl II 5)

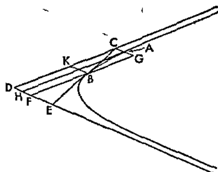
And $MP$ and $GN$ are parallel therefore

$$LM = MN$$

### Proposition 34

*If ome point is taken on one of the asymptotes of an hyperbola, and a straight line from it touches the section and through the point of contact a parallel to the asymptote is drawn then the straight line drawn from the point taken parallel to the other asymptote will be bisected by the section*

Let there be the hyperbola $AB$ and the asymptotes $CD$ and $DE$, and let a point $C$ be taken at random on $CD$, and through it let $CBE$ be drawn touching the section and through $B$ let $FBG$ be drawn parallel to $CD$ and through $C$ let $CAG$ be drawn parallel to $DE$




I say that

$$CA = AG$$

For let $AH$ be drawn through $A$ parallel to $CD$, and $BK$ through $B$ parallel to $DE$ Since then

$$CB = BE \text{ (II 3)},$$

therefore also

$$CK = KD$$

and

$$DF = FE$$

And since

$$\text{rect } KB, BF = \text{rect } CA, AH \text{ (II 12)},$$

and

$$BF = DK = CK,$$

and

$$AH = DC,$$

therefore

$$\text{rect } DC, CA = \text{rect } GC\ CK$$

Therefore

$$DC \quad CK \quad GC \quad CA$$

And

$$CD = 2CK,$$

therefore also

$$GC = 2\,CA$$

Therefore

$$CA = AG$$

### Proposition 35

*With the same things being so if from the point taken some straight line is drawn cutting the section at two points then as the whole straight line is to the straight line cut off outside so will the segments of the straight line cut off inside be to each other*

For let there be the hyperbola $AB$ and the asymptotes $CD$ and $DE$ and $CBE$ touching and $HB$ parallel and through $C$ let some straight line $CALFG$ be drawn across cutting the section at $A$ and $F$

I say that

$$FC \quad CA \quad FL \quad AL$$

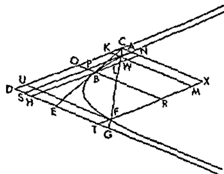
For let $CNX$ $KAM$ $OPBR$ and $FU$ be drawn through $C$ $A$ $B$ and $F$

parallel to $DE$, and $APS$ and $TFRMX$ through $A$ and $F$ parallel to $CD$. Since then

$$AC = FG \text{ (ii. 8)},$$

therefore also

$$KA = TG \text{ (Eucl. vi. 4)}.$$




But

$$KA = DS,$$

therefore also

$$TG = DS.$$

And so also

$$CK = DU.$$

And since

$$CK = DU,$$

also

$$DK = CU,$$

therefore

$$DK : CK = CU : CK.$$

And

$$CU : CK = FC : AC,$$

and

$$FC : AC = MK : KA,$$

and

$$MK : KA = \text{pllg } MD : \text{pllg } DA \text{ (Eucl. vi. 1)},$$

and

$$DK : CK = \text{pllg } HK : \text{pllg } KN,$$

therefore also

$$\text{pllg } MD : \text{pllg } DA = \text{pllg } HK : \text{pllg } KN.$$

But

$$\text{pllg } DA = \text{pllg } DB \text{ (ii. 12)} = \text{pllg } ON,$$

for

$$CB = BE \text{ (ii. 3)},$$

and

$$DO = OC,$$

therefore

$$\text{pllg } MD : \text{pllg } ON = \text{pllg } HK : \text{pllg } KN$$

and

remainder pllg $MH$ : remainder pllg $BK$ = whole pllg $MD$ : whole pllg $ON$.

And since

$$\text{pllg } DA = \text{pllg } DB,$$

let the common parallelogram $DP$ be subtracted; therefore

$$\text{pllg } KP = \text{pllg } PH.$$

Let the common parallelogram $AB$ be added; therefore

$$\text{whole pllg } BK = \text{whole pllg } AH.$$

Therefore

$$\text{pllg } MD : \text{pllg } DA = \text{pllg } MH : \text{pllg } AH.$$

But

$$\text{pllg } MD : \text{pllg } DA = MK : KA = FC : AC,$$

and

pllg $MH$ pllg $AH$ $MW$ $WA$ $FL$ $LA$,

therefore also

$FC$ $AC$ $TL$ $LA$

PROPOSITION 36

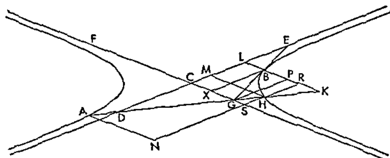
*With the same things being so if the straight line drawn across from the point neither cuts the section at two points nor is parallel to the asymptote it will meet the opposite section and as the whole straight line is to the straight line between the section and the parallel through the point of contact so will the straight line between the opposite section and the asymptote be to the straight line between the asymptote and the other section*

Let there be the opposite sections $A$ and $B$ whose center is $C$ and asymptotes $DE$ and $FG$ and let some point $G$ be taken on $CG$ and from it let $GBE$ be drawn tangent and $GH$ neither parallel to $CE$ nor cutting the section in two points (I 26)

It has been shown that $GH$ produced meets $CD$ and therefore also section $A$ Let it meet it at $A$ and let $KBL$ be drawn through $B$ parallel to $CG$

I say that

$AK$ $KH$ $AG$ $GH$




For let $HM$ and $AN$ be drawn from the points $A$ and $H$ parallel to $CG$ and $BX$ $GP$ and $RHSN$ from $B$ $G$ and $H$ parallel to $DE$ Since then

$AD = GH$ (II 16)

$AG$ $GH$ $DH$ $HG$

But

$AG$ $GH$ $NS$ $SH$

and

$DH$ $GH$ $CS$ $SG$

And therefore

$NS$ $SH$ $CS$ $SG$

But

$NS$ $SH$ pllg $NC$ pllg $CH$

and

$CS$ $SG$ pllg $RC$ pllg $RG$

therefore also

pllg $NC$ pllg $CH$ pllg $RC$ pllg $RG$

And as one is to one so are all to all therefore

pllg $NC$ pllg $CH$ whole pllg $NL$ whole pllg $CH$+pllg $RG$

And since

$$EB=BG,$$

also

$$LB=BP$$

and

pllg $L\Lambda$=pllg $BG$

And

pllg $L\Lambda$=pllg $CH$ (II 12),

therefore also

pllg $BG$=pllg $CH$

Therefore

pllg $NC$ pllg $CH$ whole pllg $NL$ whole pllg $BG$+pllg $RG$

or

pllg $NC$ pllg $CH$ pllg $NL$ pllg $R\Upsilon$

But

pllg $R\Lambda$=pllg $LH$

since also

pllg $CH$=pllg $BC$ (II 12),

and

pllg $MB$=pllg $\Lambda H$

Therefore

pllg $NC$ pllg $CH$ pllg $NL$ pllg $LH$

But

pllg $NC$ pllg $CH$ $NS$ $SH$ $AG$ $GH$,

and

pllg $NL$ pllg $LH$ $NR$ $RH$ $AK$ $KH$,

therefore also

$$AK\ KH\ \ AG\ GH$$

PROPOSITION 37

*If two straight lines touching a section of a cone or circumference of a circle or opposite sections meet and a straight line is joined to their points of contact and from the point of meeting of the tangents some straight line is drawn across cutting the line (of the section) at two points then as the whole straight line is to the straight line cut off outside so will the segments produced by the straight line joining the points of contact be to each other*

Let there be the section of a cone $AB$ and tangents $AC$ and $CB$ and let $AB$ be joined and let $CDEF$ be drawn across

I say that

$$CF\ CD\ \ FE\ FD$$

Let the diameters $CH$ and $AK$ be drawn through $C$ and $A$ and through $F$ and $D$ $DP$ $FR$ $LFM$ and $NDO$ parallel to $AH$ and $LC$ Since then $LFM$ is parallel to $NDO$

$$FC\ CD\ \ LF\ ND\ \ FM\ DO\ \ LM\ NO$$

and therefore

sq $LM$ sq $NO$ sq $FM$ sq $DO$

But

sq $LM$ sq $NO$ trgl $LMC$ trgl $NCO$ (Eucl VI 19),

and

$$\text{sq } FM \quad \text{sq } DO \quad \text{trgl } \Gamma RM \quad \text{trgl } DPO,$$

therefore also

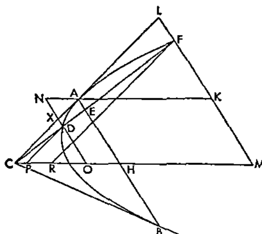
$$\text{trgl } LMC \quad \text{trgl } XCO \quad \text{trgl } FRM \quad \text{trgl } DPO$$
$$\text{remainder quadr } LCRF \quad \text{remainder quadr } XCP$$

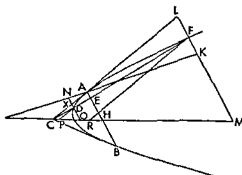
But

$$\text{quadr } LCRF = \text{trgl } ALK \text{ (III 2, III 11)},$$

and

$$\text{quadr } XCPD = \text{trgl } ANX \text{ (III 2, III 11)},$$




therefore

$$\text{sq } LM \quad \text{sq } XO \quad \text{trgl } ALK \quad \text{trgl } ANX$$

But

$$\text{sq } LM \quad \text{sq } XO \quad \text{sq } FC \quad \text{sq } CD,$$

and

$$\text{trgl } ALK \quad \text{trgl } ANX \quad \text{sq } LA \quad \text{sq } AX \quad \text{sq } FE \quad \text{sq } ED,$$

therefore also

$$\text{sq } FC \quad \text{sq } CD \quad \text{sq } FE \quad \text{sq } ED$$

And therefore

$$FC \quad CD \quad FE \quad ED$$

PROPOSITION 38

*With the same things being so, if some straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact, and a straight line drawn through the midpoint of the straight line joining the points of contact cuts the section in two points and the straight line through the point of meeting parallel to the straight line joining the points of contact then as the whole straight line drawn across is to the straight line cut off outside between the section and the parallel so will the segments produced by the straight line joined to the points of contact be to each other*

Let there be the section $AB$ and tangents $AC$ and $BC$ and $AB$ the straight line joining the points of contact and $AN$ and $CM$ diameters then it is evident that $AB$ has been bisected at $E$ (II 30 39)

Let $CO$ be drawn from $C$ parallel to $AB$ and let $FEDO$ be drawn across through $E$

I say that

$FO$ $OD$ $FE$ $ED$

For let $LFKM$ and $DHGXN$ be drawn through $F$ and $D$ parallel to $AB$, and through $F$ and $G$ $FR$ and $GP$ parallel to $LC$. Then likewise as before (III 37) it will be shown that

sq $LM$ sq $XH$ sq $LA$ sq $AX$

And

sq $LM$ sq $XH$ sq $LC$ sq $CX$
sq $FO$ sq $OD$,

and

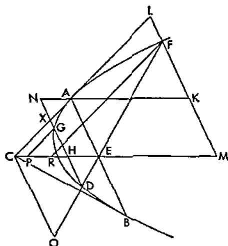
sq $LA$ sq $AX$ sq $FE$ sq $ED$,

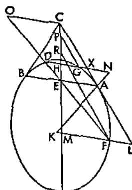
therefore

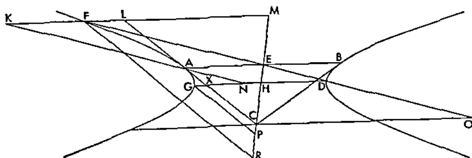
sq $FO$ sq $OD$ sq $FE$ sq $ED$,

and

$FO$ $OD$ $FE$ $ED$




PROPOSITION 39

*If two straight lines touching opposite sections meet and a straight line is produced through the points of contact and a straight line drawn from the point of meeting of the tangents cuts both of the sections and the straight line joining the points of contact then as the whole straight line drawn across is to the straight line*

*cut off outside between the section and the straight line joining the points of contact, so will the segments of the straight line produced by the segments and the point of meeting of the tangents be to each other*

Let there be the opposite sections $A$ and $B$ whose center is $C$ and tangents $AD$ and $DB$, and let $AB$ and $CD$ be joined and produced, and through $D$ let some straight line $EDFG$ be drawn across

I say that

$$FG \quad GF \quad ED \quad DF$$

For let $AC$ be joined and produced, and through $E$ and $F$ let $EHS$ and $FIMN\lambda O$ be drawn parallel to $AB$ and parallel to $AD$ $EP$ and $FR$

Since then $F\lambda$ and $ES$ are parallel and $E\Gamma$, $\lambda S$, and $HM$ have been drawn through to them,

$$EH \quad HS \quad FM \quad M\lambda$$

And alternately

$$EH \quad FM \quad HS \quad M\lambda$$

therefore also

$$\text{sq } EH \quad \text{sq } FM \quad \text{sq } HS \quad \text{sq } M\lambda$$

But

$$\text{sq } EH \quad \text{sq } FM \quad \text{trgl } EHP \quad \text{trgl } FRM$$

and

$$\text{sq } HS \quad \text{sq } M\lambda \quad \text{trgl } DHS \quad \text{trgl } \lambda MD,$$

therefore also

$$\text{trgl } EHP \quad \text{trgl } FRM \quad \text{trgl } DHS \quad \text{trgl } \lambda MD$$

And

$$\text{trgl } EHP = \text{trgl } ASK + \text{trgl } DHS \text{ (III 11)}$$

and

$$\text{trgl } FRM = \text{trgl } AXN + \text{trgl } XMD \text{ (III 11)},$$

therefore

$$\text{trgl } DHS \quad \text{trgl } XMD \quad \text{trgl } ASK + \text{trgl } DHS \quad \text{trgl } AXN + \text{trgl } XMD,$$

and

$$\text{remainder trgl } ASK \quad \text{remainder trgl } ANX \quad \text{trgl } DHS \quad \text{trgl } XMD$$

But

$$\text{trgl } ASK \quad \text{trgl } ANX \quad \text{sq } KA \quad \text{sq } AN \quad \text{sq } EG \quad \text{sq } FG,^{*}$$

and

$$\text{trgl } DHS \quad \text{trgl } XMD \quad \text{sq } HD \quad \text{sq } DM \quad \text{sq } ED \quad \text{sq } DF$$

Therefore also

$$EG \quad FG \quad ED \quad DF$$

PROPOSITION 40

*With the same things being so if a straight line is drawn through the point of meeting of the tangents parallel to the straight line joining the points of contact and if a straight line drawn from the midpoint of the straight line joining the points of contact cuts both of the sections and the straight line parallel to the straight line joining the points of contact then as the whole straight line drawn across is to the straight line cut off outside between the parallel and the section, so will the straight line s segments produced by the sections and the straight line joining the points of contact be to each other*

Let there be the opposite sections $A$ and $B$ whose center is $C$, and tangents $AD$ and $DB$, and let $AB$ and $CDE$ be joined, therefore

$$AE = EB \text{ (II 39)}$$

And from $D$ let $FDG$ be drawn parallel to $AB$, and from $E$, $LE$ at random

I say that

$$HL \quad LK \quad HE \quad EK$$

From $H$ and $K$ let $NMHX$ and $KOP$ be drawn parallel to $AB$ and $HR$ and $KS$ parallel to $AD$, and let $XACT$ be drawn through

Since then $XAU$ and $MAP$ have been drawn across the parallels $XM$ and $KP$

$$XA \quad AU \quad MA \quad AP$$

But

$$XA \quad AU \quad HE \quad EK,$$

and

$$HE \quad EK \quad HN \quad KO$$

because of the similarity of the triangles $HEN$ and $KEO$, therefore

$$HN \quad KO \quad MA \quad AP,$$

therefore also

$$\text{sq } HN \quad \text{sq } KO \quad \text{sq } MA \quad \text{sq } AP$$

For

$$EG \quad TG \quad KA \quad TA$$

and

$$TG \quad TF \quad TA \quad TN$$

and

$$TG - TF \quad TG \quad TA - TN \quad TA$$

therefore *ex aequali*

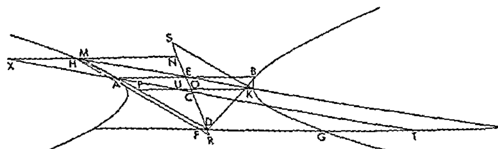
$$EG \quad FG \quad KA \quad AN$$

But

sq $HN$ sq $KO$ trgl $HRN$ trgl $KSO$,

and

sq $MA$ sq $AP$ trgl $XMA$ trgl $AUP$,




therefore also

trgl $HRN$ trgl $KSO$ trgl $XMA$ trgl $AUP$

And

trgl $HNR$ = trgl $XMA$ + trgl $MND$ (III 11),

and

trgl $KSO$ = trgl $AUP$ + trgl $DOP$ (III 11),

therefore also

trgl $XMA$ + trgl $MND$ trgl $AUP$ + trgl $DOP$ trgl $XMA$ trgl $AUP$,

therefore also

remainder trgl $NMD$ remainder trgl $DOP$ whole whole

But

trgl $XMA$ trgl $AUP$ sq $XA$ sq $AU$,

and

trgl $NMD$ trgl $DOP$ sq $MN$ sq $PO$,

therefore also

sq $MN$ sq $PO$ sq $XA$ sq $AU$

But

sq $MN$ sq $PO$ sq $ND$ sq $OD$,

and

sq $XA$ sq $AU$ sq $HE$ sq $EK$,

and

sq $ND$ sq $DO$ sq $HL$ sq $LK$,

therefore also

sq $HE$ sq $EK$ sq $HL$ sq $LK$

Therefore

$HE$ $EK$ $HL$ $LK$

## Proposition 41

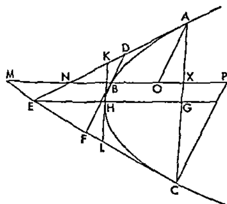
*If three straight lines touching a parabola meet each other they will be cut in the same ratio*

Let there be the parabola $ABC$, and tangents $ADE$ $EFC$ and $DBF$

I say that

$$CF \quad FE \qquad ED \quad DA \qquad FB \quad BD$$

For let $AC$ be joined and bisected at $G$




Then it is evident that the straight line from $E$ to $G$ is a diameter of the section (II 29)

If then it goes through $B$ $DF$ is parallel to $AC$, (II 5) and will be bisected by $EG$ and therefore

$$AD = DE \text{ (I 35)}$$

and

$$CF = FE \text{ (I 35)},$$

and what was sought is apparent

Let it not go through $B$ but through $H$, and let $KHL$ be drawn through $H$ parallel to $AC$ therefore it will touch the section at $H$ (I 32), and because of things already said (I 35),

$$AK = KE$$

and

$$LC = LE$$

Let $MNBX$ be drawn through $B$ parallel to $EG$ and $AO$ and $CP$ through $A$ and $C$ parallel to $DF$ Since then $MB$ is parallel to $EH$, $MB$ is a diameter (I 40, I 51 end), and $DF$ touches at $B$, therefore $AO$ and $CP$ have been dropped ordinatewise (II 5, First Def I 4) And since $MB$ is a diameter, and $CM$ a tangent and $CP$ an ordinate

$$MB = BP \text{ (I 35)},$$

and so also

$$MF = FC$$

And since

$$MF = FC$$

and

$$EL = LC,$$

$$MC \quad CF \qquad EC \quad CL,$$

and alternately

$$MC \quad EC \qquad CF \quad CL$$

But

$$MC \quad EC \qquad XC \quad CG$$

therefore also

$$CF \quad CL \qquad XC \quad CG$$

And

$$CL \quad EC \qquad CG \quad CA$$

therefore *ex aequali*

$$CA \quad XC \qquad EC \quad CF$$

and *convertendo*

$$EC \quad FE \qquad CA \quad AX,$$

*separando*

$$CF \quad FE \qquad XC \quad AX$$

Again since $MB$ is a diameter and $AN$ a tangent and $AO$ an ordinate,

$$NB = BO \text{ (I 35)},$$

and

$$ND = DA$$

And also

$$EK = KA,$$

therefore

$$AE\ \ KA\quad NA\ \ DA,$$

alternately

$$AE\ \ NA\quad KA\ \ DA$$

But

$$AE\ \ NA\quad GA\ \ AX,$$

therefore also

$$KA\ \ DA\quad GA\ \ AX$$

And also

$$AE\ \ KA\quad CA\ \ GA,$$

therefore, *ex aequali*,

$$AE\ \ DA\quad CA\ \ AX,$$

*separando*

$$ED\ \ DA\quad XC\ \ AX$$

And it was also shown

$$XC\ \ AX\quad C\Gamma\ \ \Gamma E,$$

therefore

$$CF\ \ \Gamma F\quad FD\ \ DA$$

Again since

$$XC\ \ AX\quad CP\ \ AO,$$

and

$$CP = 2\ BF,$$

and

$$CM = 2\ MF,$$

and

$$AO = 2\ BD,$$

and

$$AN = 2\ ND,$$

therefore

$$XC\ \ AX\quad FB\ \ BD\quad C\Gamma\ \ \Gamma E\quad ED\ \ DA$$

PROPOSITION 42

*If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the diameter parallel to an ordinate and some other straight line at random is drawn tangent it will cut off from them straight lines containing a rectangle equal to the fourth part of the figure to the same diameter*

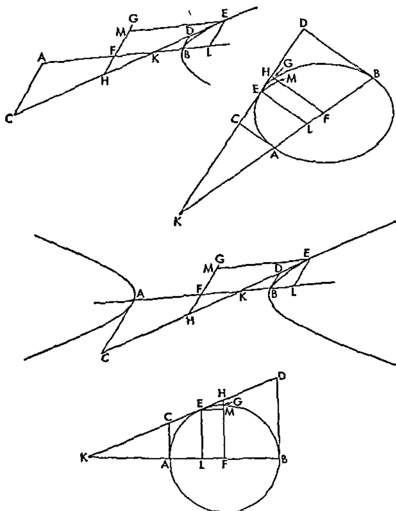
For let there be some one of the aforesaid sections whose diameter is $AB$ and from $A$ and $B$ let $AC$ and $DB$ be drawn parallel to an ordinate, and let some other straight line $CED$ be tangent at $E$

I say that

$$\text{rect } AC\ BD = \text{fourth part of figure to } AB$$

For let its center be $F$ and through it let $FG$ be drawn parallel to $AC$ and $BD$. Since then $AC$ and $BD$ are parallel and $FG$ is also parallel, therefore it is the diameter conjugate to $AB$ (First Def, I 6), and so

sq $FG$ = fourth part of figure to $AB$ (Sec. Def. I 3)




If then $FG$ goes through $E$ in the case of the ellipse and circle

$AC = FG = BD$ (II 7)

and it is immediately evident that

rect $AC$, $BD$ = sq $FG$ or fourth part of figure to $AB$

Then let it not go through it and let $DC$ and $BA$ produced meet at $K$ and let $EL$ be drawn through $E$ parallel to $AC$ and $FM$ parallel to $AB$. Since then

rect $KF, FL$ = sq $AF$ (I 37)

$KF$ $AF$ $AF$ $FL$

and

$$KA \quad AL \quad KF \quad AF \text{ or } FB \text{ (Eucl v 19)},$$

inversely

$$FB \quad K\Gamma \quad AL \quad KA,$$

*componendo* or *separando*

$$BK \quad K\Gamma \quad LK \quad KA$$

Therefore also

$$DB \quad FH \quad EL \quad CA$$

Therefore

$$\text{rect } DB, CA = \text{rect } FH, EL,$$
$$= \text{rect } HF, FM$$

But

$$\text{rect } HF, FM = \text{sq } FG \text{ (I 38)},$$
$$= \text{fourth figure to } AB \text{ (See Def I 11)},$$

therefore also

$$\text{rect } DB\ CA = \text{fourth figure to } AB$$

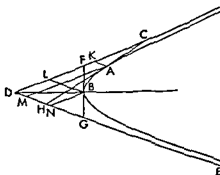
## Proposition 43

*If a straight line touch an hyperbola it will cut off from the asymptotes beginning with the center of the section straight lines containing a rectangle equal to the rectangle contained by the straight lines cut off by the tangent at the section's vertex at its axis*

Let there be the hyperbola $AB$, and asymptotes $CD$ and $DE$, and axis $BD$, and let $FBG$ be drawn through $B$ tangent, and some other tangent at random $CAH$

I say that

$$\text{rect } FD\ DG = \text{rect } CD\ DH$$




For let $AK$ and $BL$ be drawn from $A$ and $B$ parallel to $DG$ and $AM$ and $BN$ parallel to $CD$ Since then $CAH$ touches

$$CA = AH \text{ (II 3)},$$

and so

$$CH = 2AH$$

and

$$CD = 2AM$$

and

$$DH = 2AK$$

Therefore

$$\text{rect } CD\ DH = 4 \text{ rect } KA\ AM$$

Then likewise it could be shown

$$\text{rect } FD\ DG = 4 \text{ rect } LB\ BN$$

But

$$\text{rect } KA\ AM = \text{rect } LB\ BN \text{ (II 12)}$$

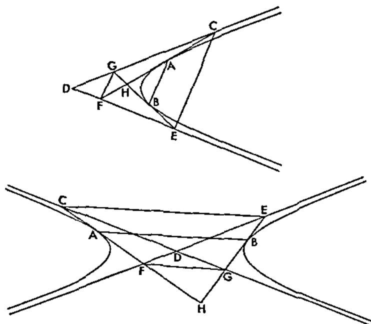
Therefore also

$$\text{rect } CD\ DH = \text{rect } FD\ DG$$

Then likewise it could be shown even if $DB$ were some other diameter and not the axis

## Proposition 44

*If two straight lines touching an hyperbola or opposite sections meet the asymptotes then the straight lines drawn to the sections will be parallel to the straight line joining the points of contact*




For let there be either the hyperbola or the opposite sections $AB$ and asymptotes $CD$ and $DE$ and tangents $CAHF$ and $EBHG$, and let $AB$ $FG$, and $CE$ be joined

I say that they are parallel

For since

$$\text{rect } CD\ DF = \text{rect } GD, DE \quad (\text{III } 43),$$

therefore

$$CD \quad DE \qquad GD \quad DF,$$

therefore $CE$ is parallel to $FG$ And therefore

$$HF \quad FC \qquad HG \quad GE$$

And

$$FC \quad AF \qquad GE \quad GB,$$

for each is double (II 3) therefore *ex aequali*

$$HG \quad GB \qquad HF \quad FA$$

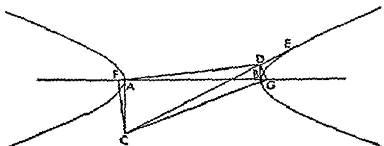
Therefore $FG$ is parallel to $AB$

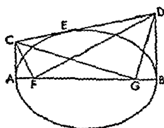
## Proposition 45

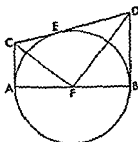
*If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the axis at right angles and a rectangle equal to the fourth part of the figure is applied to the axis on each side and exceed*

*ing by a square figure in the case of the hyperbola and opposite sections but deficient in the case of the ellipse and some straight line is drawn tangent to the section and meeting the perpendicular straight lines then the straight lines drawn from the points of meeting to the points produced by the application make right angles at the aforesaid points* [1]

Let there be one of the sections mentioned whose axis is $AB$, and $AC$ and $BD$




[1] The points of application are in modern terminology the foci of the conics The circle is seen here as an ellipse whose two *foci or focal points coincide with the center* This theory is of course a special application of Euclid vi 28 and 29 two theorems on which depends one whole side of Greek geometry

Apollonius never speaks of the focus of the parabola but it can be found by analogy with the ellipse

Thus in the ellipse above

$$\text{rect } AF\ FB = \text{fourth rect } AB\ R$$

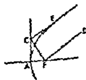
where $R$ is the parameter Or

$$\text{rect } AF\ (AB - AF) = \text{fourth rect } AB\ R$$

or

$$AF \quad \text{fourth } R \quad AB \quad (AB - AF)$$

Then if we consider the ellipse as its axis $AB$ gets as large as we please we can think of it as *approaching as near as we please to a parabola with parameter $R$* The ratio $AB$ $(AB - AF)$ approaches as near as we please to equality and hence also the ratio $AF$ fourth $R$ At the limit we can think of the ellipse as the parabola its axis $AB$ as infinite and $AB$ as equal to $AB - AF$ Then $AF$ will be equal to a fourth $R$ Thus the focus of a parabola will be defined as the point on its axis at a distance from the vertex equal to one quarter of the parameter Then many of the properties of the foci of the ellipse can be proved analogously for the parabola Thus in the case of this proposition, $FD$ will become parallel to $CE$ Hence any straight line from the focus of a parabola parallel to a tangent will make a right angle with the straight line drawn from the focus to the intersection of the tangent and the perpendicular to the axis at the vertex



at right angles and $CED$ tangent and let the rectangle $AF$ $FB$ and the rectangle $AG$ $GB$ equal to the fourth part of the figure be applied on each side (Eucl VI 28 29), as has been said and let $CF$ $CG$, $DF$, and $DG$ be joined

I say that angle $CFD$ and angle $CGD$ are each a right angle

For since it has been shown

rect $AC$ $BD$ = fourth figure on $AB$ (III 42),

and since also

rect $AF$ $FB$ = fourth figure on $AB$,

therefore

rect $AC, BD$ = rect $AF$, $FB$

Therefore

$AC$ $AF$ $FB$ $BD$

And the angles at points $A$ and $B$ are right therefore

angle $ACF$ = angle $BFD$ (Eucl VI 6),

and

angle $AFC$ = angle $FDB$

And since angle $CAF$ is right, therefore

angle $ACF$ + angle $AFC$ = 1 rt angle

And it has also been shown that

angle $ACF$ = angle $DFB$,

therefore

angle $AFC$ + angle $DFB$ = 1 right angle

Therefore

angle $DFC$ = 1 right angle

Then likewise it could also be shown

angle $CGD$ = 1 right angle

PROPOSITION 46

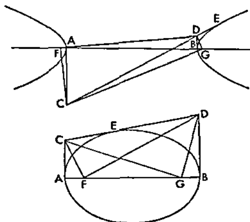
*With the same things being so the straight lines joined make equal angles with the tangents*

For with the same things supposed I say that

angle $ACF$ = angle $DCG$

and

angle $CDF$ = angle $BDG$




For since it has been shown that both angle $CFD$ and angle $CGD$ are right angles (III 45) the circle described about $CD$ as a diameter will pass through points $F$ and $G$ therefore

angle $DCG$ = angle $DFG$

for they are on the same segment of the circle And it was shown

angle $DFG$ = angle $ACF$ (III 45),

and so

angle $DCG$ = angle $ACF$

And likewise also

angle $CDF$ = angle $BDG$

### Proposition 47

*With the same things being so the straight line drawn from the point of meeting of the joined straight lines to the point of contact will be perpendicular to the tangent*

For let the same things as before be supposed and let $CG$ and $FD$ meet each other at $H$ and let $CD$ and $BA$ produced meet at $K$, and let $EH$ be joined

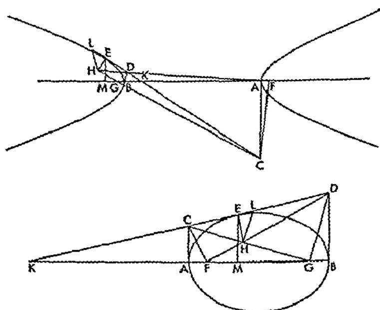
I say that $EH$ is perpendicular to $CD$

For if not let $HL$ be drawn from $H$ perpendicular to $CD$ Since then

$$\text{angle } CDF = \text{angle } GBD \text{ (III 46)},$$

and also

$$\text{rt angle } DBC = \text{rt angle } DLH,$$




therefore triangle $DGB$ is similar to triangle $LHD$ Therefore

$$GD \quad DH \qquad BD \quad DL$$

But

$$GD \quad DH \qquad FC \quad CH$$

because the angles at $F$ and $G$ are right angles (III 45) and the angles at $H$ are equal but

$$FC \quad CH \qquad AC \quad CL$$

because of the similarity of the triangles $AFC$ and $LCH$ (III 46), therefore also

$$BD \quad DL \qquad AC \quad CL$$

Alternately

$$BD \quad AC \qquad DL \quad CL$$

But

$$BD \quad AC \qquad BK \quad KA$$

therefore also

$$DL \quad CL \qquad BK \quad KA$$

Let $EM$ be drawn from $E$ parallel to $AC$, therefore it will have been dropped ordinatewise to $AB$ (II 7), and

$$BK \quad KA \qquad BM \quad MA$$

And

$$BM \quad MA \qquad DE \quad EC$$

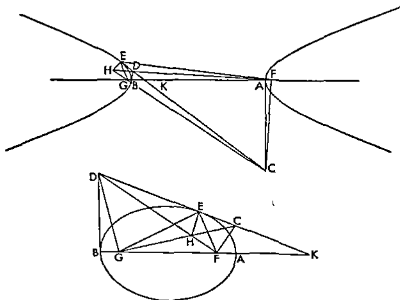
therefore also

$$DL \quad CL \qquad DE \quad EC,$$

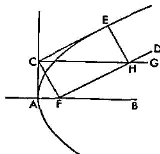
and this is absurd Therefore $HL$ is not perpendicular nor is any other straight line except $HE$ *

PROPOSITION 48

*With the same things being so, it must be shown that the straight lines drawn from the point of contact to the points produced by the application make equal angles with the tangent*




*There is the analogous theorem for the parabola $FD$ becomes a straight line parallel to




$CE$ and $CG$ a straight line parallel to $AB$ Again $HE$ is perpendicular to $CE$ and this can be proved rigorously as well as understood by analogy

For let the same things be supposed, and let $EF$ and $EG$ be joined I say that

angle $CEF$ = angle $GED$

For since angles $DGH$ and $DEH$ are right angles (III 45 47), the circle described about $DH$ as a diameter will pass through the points $E$ and $G$ (Eucl III 31), and so

angle $DHG$ = angle $DEG$ (Eucl III 21),

for they are in the same segment Likewise then also

angle $CEF$ = angle $CHF$

But

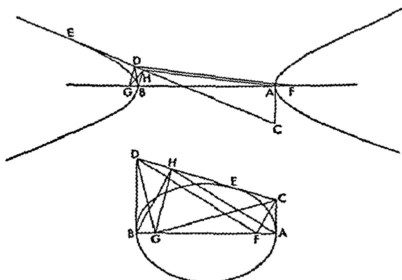
angle $CHF$ = angle $DHG$,

for they are vertical angles, therefore also

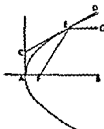
angle $CEF$ = angle $DEG$ *

## Proposition 49

*With the same things being so if from one of the points (of application) a perpendicular is drawn to the tangent then the straight lines from that point to the ends of the axis make a right angle*




Here there is another and important analogous theorem for the parabola $EG$ becomes parallel to $AB$ and




angle $DEG$ = angle $CEF$

For let the same things be supposed, and let the perpendicular $GH$ be drawn from $G$ to $CD$, and let $AH$ and $BH$ be joined

I say that angle $AHB$ is a right angle

For since angle $DBG$ is a right angle and also angle $DHG$ the circle described about $DG$ as a diameter will pass through $H$ and $B$, and

$$\text{angle } BHG = \text{angle } BDG$$

But it was shown

$$\text{angle } AGC = \text{angle } BDG \text{ (III 46)},$$

therefore also

$$\text{angle } BHG = \text{angle } AGC = \text{angle } AHC \text{ (Eucl III 21)}$$

And so also

$$\text{angle } CHG = \text{angle } AHB$$

But angle $CHG$ is a right angle, therefore also angle $AHB$ is a right angle

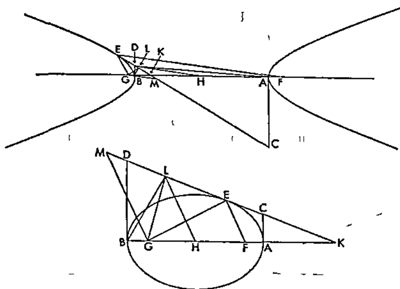
PROPOSITION 50

*With the same things being so if from the center of the section there falls to the tangent a straight line parallel to the straight line drawn through the point of contact and one of the points (of application), then it will be equal to one half the axis*

For let there be the same things as before and let $H$ be the center, and let $EF$ be joined and let $DC$ and $BA$ meet at $K$ and through $H$ let $HL$ be drawn parallel to $EF$

I say that

$$HL = HB$$




For let $EG$ $AL$, $LG$ be joined and through $G$ let $GM$ be drawn parallel to $EF$ Since then

$$\text{rect } AF\ FB = \text{rect } AG\ GB \text{ (See III 45)},$$

therefore

$$AF = GB$$

But also

$$AH = HB,$$

therefore also

$$FH = HG$$

And so also

$$EL = LM$$

And since it was shown (III 48)

$$\text{angle } CEF = \text{angle } DEG,$$

and

$$\text{angle } CEF = \text{angle } EMG,$$

therefore also

$$\text{angle } EMG = \text{angle } DEG$$

And therefore

$$EG = GM$$

But it was also shown

$$EL = LM$$

therefore $GL$ is perpendicular to $EM$ And so through what was shown before (III 49) angle $ALB$ is a right angle and the circle described about $AB$ as a diameter will pass through $L$ And

$$HA = HB$$

therefore also since $HL$ is a radius of the semicircle,

$$HL = HB$$

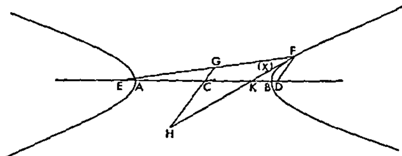
## Proposition 51

*If a rectangle equal to the fourth part of the figure is applied from both sides to the axis of an hyperbola or opposite sections and exceeding by a square figure and straight lines are deflected from the resulting points of application to either one of the sections, then the greater of the two straight lines exceeds the less by exactly as much as the axis*

For let there be an hyperbola or opposite sections whose axis is $AB$ and center $C$ and let each of the rectangles $AD$ $DB$ and $AE$, $EB$ be equal to the fourth part of the figure and from points $E$ and $D$ let the straight lines $EF$ and $FD$ be deflected to the line of the section

I say that

$$EF = FD + AB$$



Let $FKH$ be drawn tangent through $F$, and $GCH$ through $C$ parallel to $FD$ therefore

$$\text{angle } KHG = \text{angle } KFD,$$

for they are alternate And

$$\text{angle } KFD = \text{angle } GFH \text{ (III 48)},$$

therefore

$$GF = GH$$

But

$$GF = GE,$$

since also

$$AE = BD$$

and

$$AC = CB$$

and

$$EC = CD,$$

and therefore

$$GH = EG$$

And so

$$FE = 2GH$$

And since it has been shown (III 50)

$$CH = CB,^*$$

therefore

$$FE = 2(GC\ CB)$$

But

$$FD = 2GC,$$

and

$$AB = 2CB,$$

therefore

$$FE = FD + AB$$

And so $EF$ is greater than $FD$ by $AB$

PROPOSITION 52

*If in an ellipse a rectangle equal to the fourth part of the figure is applied from both sides to the major axis and deficient by a square figure, and from the points resulting from the application straight lines are deflected to the line of the section then they will be equal to the axis*

Let there be an ellipse whose major axis is $AB$ and let each of the rectangles $AC\ CB$ and $AD\ DB$ be equal to the fourth of the figure and from $C$ and $D$ let the straight lines $CE$ and $ED$ have been deflected to the line of the section

I say that

$$CE + ED = AB$$

*For

$$GF = GH$$

and by III 50 a line $C(\lambda)$ drawn parallel to $GF$ is equal to $CB$ But also

$$C(\lambda) = CH$$

Hence

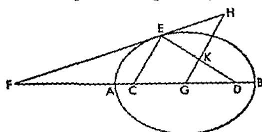
$$CH = CB$$

Let $FEH$ be drawn tangent and $G$ be center and through it let $GKH$ be drawn parallel to $CE$ Since then

$$\text{Angle } CEF = \text{angle } HFK \text{ (III 48)},$$

and

$$\text{angle } CEF = \text{angle } EHK,$$




therefore also

$$\text{angle } EHK = \text{angle } HEK$$

Therefore also

$$HK = KE$$

And since

$$AG = GB$$

and

$$AC = DB,$$

therefore also

$$CG = GD,$$

and so also

$$FK = KD$$

And for this reason

$$ED = 2HK,$$

and

$$EC = 2KG$$

and

$$ED + EC = 2GH$$

But also

$$AB = 2GH \text{ (III 50)},$$

therefore

$$AB = FD + EC$$

Proposition 53

*If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of a diameter parallel to an ordinate, and straight lines drawn from the same ends to the same point on the line of the section cut the parallels then the rectangle contained by the straight lines cut off is equal to the figure on that same diameter*

Let there be one of the aforesaid sections $ABC$ whose diameter is $AC$, and let $AD$ and $CF$ be drawn parallel to an ordinate and let $ABE$ and $CBD$ be drawn across

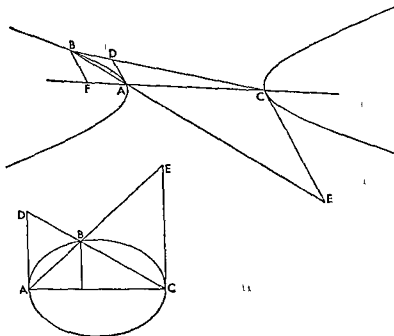
I say that

$$\text{rect } AD\ FC = \text{figure on } AC$$

For let $BF$ be drawn from $B$ parallel to an ordinate

Therefore

rect $A\Gamma, FC$ sq $FB$ transverse side upright side
sq $AC$ the figure (I 21)




But

rect $AF$ $FC$ sq $FB$ comp $AF$ $FB$ $FC$ $FB$,

therefore

figure sq $AC$ comp $FB$ $AF, \Gamma B$ $FC$

But

$AF$ $\Gamma B$ $AC$ $CE$

and

$\Gamma C$ $FB$ $AC$ $AD$

therefore

figure sq $AC$ comp $CE$ $AC$ $AD$ $AC$

And also

rect $AD$ $CE$ sq $AC$ comp $CE$ $AC$, $AD$ $AC$,

therefore

figure sq $AC$ rect $AD$ $CE$ sq $AC$

Therefore

rect $AD$ $CE$=figure on $AC$

PROPOSITION 54

*If two tangents to a section of a cone or to a circumference of a circle meet and through the points of contact parallels to the tangents are drawn and from the points of contact to the same point of the line of the section straight lines are drawn across cutting the parallels then the rectangle contained by the straight lines cut*

*off to the square on the straight line joining the points of contact has a ratio compounded of the ratio which the inside segment line joining the point of meeting of the tangents and the midpoint of the straight line joining the points of contact has in square to the remainder and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact*

Let there be a section of a cone or circumference of a circle $ABC$ and tangents $AD$ and $CD$, and let $AC$ be joined and bisected at $E$, and let $DBE$ be joined, and let $AF$ be drawn from $A$ parallel to $CD$, and $CG$ from $C$ parallel to $AD$ and let some point $H$ on the section be taken and let the straight lines $AH$ and $CH$ be joined and produced to $G$ and $F$

I say that

rect $AF, CG$ sq $AC$ comp sq $EB$ sq $BD$, rect $AD, DC$ fourth sq $AC$ or rect $AE, EC$

For let $KHOXL$ be drawn from $H$ parallel to $AC$ and from $B$, $MBN$ parallel to $AC$, then it is evident that $MN$ is tangent (II 29 5, 6) Since then

$$AE = EC,$$

also

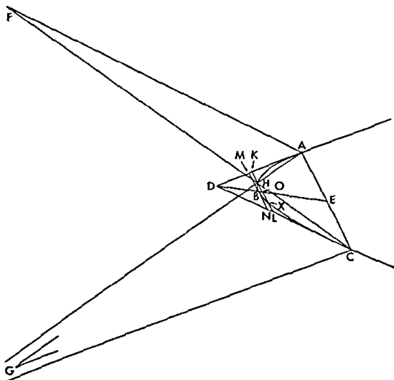
$$MB = BN$$

and

$$KO = OL$$

and

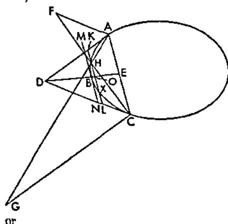
$$HO = OX \text{ (II 7)}$$



and

$$KH = KL$$

Since then $MB$ and $MA$ are tangents and $KHL$ has been drawn parallel to $MB$,

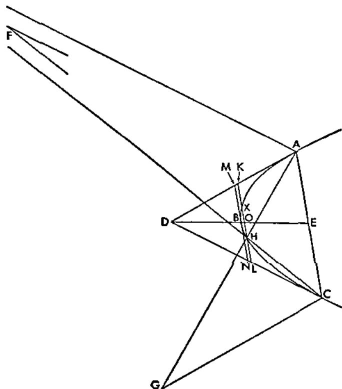



sq $AM$ sq $MB$ sq $AK$ rect $XK$ $KH$ (III 16)
or
sq $AM$ rect $MB,BN$ sq $AK$ rect $LH,HK$
And
rect $NC,AM$ sq $AM$ rect $LC,AK$ sq $AK$ (Eucl VI 2, V, 18),
therefore *ex aequali*
rect $NC,AM$ rect $MB$ $BN$ rect $LC,AK$ rect $LH,HK$
But
rect $LC$ $AK$ rect $LH,HK$ comp $LC$ $LH,AK$ $HK$
or

rect $LC,AK$ rect $LH,HK$ comp $FA$ $AC,GC$ $CA$

which is the same as

rect $GC$ $FA$ sq $CA$



Therefore

rect $NC, AM$ rect $MB, BN$ rect $GC, FA$ sq $CA$

But with the rectangle $ND, DM$ taken as a mean

rect $NC, AM$ rect $MB\ BN$ comp

rect $NC, AM$ rect $ND, DM$ rect $ND\ DM$ rect $MB, BN$,

therefore

rect $GC\ FA$ sq $CA$ comp

rect $NC, AM$ rect $ND, DM$ rect $ND, DM$ rect $MB\ BN$

But

rect $NC, AM$ rect $ND\ DM$ sq $EB$ sq $BD$,

and

rect $ND, DM$ rect $NB, BM$ rect $CD\ DA$ rect $CE, EA$,

therefore

rect $GC, FA$ sq $CA$ comp sq $BE$ sq $BD$, rect $CD, DA$ rect $CE, EA$

## Proposition 55

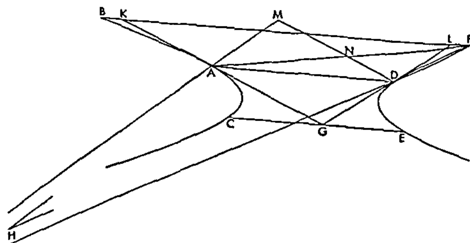
*If two straight lines touching opposite sections meet and through the point of meeting a straight line is drawn parallel to the straight line joining the points of contact and from the points of contact parallels to the tangents are drawn across, and straight lines are produced from the points of contact to the same point of one of the sections cutting the parallels then the rectangle contained by the straight lines cut off will have to the square on the straight line joining the points of contact the ratio which the rectangle contained by the tangents has to the square on the straight line drawn through the point of meeting parallel to the straight line joining the points of contact as far as the section*

Let there be the opposite sections $ABC$ and $DEF$, and tangents to them $AG$ and $GD$ and let $AD$ be joined and from $G$ let $CGE$ be drawn parallel to $AD$, and from $A$ $AM$ parallel to $DG$ and from $D$ $DM$ parallel to $AG$ and let some point $F$ be taken on the section $DF$, and let $ANF$ and $FDH$ be joined

I say that

sq $CG$ rect $AG\ GD$ sq $AD$ rect $HA, DN$

For let $FLKB$ be drawn through $F$ parallel to $AD$



Since then it has been shown that

$$\text{sq } EG : \text{sq } GD = \text{rect } BL, LF : \text{sq } DL \text{ (III 20)},$$

and

$$CG = EG \text{ (II 38)},$$

and

$$BK = L\Gamma \text{ (II 38)},$$

therefore

$$\text{sq } CG : \text{sq } GD = \text{rect } KF, FL : \text{sq } DL$$

And also

$$\text{sq } GD : \text{rect } AG, GD = \text{sq } DL : \text{rect } DL, AK \text{ (Eucl VI 2, 1)},$$

therefore *ex aequali*

$$\text{sq } GC : \text{rect } AG, GD = \text{rect } K\Gamma, FL : \text{rect } DL, AK$$

But

$$\text{rect } KF, FL : \text{rect } DL, AK = \text{comp } KF : AK, FL : DL$$

But

$$KF : AK = AD : DN,$$

and

$$\Gamma L : DL = AD : HA,$$

therefore

$$\text{sq } CG : \text{rect } AG, GD = \text{comp } AD : DN, AD : HA$$

And also

$$\text{sq } AD : \text{rect } HA, DN = \text{comp } AD : DN, AD : HA,$$

therefore

$$\text{sq } CG : \text{rect } AG, GD = \text{sq } AD : \text{rect } HA, DN$$

## Proposition 56

*If two straight lines touching one of the opposite sections meet and parallels to the tangents are drawn through the points of contact and straight lines cutting the parallels are drawn from the points of contact to the same point of the other section, then the rectangle contained by the straight lines cut off will have to the square on the straight line joining the points of contact the ratio compounded of the ratio which of the straight line joining the point of meeting and the midpoint that part between the midpoint and the other section has in square to that part between the same section and the point of meeting and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact*

Let there be the opposite sections $AB$ and $CD$ whose center is $O$ and tangents $AEFG$ and $BEHK$ and let $AB$ be joined and bisected at $L$ and let $LE$ be joined and drawn across to $D$, and let $AM$ be drawn from $A$ parallel to $BE$ and $BN$ from $B$ parallel to $AE$ and let some point $C$ be taken on the section $CD$ and let $CBM$ and $CAN$ be joined

I say that

$$\text{rect } MA, BN : \text{sq } AB = \text{comp sq } LD : \text{sq } DE, \text{rect } AE, EB : \text{fourth sq } AB \text{ or rect } AL, LB$$

For let $GCK$ and $HDF$ be drawn from $C$ and $D$ parallel to $AB$ then it is evident that

$$HD = DF$$

and

$$KY = YG$$

and also

$$XC = XP,$$

and so also

$$CK = GP$$

And since $AB$ and $DC$ are opposite sections and $BEH$ and $HD$ are tangents, and $KG$ is parallel to $DH$ therefore

sq $BH$ sq $HD$ sq $BK$ rect $PK, KC$ (III 18 note)

But

sq $HD$ = rect $HD$ $DF$,
rect $PK, KC$ = rect $KC$ $CG$

Therefore

sq $BH$ rect $HD, DF$ sq $BK$ rect $KC, CG$

And also

rect $FA$ $BH$ sq $BH$ rect $GA$ $BK$ sq $BK$,

therefore *ex aequali*

rect $FA$ $BH$ rect $HD$ $DF$ rect $GA$ $BK$ rect $KC$ $CG$

And with rectangle $HE$ $EF$ taken as a mean

rect $FA$ $BH$ rect $HD$ $DF$ comp
rect $FA$ $HB$ rect $HE$ $EF$ rect $HE$ $EF$ rect $HD$ $DF$,

and

rect $FA$ $HB$ rect $HE$ $EF$ sq $LD$ sq $DE$

and

rect $HE$ $EF$ rect $HD$ $DF$ rect $AE$ $EB$ rect $AL$ $LB$

therefore

rect $GA$ $BK$ rect $KC, CG$ comp sq $LD$ sq $DE$ rect $AF, FB$
rect $AL$ $LB$

And

rect $GA$ $BK$ rect $KC$ $CG$ comp $BK$ $KC, GA$ $CG$

But

$BK$ $KC$ $MA$ $AB$

and

$GA$ $CG$ $BN$ $AB$

therefore

rect $MA$ $BN$ sq $AB$ comp $MA$ $AB$ $BN$ $AB$
comp sq $LD$ sq $DE$ rect $AE$ $LB$ rect $AL$ $LB$

## Translator's Appendix on Three and Four Line Loci

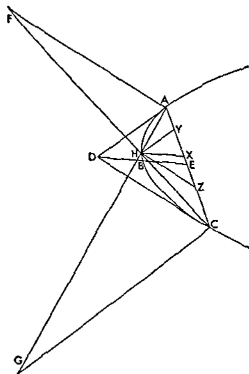
The three-line locus property of conics is easily deduced for the ellipse hyperbola parabola and circle from III 54, and for the opposite sections from III 55 and 56 The three-line locus property of conics can be stated thus Any conic section or circle or pair of opposite sections can be considered as the locus of points whose distances from three given fixed straight lines (the distances being either perpendicular or at a given constant angle to each of the given straight lines although the constant angle may be different for each of the three straight lines) are such that the square of one of the distances is always in a constant ratio to the rectangle contained by the other two distances

It is shown in III 54 that in the case of conic sections and circles

rect $AF$ $CG$ sq $AC$ comp sq $EB$ sq $BD$, rect $AD$, $DC$ fourth sq $AC$

Now if we consider the straight lines $AD$ $DC$ and $AC$ as fixed and given and therefore straight line $DE$ fixed and given as bisecting $AC$, then it is evident that the straight lines $AC$ $EB$ $BD$ $AD$ $DC$, and therefore the squares on them and the rectangles contained by them are also fixed and given Then although as the point $H$ is taken at different points along the conic the straight lines $AF$ and $CG$ change in magnitude nevertheless the magnitude of the rectangle $AF$ $CG$ because of the above proportion remains constant

For let $HX$ be drawn parallel to $BE$ and $HY$ to $AD$ and $HZ$ to $DC$ Then $HX$ is the distance from $H$ to $AC$ at a given angle and $AY$ because of parallels represents the distance from $H$ to $AD$ at another given angle and $ZC$ represents the distance from $H$ to $DC$ at another given angle Then by similar triangles




$CZ$ $ZH$ $AC$ $AF$

$AY$ $YH$ $AC$ $CG$,

therefore compounding

rect $CZ, AY$ rect $ZH$ $YH$ sq $AC$ rect $AF, CG$

Now we have seen that the rectangle $AF$ $CG$ is a constant magnitude as the point $H$ changes and the square on $AC$ is constant therefore their ratio is constant Therefore

rect $CZ, AY$ rect $ZH$ $YH$ is a constant ratio (1)

Again by similar triangles

$ZH$ $HX$ $CD$ $DE$,

$YH$ $HX$ $AD$ $DE$,

therefore compounding

rect $ZH$ $YH$ sq $HX$ rect $CD$, $AD$ sq $DE$

But rectangle $CD, AD$ and the square on $DE$ are constant magni-

tudes as the point $H$ changes therefore their ratio is constant Therefore

rect $ZH, YH$ sq $HX$ is a constant ratio (2)

Compounding (1) and (2) we get a constant ratio that is

rect $CZ\ AY$ sq $HX$ is a constant ratio

In other words, as the point $H$ changes the rectangle contained by the distances from $H$ to two of the given straight lines (at given angles to those straight lines) has a constant ratio to the square on the distance to the third straight line (at a given angle to that straight line) And it can easily be proved by means of similar triangles that *if any other three angles are* chosen for the distances than those chosen here for the demonstration then the corresponding ratio will be constant although not equal

The four line locus property can be easily deduced from the three-line If to any conic section we construct four tangents $AG$ $BE$, $AI$, and $EC$ and the straight lines $FG$ $GI$ $ID$ and $DF$, joining the points of contact, and draw the distances from any point $H$ on the conic to these straight lines at any given angles (perpendiculars are convenient) then by the three line locus property with respect to triangle $FBG$ for any point $H$ on the conic

rect $HX, HV$ sq $HP$ is constant, ($\alpha$)

with respect to triangle $AIG$

rect $HX\ HY$ sq $HR$ is constant ($\beta$)

with respect to triangle $DCI$

rect $HY\ HZ$ sq $HS$ is constant ($\gamma$)

with respect to triangle $IFD$

rect $HZ\ HV$ sq $HQ$ is constant ($\delta$)

It will be noticed that we have taken in succession a pair of adjacent tangents and the *straight line joining their points of contact It will also be noticed that* the rectangles in the four ratios present a cyclical arrangement so that if the

inverse of ($\alpha$) is compounded with ($\beta$), and the inverse of ($\gamma$) with ($\delta$) we would have two constant ratios

$$\text{pllpd } HY, HP\ HP \quad \text{pllpd } HV, HR, HR, \qquad (\epsilon)$$
$$\text{pllpd } HV, HS\ HS \quad \text{pllpd } HY, HQ, HQ \qquad (\zeta)$$

Again compounding the first of these with the second we would have finally

$$\text{rect } HP, HS \quad \text{rect } HQ\ HR, \text{ a constant ratio}$$

And this is the property of the four line locus namely the locus of points $H$ such that the rectangle contained by the distances from points $H$ to any two given fixed straight lines $FG$ and $ID$ has to the rectangle contained by the distances from $H$ to two other fixed straight lines $IG, FD$, a constant ratio

The rigorous method of effecting these compoundings is as follows For inverting ($\alpha$), by Eucl XI 32 we have the constant ratios

$$\text{sq } HP \quad \text{rect } HX, HV \quad \text{pllpd } HP\ HP, HY \quad \text{pllpd } HX, HV, HY,$$
$$\text{rect } HX\ HY \quad \text{sq } HR \quad \text{pllpd } HX\ HY, HV \quad \text{pllpd } HR, HR\ HV$$

Hence, by definition the ratio (a constant one) compounded of these two is

$$\text{pllpd } HY\ HP, HP \quad \text{pllpd } HV, HR\ HR$$

And in the same way we find the constant ratio compounded of the inverse of ($\gamma$) and ($\delta$) Now

$$\text{pllpd } HY\ HP, HP \quad \text{pllpd } HV, HR\ HR \text{ comp } HY \quad HV, \text{sq } HP \quad \text{sq } HR$$
$$\text{pllpd } HV, HS\ HS \quad \text{pllpd } HY, HQ\ HQ \text{ comp } HV \quad HY \text{ sq } HS \quad \text{sq } HQ$$

If then we take two lines $M$ and $N$ such that

$$HP \quad HR \quad HR \quad M, \qquad (\eta)$$
$$HS \quad HQ \quad HQ \quad N, \qquad (\theta)$$

then

$$\text{sq } HP \quad \text{sq } HR \quad HP \quad M,$$
$$\text{sq } HS \quad \text{sq } HQ \quad HS \quad N$$

Hence

$$\text{ratio comp } HY \quad HV \text{ sq } HP \quad \text{sq } HR \text{ ratio comp } HY \quad HV\ HP \quad M$$
$$\text{ratio comp } HV \quad HY, \text{sq } HS \quad \text{sq } HQ \text{ ratio comp } HV \quad HY, HS \quad N$$

But

$$\text{rect } HY\ HP \quad \text{rect } HV\ M \text{ comp } HY \quad HV, HP \quad M,$$
$$\text{rect } HV, HS \quad \text{rect } HY\ N \text{ comp } HV \quad HY, HS \quad N$$

and

$$\text{pllpd } HY\ HP\ HS \quad \text{pllpd } HV\ M, HS \quad \text{rect } HY\ HP \quad \text{rect } HY\ M,$$
$$\text{pllpd } HV\ HS, M \quad \text{pllpd } HY\ N\ M \quad \text{rect } HV, HS \quad \text{rect } HY\ N,$$

and these are constant ratios Hence compounding we get the constant ratio

$$\text{pllpd } HY, HP, HS \quad \text{pllpd } HY, N, M,$$

which is the same as the constant ratio

$$\text{rect } HP, HS \quad \text{rect } N, M$$

Now taking $L$ and $O$ as some constants

$$\text{rect } HP\ HS \quad \text{rect } N\ M \quad L \quad O$$

and

$$\text{rect } HP\ HS \quad \text{rect } HR\ HQ \quad \text{rect } HR\ HQ \quad \text{rect } M\ N$$

by compounding ($\eta$) and ($\theta$) But equal ratios have equal duplicate ratios (Heath s note to Euclid VI 22) and hence

$$\text{rect } HP\ HS \quad \text{rect } HR\ HQ \text{ is constant}$$

In the case of opposite sections it is shown in III 56

$$\text{rect } MA\ BV \quad \text{sq } AB \text{ comp sq } LD \quad \text{sq } DE \text{ rect } AE\ FB \quad \text{fourth sq } AB$$

Then it is evident for the same reasons as before that for different points $C$ the

magnitudes $MA$ and $BN$ may change, but the rectangle $MA$ $BN$ is a constant magnitude

For as before, let $CY$ be drawn parallel to $DE$, $CY$ to $EA$, $CZ$ to $EB$ By similar triangles

$$AY \quad YC \quad AB \quad BN,$$
$$BZ \quad ZC \quad AB \quad MA,$$

therefore compounding

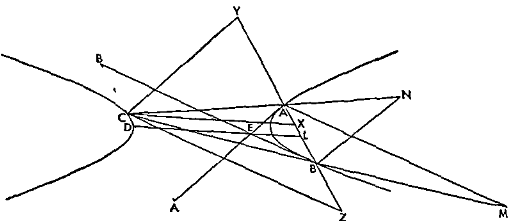
$$\text{rect } AY,BZ \quad \text{rect } YC,ZC \quad \text{sq } AB \quad \text{rect } MA\ BN$$

Since rectangle $MA,BN$ is constant as $C$ changes, and also the square on $AB$ is constant therefore

$$\text{rect } AY,BZ \quad \text{rect } YC\ ZC \text{ is a constant ratio} \qquad (1)$$

Again by similar triangles

$$ZC \quad CX \quad EB \quad EL,$$




$$YC \quad CX \quad EA \quad EL$$

therefore compounding

$$\text{rect } YC\ ZC \quad \text{sq } CX \quad \text{rect } EB\ EA \quad \text{sq } EL$$

Hence

$$\text{rect } YC\ ZC \quad \text{sq } CX \text{ is a constant ratio} \qquad (2)$$

Compounding (1) and (2) we have a constant ratio

$$\text{rect } AY\ BZ \quad \text{sq } CX$$

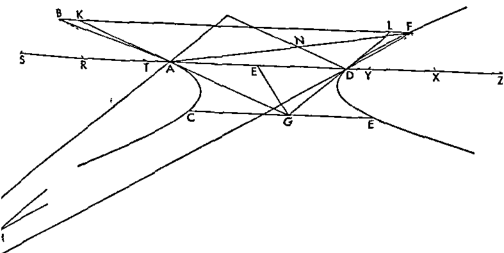
But $AY$ and $BZ$ are equal to $CA$ and $CB$ the distances from $C$ This is the property of the three line locus of section C with respect to the straight lines $EA$ and $EB$ tangents to the other section and $LB$ the straight line joining their points of contact And so one opposite section is a three line-locus to the tangents to the other of the opposite sections That it is also a four line locus could be shown in the same way as before

Again from III 55 we can conclude that both of the opposite sections together are a three line locus to the triangle formed by a tangent to each of the sections and the straight line joining their points of contact For by III 55

$$\text{rect } HA\ DN \quad \text{sq } AD \quad \text{rect } AG\ GD \quad \text{sq } CG$$

Now since the three last terms of this proportion are evidently constants as the point $F$ changes therefore also although $HA$ and $DN$ change with $F$ yet rectangle $HA$, $DN$ remains constant in magnitude Then reproducing the figure of

in 55 we drop $YF$ parallel to $DL$ and $FZ$ to $KA$, and $FX$ to $GE$, where $E$ is the midpoint of $AD$ Then by similar triangles




$$YD \quad FY \quad AD \quad HA,$$
$$AZ \quad FZ \quad AD \quad DN$$

therefore compounding

rect $YD$ $AZ$ rect $FY, FZ$ sq $AD$ rect $HA, DN$

But the last two terms are constant, therefore

rect $YD$ $AZ$ rect $FY, FZ$ is a constant ratio (1)

Again by similar triangles

$$FY \quad FX \quad DG \quad EG$$
$$FZ \quad FX \quad AG \quad EG,$$

therefore compounding

rect $FY$ $FZ$ sq $FX$ rect $DG, AG$ rect $ED, EG$

But the last two terms are constant therefore

rect $FY$ $FZ$ sq $FX$ is a constant ratio (2)

Compounding (1) and (2), we see that

rect $YD$ $AZ$ sq $FX$ is a constant ratio

But this is the definition of a three line locus that the rectangle contained by the distances from any point on the locus to two fixed straight lines have to the square on the distance to a third fixed straight line a constant ratio But

$$DY = LF$$
$$AZ = KF$$

and $FX$ is the distance from $F$ to $AD$ And so the ratio fulfills the definition

Furthermore if we consider $B$ the point of intersection of the straight line $KF$ drawn parallel to $AD$ with the other opposite section and draw $BS$ parallel to $FY$ $BR$ to $FX$, and $BT$ to $FZ$ since they are parallels between parallels,

$$BR = FX$$
$$KF = AZ$$
$$TA = BK$$

magnitudes $MA$ and $BN$ may change, but the rectangle $MA, BN$ is a constant magnitude

For as before let $CX$ be drawn parallel to $DE$, $CY$ to $EA$, $CZ$ to $EB$ By similar triangles

$$AY \quad YC \quad AB \quad BN,$$
$$BZ \quad ZC \quad AB \quad MA,$$

therefore compounding

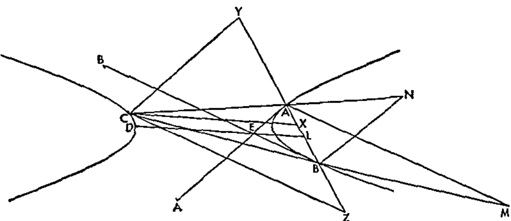
$$\text{rect } AY, BZ \quad \text{rect } YC, ZC \quad \text{sq } AB \quad \text{rect } MA \ BN$$

Since rectangle $MA \ BN$ is constant as $C$ changes and also the square on $AB$ is constant therefore

$$\text{rect } AY \ BZ \quad \text{rect } YC, ZC \text{ is a constant ratio} \qquad (1)$$

Again by similar triangles

$$ZC \quad CX \quad EB \quad EL,$$




$$YC \quad CX \quad EA \quad EL$$

therefore compounding

$$\text{rect } YC \ ZC \quad \text{sq } CX \quad \text{rect } EB, EA \quad \text{sq } EL$$

Hence

$$\text{rect } YC \ ZC \quad \text{sq } CX \text{ is a constant ratio} \qquad (2)$$

Compounding (1) and (2) we have a constant ratio

$$\text{rect } AY \ BZ \quad \text{sq } CX$$

But $AY$ and $BZ$ are equal to $CA'$ and $CB$ the distances from $C$ This is the property of the three line locus of section C with respect to the straight lines $EA$ and $EB$ tangents to the other section and $EB$ the straight line joining their points of contact And so one opposite section is a three line-locus to the tangents to the other of the opposite sections That it is also a four line locus could be hown in the same way as before

Again from III 55 we can conclude that both of the opposite sections together are a three line locus to the triangle formed by a tangent to each of the sections and the straight line joining their points of contact For by III 55

$$\text{rect } HA \ DN \quad \text{sq } AD \quad \text{rect } AG \ GD \quad \text{sq } CG$$

Now since the three last terms of this proportion are evidently constants as the point $F$ changes therefore also although $HA$ and $DN$ change with $F$, yet rectangle $HA, DN$ remains constant in magnitude Then reproducing the figure of

INTRODUCTION TO ARITHMETIC

But it was shown in the course of III 55 that

$$BK = LF,$$
$$BL = KF$$

Hence

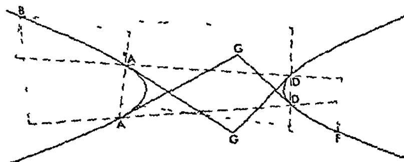
$$TA = BK = YD = LF,$$
$$AZ = KF = BL$$

Therefore

rect $LF, KF$ sq $FY$ rect $BK, BL$ sq $BR$

Hence any point $B$ on one opposite section fulfills the same constant ratio with respect to its distances from the three fixed lines $AD$, $GD$, and $AG$ as any point $F$ on the other opposite section

It can be similarly deduced that the opposite sections are together a four line locus with respect to any four fixed straight lines joining points, two on each section (the points being four points of contact of four tangents and the straight lines the straight lines joining them)




To sum up a parabola ellipse circle and hyperbola are three-line loci with respect to any two tangents to them and a straight line joining the points of contact One opposite section is also a three line locus with respect to any two tangents to the other section together with the straight line joining their points of contact The two opposite sections together are a three line-locus with respect to two tangents each to one of the sections together with the straight line joining their points of contact

The parabola ellipse circle and hyperbola are four line loci with respect to any inscribed quadrilateral One opposite section is also a four line locus with respect to any quadrilateral inscribed in the other section The two opposite sections together are a four line locus to any four straight lines joining four points, two lying on each opposite section

INTRODUCTION TO ARITHMETIC

# BIOGRAPHICAL NOTE

## NICOMACHUS, *fl* c A D 100

NICOMACHUS OF GERASA flourished around the end of the first century of our era In one of his surviving books the *Introduction to Harmonics*, he mentions a certain Thrasyllus presumably Thrasyllus of Mendes, a writer on music who lived in the reign of Tiberius Another book by Nicomachus, the *Introduction to Arithmetic* was translated into Latin by Apuleius under the Antonines This places the life of Nicomachus somewhere between the middle of the first century and the middle of the second century Perhaps the fact that Ptolemy, whose recorded astronomical observations were made between A D 127 and 151, is not mentioned in the *Introduction to Harmonics* makes it probable that he was not yet famous at the time Nicomachus was writing

The manuscripts of Nicomachus books and the scholia call him "of Gerasa " The best known city of that name was in Palestine and was primarily Greek However it can hardly be supposed that Nicomachus received all of his philosophical and mathematical education at Gerasa He probably studied at Alexandria at this time the center of mathematical studies and of Neo-Pythagoreanism Jamblichus says of Nicomachus ' The man is great in mathematics, and had as instructors those that were most skilled in the subject

Nothing is known of the personal life of Nicomachus except what is said or implied in the dedication of the *Introduction to Harmonics* to an unknown lady 'But I must spur on all my zeal most noble and august lady since it is you that bid me And if the gods are willing just as soon as I shall have leisure and a rest from my journeyings I will compile for you a better and more detailed *Introduction* dealing with this very subject and so that you may the more easily follow the argument I will take my beginning say from the same point as that at which I began your instruction when I was expounding the subject to you

Nicomachus appears to have been an important member of the Neo-Pythagorean group though his extant writings would seem to indicate that he was a popularizer and a compiler of manuals and not the head of a school Besides the *Introduction to Arithmetic* and the *Introduction to Harmonics* he also wrote a book on the mystical doctrine of number called *Theologoumena Arithmeticae*, which is one of the best sources on Neo-Pythagoreanism extracts and paraphrases of this work survive in a later anonymous work of the same name and in the *Bibliotheca* a collection of extracts from ancient works made in the ninth century by Photius patriarch of Constantinople Nicomachus also wrote an *Introduction to Geometry* and a *Life of Pythagoras* which have not survived and a larger work on music possibly that promised in the dedication to the *Introduction to Harmonics* of which we have only fragments He may have written a book on the interpretation of Plato though the evidence for it is slight and also an *Introduction to Astronomy* thereby completing the quadrivial series

The success of the *Introduction to Arithmetic* must have been immediate It

was used as a text book throughout later antiquity and in the Latin paraphrase of Boethius throughout the Middle Ages It had a host of commentators In the *Philopatris* attributed to Lucian, a character says "You reckon like Nicomachus This remark lends itself to more than one interpretation, but in any case it is evidence of his fame Nicomachus also appears to have been considered one of the "golden chain" or succession of true philosophers, for Proclus the fifth century Neo-Platonist who belonged to that "chain," claimed, on the basis of a dream, that he had within him the soul of Nicomachus

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# BOOK ONE

## CHAPTER I

[1] The ancients who under the leadership of Pythagoras first made science systematic defined philosophy as the love of wisdom In deed the name itself means this and before Pythagoras all who had knowledge were called wise indiscriminately—a carpenter for ex ample a cobbler a helmsman and in a word anyone who was versed in any art or handi craft Pythagoras however restricting the title so as to apply to the knowledge and comprehension of reality and calling the knowledge of the truth in this the only wisdom naturally designated the desire and pursuit of this knowl ed e philosophy as being desire for wisdom

[2] He is more worthy of credence than those who have given other definitions since he makes clear the sense of the term and the thing defined This wisdom he defined as the knowl edge or science of the truth in real things con ceiving science to be a steadfast and firm apprehension of the underlying substance and real things to be those which continue uni formly and the same in the universe and never depart even briefly from their existence these real things would be things immaterial by shar ing in the substance of which everything else that exists under the same name and is so called is said to be this particular thing and exists

[3] For bodily material things are to be sure forever involved in continuous flow and change—in imitation of the nature and pecul iar quality of that eternal matter and substance which has been from the beginning and which was all changeable and variable throughout The bodiless things however of which we con ceive in connection with or together with matter such as qualities quantities configurations largeness smallness equality relations actu alities dispositions places times all those things in a word whereby the qualities found in each body are comprehended—all these are of themselves immovable and unchangeable but accidentally they share in and partake of the affections of the body to which they belong

[4] Now it is with such things that 'wisdom' is particularly concerned but accidentally also with things that share in them that is bodies

## CHAPTER II

[1] Those things however are immaterial eternal without end and it is their nature to persist ever the same and unchanging abiding by their own essential being and each one of them is called real in the proper sense But what are involved in birth and destruction growth and diminution all kinds of change and participation are seen to vary continually, and while they are called real things by the same term as the former so far as they partake of them they are not actually real by their own nature for they do not abide for even the shortest moment in the same condition but are al ways passing over in all sorts of changes [2] To quote the words of Timaeus in Plato [1] What is that which always is and has no birth and what is that which is always becom ing but never is? The one is apprehended by the mental processes with reasoning and is ever the same the other can be guessed at by opinion in company with unreasoning sense a thing which becomes and passes away but never really is

[3] Therefore if we crave for the goal that is worthy and fitting for man namely happi ness of life[2]—and this is accomplished by phil osophy alone and by nothing else and philosophy as I said means for us desire for wisdom and wisdom the science of the truth in things and of things some are properly so called others merely share the name—it is reasonable and most necessary to distinguish and systematize the accidental qualities of things

[4] Things then both those properly so called and those that simply have the name are some of them unified and continuous for example an animal the universe a tree and

[1] *Timaeus* 27

[2] The word used by Nicomachus εὐζωΐα is once employed by Aristotle in the *Ethics* $1098^b$ 20 ff

the like which are properly and peculiarly called magnitudes others are discontinuous in a side-by side arrangement and, as it were in heaps which are called multitudes a flock for instance a people a heap a chorus and the like

[5] Wisdom then must be considered to be the knowledge of these two forms Since however all multitude and magnitude are by their own nature of necessity infinite—for multitude starts from a definite root and never ceases increasing and magnitude when division beginning with a limited whole is carried on cannot bring the dividing process to an end but proceeds therefore to infinity—and since sciences are always sciences of limited things and never of infinites it is accordingly evident that a science dealing either with magnitude per se or with multitude per se could never be formulated for each of them is limitless in itself multitude in the direction of the more and magnitude in the direction of the less A science however would arise to deal with something separated from each of them with quantity set off from multitude and size set off from magnitude

## CHAPTER III

[1] Again to start afresh since of quantity one kind is viewed by itself having no relation to anything else as even odd perfect and the like and the other is relative to some thing else and is conceived of together with its relationship to another thing like double greater smaller half one and one-half times one and one-third times and so forth it is clear that two scientific methods will lay hold of and deal with the whole investigation of quantity arithmetic absolute quantity and music relative quantity [1]

[2] And once more inasmuch as part of size is in a state of rest and stability and another part in motion and revolution two other sciences in the same way will accurately treat of size geometry the part that abides and is at rest astronomy that which moves and revolves

[3] Without the aid of these then it is not possible to deal accurately with the forms of being nor to discover the truth in things knowledge of which is wisdom and evidently not even to philosophize properly for just as painting contributes to the menial arts toward correctness of theory so in truth lines numbers harmonic intervals and the revolutions of circles bear aid to the learning of the doctrines of wisdom' says the Pythagorean Androcydes [4] Likewise Archytas of Tarentum at the beginning of his treatise *On Harmony* says the same thing in about these words It seems to me that they do well to study mathematics and it is not at all strange that they have correct knowledge about each thing what it is For if they knew rightly the nature of the whole, they were also likely to see well what is the nature of the parts About geometry indeed, and arithmetic and astronomy they have handed down to us a clear understanding and not least also about music For these seem to be sister sciences for they deal with sister subjects the first two forms of being '

[5] Plato too at the end of the thirteenth book of the *Laws* to which some give the title *The Philosopher* because he investigates and defines in it what sort of man the real philosopher should be in the course of his summary of what had previously been fully set forth and established adds Every diagram system of numbers every scheme of harmony and every law of the movement of the stars ought to appear one to him who studies rightly and what we say will properly appear if one studies all things looking to one principle for there will be seen to be one bond for all these things and if any one attempts philosophy in any other way he must call on Fortune to assist him For there is never a path without these this is the way these the studies be they hard or easy by this course must one go and not neglect it The one who has attained all these things in the way I describe him I for my part call wisest and this I maintain through thick and thin [6] For it is clear that these studies are like ladders and bridges that carry our minds from things apprehended by sense and opinion to those comprehended by the mind and understanding and from those material physical things our foster brethren known to us from childhood to the things with which we are unacquainted foreign to our senses but in their immateriality and eternity more akin to our souls and above all to the reason which is in our souls

[7] And likewise in Plato s *Republic* when the interlocutor of Socrates appears to bring certain plausible reasons to bear upon the mathematical sciences to show that they are useful to human life arithmetic for reckoning distributions contributions exchanges and partnerships geometry for sieges the founding of cities and sanctuaries and the partition

[1] Cf Aristotle *Met* IV 15 1020b 26

of land music for festivals entertainment and the worship of the gods and the doctrine of the spheres or astronomy, for farming navigation and other undertakings, revealing beforehand the proper procedure and suitable season Socrates, reproaching him, says You amuse me because you seem to fear that these are useless studies that I recommend but that is very difficult nay impossible For the eye of the soul blinded and buried by other pursuits is rekindled and aroused again by these and these alone and it is better that this be saved than thousands of bodily eyes for by it alone is the truth of the universe beheld [1]

## CHAPTER IV

*[1]* Which then of these four methods must we first learn? Evidently the one which naturally exists before them all is superior and takes the place of origin and root and as it were of mother to the others *[2]* And thus is arithmetic [2] not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends but also because it is naturally prior in birth inasmuch as it abolishes other sciences with itself [3] but is not abolished together with them For example animal is naturally antecedent to man for abolish animal and man is abolished but if man be abolished it no longer follows that animal is abolished at the same time And again man is antecedent to schoolteacher for if man does not exist neither does ' schoolteacher but if schoolteacher is nonexistent it is still possible for man to be Thus since it has the property of abolishing the other ideas with it self it is likewise the older

*[3]* Conversely that is called younger and posterior which implies the other thing with itself [4] but is not implied by it like musician for this always implies man Again take horse animal is always implied along with horse but not the reverse for if animal exists it is not necessary that horse should exist nor if man exists must musician also be implied

[1] *Republic* 527 ff
[2] Plato *Rep* 522
[3] Cf below II 22 3 Cf Aristotle, *Met* 1019ª 1 ff
[4] Cf Aristotle e g *Top* VI 6 144ᵇ 17 also *Top* II 4 111 25 ff

*[4]* So it is with the foregoing sciences if geometry exists arithmetic must also needs be implied for it is with the help of this latter that we can speak of triangle quadrilateral octahedron, icosahedron double eightfold or one and one-half times or anything else of the sort which is used as a term by geometry, and such things cannot be conceived of without the numbers that are implied with each one For how can ' triple exist, or be spoken of unless the number 3 exists beforehand or eightfold without 8? But on the contrary 3 4 and the rest might be without the figures existing to which they give names *[5]* Hence arithmetic abolishes geometry along with itself but is not abolished by it and while it is implied by geometry it does not itself imply geometry

## CHAPTER V

*[1]* And once more is this true in the case of music not only because the absolute is prior to the relative as great to greater and rich to richer ' and man to ' father ' but also because the musical harmonies diatessaron diapente and diapason are named for numbers similarly all of their harmonic ratios are arithmetical ones for the diatessaron is the ratio of 4 3 the diapente that of 3 2 and the diapason the double ratio and the most perfect the didiapason is the quadruple ratio

*[2]* More evidently still astronomy attains through arithmetic the investigations that pertain to it not alone because it is later than geometry[5] in origin—for motion naturally comes after rest—nor because the motions of the stars have a perfectly melodious harmony but also because risings settings progressions retrogressions increases and all sorts of phases are governed by numerical cycles and quantities

*[3]* So then we have rightly undertaken first the systematic treatment of this as the science naturally prior more honorable and more venerable and as it were mother and nurse of the rest and here we will take our start for the sake of clearness

## CHAPTER VI

*[1]* All that has by nature with systematic method been arranged in the universe seems both in part and as a whole to have been determined and ordered in accordance with number by the forethought and the mind of him that

[5] Cf Plato *Rep* 528

created all things for the pattern was fixed like a preliminary sketch by the domination of number preexistent in the mind of the world creating God number conceptual only and immaterial in every way, but at the same time the true and the eternal essence so that with reference to it as to an artistic plan should be created all these things time motion the heavens the stars all sorts of revolutions

[2] It must needs be then that scientific number being set over such things as these should be harmoniously constituted in accordance with itself not by any other but by itself [3] Everything that is harmoniously constituted is knit together out of opposites and of course out of real things for neither can non existent things be set in harmony nor can things that exist but are like one another nor yet things that are different but have no relation one to another It remains accordingly that those things out of which a harmony is made are both real different and things with some relation to one another

[4] Of such things therefore scientific number consists for the most fundamental species in it are two embracing the essence of quantity different from one another and not of a wholly different genus odd and even and they are reciprocally woven into harmony with each other inseparably and uniformly, by a wonderful and divine Nature as straightway we shall see

## CHAPTER VII

[1] Number is limited multitude or a combination of units or a flow of quantity made up of units and the first division of number is even and odd

[2] The even[1] is that which can be divided into two equal parts without a unit intervening in the middle and the odd is that which cannot be divided into two equal parts because of the aforesaid intervention of a unit

[3] Now this is the definition after the ordinary conception by the Pythagorean doctrine however the even number is that which admits of division into the greatest and the smallest parts at the same operation greatest in size and smallest in quantity in accordance with the natural contrariety of these two genera and the odd is that which does not allow this to be done to it but is divided into two unequal parts

[4] In still another way by the ancient definition the even is that which can be divided alike into two equal and two unequal parts except that the dyad which is its elementary form admits but one division that into equal parts and in any division whatsoever it brings to light only one species of number, however it may be divided, independent of the other The odd is a number which in any division whatsoever, which necessarily is a division into unequal parts shows both the two species of number together never without intermixture one with another but always in one another s company

[5] By the definition in terms of each other, the odd is that which differs by a unit from the even in either direction that is toward the greater or the less and the even is that which differs by a unit in either direction from the odd that is, is greater by a unit or less by a unit

## CHAPTER VIII

[1] Every number is at once half the sum of the two on either side of itself and similarly half the sum of those next but one in either direction and of those next beyond them and so on as far as it is possible to go [2] Unity alone because it does not have two numbers on either side of it is half merely of the adjoining number hence unity is the natural starting point of all number

[3] By subdivision of the even there are the even times even the odd times even and the even times odd The even times even and the even times odd are opposite to one another like extremes and the odd times even is common to them both like a mean term

[4] Now the even times even[2] is a number which is itself capable of being divided into two equal parts in accordance with the properties of its genus and with each of its parts similarly capable of division and again in the same way each of their parts divisible into two equals until the division of the successive subdivisions reaches the naturally indivisible unit [5] Take for example 64 one half of this is 32 and of this 16 and of this the half is 8 and of this 4 and of this 2 and then finally unity is half of the latter and this is naturally indivisible and will not admit of a half

[6] It is a property of the even times even that whatever part of it be taken it is always

[1] Cf Euclid, VII Def 6

[2] Euclid s definition is The even times even number is that which is measured by an even number an even number of times *Elements* VII Def 8

even times even in designation and at the same time by the quantity of the units in it even times even in value and that neither of these two things will ever share in the other class [7] Doubtless it is because of this that it is called even times even because it is itself even and always has its parts and the parts of its parts down to unity even both in name and in value in other words every part that it has is even times even in name and even times even in value

[8] There is a method of producing the even times even so that none will escape but all successively fall under it if you do as follows [9] As you proceed from unity as from a root by the double ratio to infinity as many terms as there are will all be even times even and it is impossible to find others besides these for instance 1 2 4 8 16 32 64 128 256 512

[10] Now each of the numbers set forth was produced by the double ratio, beginning with unity and is in every respect even times even and every part that it may be found to have is always named from some one of the numbers before it in the series and the sum of units in this part is the same as one of the numbers before it, by a system of mutual correspondence indeed and interchange If there is an even number of terms of the double ratio from unity, not one mean term can be found but always two from which the correspondence and interchange of factors and values values and factors will proceed in order going first to the two on either side of the means then to the next on either side until it comes to the extreme terms so that the whole will correspond in value to unity and unity to the whole For example if we set down 128 as the largest term the number of terms will be even for there are eight in all up to this number and they will not have one mean term for this is impossible with an even number but of necessity two 8 and 16 These will correspond to each other as factors for of the whole 128 16 is one eighth and conversely 8 is one sixteenth Thence proceeding in either direction we find that 32 is one fourth and 4 one thirty second and again 64 is one half and 2 one sixty fourth and finally at the extremes unity is one one hundred twenty eighth and conversely 128 is the whole to correspond with unity

[11] If however the series consists of an odd number of terms seven for example and we deal with 64 there will be of necessity one mean term in accordance with the nature of the odd the mean term will correspond to itself because it has no partner and those on either side of it in turn will correspond to one another until this correspondence ends in the extremes Unity for example will be one sixty fourth and 64 the whole corresponding to unity 32 is one half and 2 one thirty-second 16 is one fourth and 4 one sixteenth and 8 the eighth part with nothing else to correspond to it

[12] It is the property of all these terms when they are added together successively to be equal to the next in the series lacking one unit so that of necessity their summation in any way whatsoever will be an odd number, for that which fails by a unit of being equal to an even number is odd [13] This observation will be of use to us very shortly in the construction of perfect numbers [1] But to take an example the terms from unity preceding 256 in the series when added together, are within 1 of equaling 256 and all the terms before 128 the term immediately preceding are similarly equal to 128 save for one unit and to the next terms the sums of those below them are similarly related Thus unity itself[2] is within one unit of equaling the next term which is 2 and these two together fail by 1 of equaling the next and the three together are within 1 of the next in order and you will find that this goes on without interruption to infinity

[14] This too it is very needful to recall If the number of terms of the even times even series dealt with is even the product of the extremes will always be equal to the product of the means if there is an odd number of terms the product of the extremes will be equal to the square of the mean For in the case of an even number of terms 1 times 128 is equal to 8 times 16 and further to 2 times 64 and again to 4 times 32 and this is so in every case and with an odd number of terms 1 times 64 equals 2 times 32 and this equals 4 times 16 and this again equals 8 times 8 the mean term alone multiplied by itself

## CHAPTER IX

[1] The even times odd number is one which is by its genus itself even but is specifically opposed to the aforesaid even times even It is a number of which though it admits of the division into two equal halves after the fashio the genus common to it and the eve even the halves are not

[1] See Chapter 16
[2] Cf on I 8 7

into two equals for example 6 10 14 18 22 26 and the like for after these have been divided their halves are found to be indivisible

*[2]* It is the property of the even times odd that whatever factor it may be discovered to have is opposite in name to its value and that the quantity of every part is opposite in value to its name and that the numerical value of its part never by any means i of the same genus as its name To take a single example the number 18 its half with an even name is 9 odd in value its third part again with an odd designation is 6 even in value conversely, the sixth part is 3 and the ninth part 2 and in other numbers the same peculiarity will be found

*[3]* It is possibly for this reason that it received such a name that is because although it is even its halves are at once odd

*[4]* This number is produced from the series beginning with unity with a difference of 2, namely the odd numbers set forth in proper order as far as you like and then multiplied by 2 The numbers produced would be in order the e 6 10 14 18 22 26 30 and so on as far as you care to proceed The greater terms always differ by 4 from the next smaller ones the reason for which is that their original basic forms the odd numbers exceed one another by 2 and were multiplied by 2 to make this series and 2 times 2 makes 4

*[5]* Accordingly in the natural series of numbers the even times odd numbers will be found fifth from one another exceeding one another by a difference of 4 passing over three terms and produced by the multiplication of the odd numbers by 2

*[6]* They are said to be opposite in properties to the even times even because of these the greatest extreme term alone is divisible while of these former the smallest only proved to be indivisible and in particular because in the former case the reciprocal arrangement of parts[1] from extremes to mean term or terms makes the product of the former equal to the square or product of the latter but in this case by the same correspondence and comparison the mean term is one half the sum of the extremes or if there should be two means their sum equals that of the two extremes

## CHAPTER X

*[1]* The odd times even number is the one which displays the third form of the even belonging in common to both the previously mentioned species like a single mean between two extremes for in one respect it resembles the even times even and in another the even times odd and that property wherein it varies from the one it shares with the other, and by that property which it shares with the one it differs from the other

*[2]* The odd times even number is an even number which can be divided into two equal parts whose parts also can so be divided and sometimes even the parts of its parts but it cannot carry the division of its parts as far as unity Such numbers are 24 28 40 for each of these has its own half and indeed the half of its half and sometimes one is found among them that will allow the halving to be carried even farther among its parts There is none however that will have its parts divisible into halves as far as the naturally indivisible unit

*[3]* Now in admitting more than one division the odd times even is like the even times even and unlike the even times odd but in that its subdivision never ends with unity it is like the even times odd and unlike the even times even

*[4]* It alone has at once the proper qualities of each of the former two[1] and then again properties which belong to neither of them for of them one had only the highest term divisible, and the other only the smallest indivisible but this neither for it is observed to have more divisions than one in the greater term, and more than one indivisible in the lesser

*[5]* Furthermore there are in it certain parts whose names are not opposed to their values nor of the opposite genus[2] after the fashion of the even times even and there are also always other parts of a name opposite and contrary in kind to their values after the fashion of the even times odd For example in 24 there are parts not opposed in name to their values the fourth part 6 the half 12 the sixth 4 and the twelfth 2 but the third part 8 the eighth 3 and the twenty fourth 1 are opposed and so it is with the rest

*[6]* This number is produced by a somewhat complicated method and shows after a fashion even in its manner of production that it is a mixture of both other kinds For whereas the even times even is made from even numbers the doubles from unity to infinity and the even times odd from the odd numbers from 3 progressing to infinity this must be woven

[1] Cf I 8 10

[1] Cf I 9 6

[2] Cf I 8 7, 9 2

together out of both classes as being common to both [ ] Let us then set forth the odd numbers from 3 by themselves in due order in one series

3 5 7 9 11 13 15 17 19

and the even times even, beginning with 4 again one after another in a second series after their own order

4 8 16 32 64 128 256

as far as you please [8] Now multiply by the first number of either series—it makes no difference which—from the beginning and in order all those in the remaining series and note down the resulting numbers then again multiply by the second number of the same series the same numbers once more as far as you can and write down the results then with the third number again multiply the same terms anew and however far you go you will get nothing but the odd times even numbers

[9] For the sake of illustration let us use the first term of the series of odd numbers and multiply by it all the terms in the second series in order thus 3 × 4 3 × 8 3 × 16 3 × 32 and so on to infinity The results will be 12 24 48 96 which we must note down in one line Then taking a new start do the same thing with the second number 5×4 5×8, 5×16 5×32 The results will be 20 40 80 160 Then do the same thing once more with 7 the third number 7×4 7×8 7×16 7×32 The results are 28 56 112 224 and in the same way as far as you care to go you will get similar results

[10] Now when you arrange the products of multiplication by each term in its proper line making the lines parallel in marvelous fashion there will appear along the breadth of the table *the peculiar property of the even times odd* that the mean term is always half the sum of the extremes if there should be one mean and the sum of the means equals the sum of the extremes if two But along the length of the table the property of the even times even will appear for the product of the extremes is equal to the square of the mean should there be one mean term or their product should there be two Thus this one species has the peculiar properties of them both because it is a natural mixture of them both

| | | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|---|---|
| Odd numbers | | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
| Even times even | | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
| Odd times even numbers | *Breadth* | 12 | 24 | 48 | 96 | 192 | 384 | 768 |
| | | 20 | 40 | 80 | 160 | 320 | 640 | 1280 |
| | | 28 | 56 | 112 | 224 | 448 | 896 | 1792 |
| | | 36 | 72 | 144 | 288 | 576 | 1152 | 2304 |
| | | 44 | 88 | 176 | 352 | 704 | 1408 | 2816 |

*Length*

## CHAPTER XI

[1] Again while the odd is distinguished over against the even in classification and has nothing in common with it since the latter is divisible into equal halves and the former is not thus divisible nevertheless there are found three species of the odd [1] differing from one another of which the first is called the prime and incomposite [2] that which is opposed to it the secondary and composite and that which is midway between both of these and is viewed as a mean among extremes namely the variety which in itself is secondary and composite, but relatively is prime and incomposite

[2] Now the first species *the prime and incomposite* is found whenever an odd number admits of no other factor save the one with the number itself as denominator which is always unity for example 3, 5 7 11, 13 17 19, 23 29 31 None of these numbers will by any chance be found to have a fractional part with a denominator different from the number itself but only the one with this as denominator and this part will be unity in each case for 3 has only a third part, which has the same denominator as the number and is of course unity 5 a fifth 7 a seventh and 11 only an eleventh part and in all of them these parts are unity

[3] It has received this name because it can be measured only by the number which is first and common to all unity and by no other *moreover because it is produced by no other* number combined with itself save unity alone for 5 is 5×1 and 7 is 7×1 and the others in accordance with their own quantity To be sure when they are combined with themselves other

[1] Cf Euclid *Elem* VII D ff 11 14
[2] Cf Euclid *Elem* VII Def 11 Aristotle VIII 2 157 39

numbers might be produced originating from them as from a fountain and a root wherefore they are called 'prime because they exist beforehand as the beginnings of the others For every origin is elementary and incomposite into which everything is resolved and out of which everything is made but the origin itself cannot be resolved into anything or constituted out of anything

## CHAPTER XII

*[1]* The secondary composite number[1] is an odd number indeed because it is distinguished as a member of this same class but it has no elementary quality, for it gets its origin by the combination of something else For this reason it is characteristic of the secondary number to have in addition to the fractional part with the number itself as denominator yet another part or parts with different denominators the former always as in all cases unity the latter never unity but always either that number or those numbers by the combination of which it was produced For example 9 15 21 25 27 33 35 39 each one of these is measured by unity as other numbers are and like them has a fractional part with the same denominator as the number itself by the nature of the class common to them all but by exception and more peculiarly they also employ a part or parts with a different denominator 9 in addition to the ninth part has a third part besides 15 a third and a fifth besides a fifteenth 21 a seventh and a third besides a twenty first and 25 in addition to the twenty fifth which has as a denominator 25 itself also a fifth with a different denominator

*[2]* It is called secondary then because it can employ yet another measure along with unity and because it is not elementary but is produced by some other number combined with itself or with something else in the case of 9 3 in the case of 15 5 or by Zeus 3 and those following in the same fashion And it is called composite for this or some such reason that it may be resolved into those numbers out of which it was made since it can also be measured by them For nothing that can be broken down is incomposite but by all means composite

## CHAPTER XIII

*[1]* Now while these two species of the odd are opposed to each other a third one[2] is conceived of between them deriving as it were its specific form from them both namely the number which is in itself secondary and composite but relatively to another number is prime and incomposite This exists when a number, in addition to the common measure unity is measured by some other number and is therefore able to admit of a fractional part or parts with denominator other than the number itself as well as the one with itself as denominator When this is compared with another number of similar properties it is found that it cannot be measured by a measure common to the other nor does it have a fractional part with the same denominator as those in the other As an illustration let 9 be compared with 25 Each in itself is secondary and composite but relatively to each other they have only unity as a common measure and no factors in them have the same denominator for the third part in the former does not exist in the latter nor is the fifth part in the latter found in the former

*[2]* The production of these numbers is called by Eratosthenes the sieve because we take the odd numbers mingled together and indiscriminate and out of them by this method of production separate as by a kind of instrument or sieve the prime and incomposite by themselves and the secondary and composite by themselves and find the mixed class by themselves

*[3]* The method of the sieve is as follows I set forth all the odd numbers in order beginning with 3 in as long a series as possible and then starting with the first I observe what ones it can measure and I find that it can measure the terms two places apart as far as we care to proceed And I find that it measures not as it chances and at random but that it will measure the first one that is the one two places removed by the quantity of the one that stands first in the series that is by its own quantity for it measures it 3 times and the one two places from this by the quantity of the second in order for this it will measure 5 times and again the one two places further on by the quantity of the third in order or 7 times and the one two places still farther on by the quantity of the fourth in order or 9 times and so *ad infinitum* in the same way

*[4]* Then taking a fresh start I come to the second number and observe what it can measure and find that it measures all the terms four places apart the first by the quantity of the first in order or 3 times the second by that of the second or 5 times the third by that of

[1] Cf Euclid *Elements* VII Def 14
[2] Cf Euclid *Elements* VII Def 13

the third or 7 times and in this order *ad infinitum*

[5] Again as before the third term 7 taking over the measuring function will measure terms six places apart and the first by the quantity of 3 the first of the series the second by that of 5 for this is the second number and the third by that of 7, for this has the third position in the series

[6] And analogously throughout this process will go on without interruption so that the numbers will succeed to the measuring function in accordance with their fixed position in the series the interval separating terms measured is determined by the orderly progress of the even numbers from 2 to infinity or by the doubling of the position in the series occupied by the measuring term and the number of times a term is measured is fixed by the orderly advance of the odd numbers in series from 3

[7] Now if you mark the numbers with certain signs you will find that the terms which succeed one another in the measuring function neither measure all the same number—and sometimes not even two will measure the same one—nor do absolutely all of the numbers set forth submit themselves to a measure but some entirely avoid being measured by any number whatsoever some are measured by one only and some by two or even more [8] Now these that are not measured at all but avoid it are primes and incomposites sifted out as it were by a sieve those measured by only one measure in accordance with its own quantity will have but one fractional part with denominator different from the number itself in addition to the part with the same denominator and those which are measured by one measure only but in accordance with the quantity of some other number than the measure and not its own, or are measured by two measures at the same time will have several fractional parts with other denominators besides the one with the same as the number itself these will be secondary and composite

[9] The third division the one common to both the former which is in itself secondary and composite but primary and incomposite in relation to another will consist of the numbers produced when some prime and incomposite number measures them in accordance with its own quantity if one thus produced be compared to another of similar origin For example if 9 which was produced by 3 measuring by its own quantity for it is 3 times 3 be compared with 25 which was produced from 5 measuring by its own quantity for it is 5 times 5 these numbers have no common measure except unity

[10] We shall now investigate how we may have a method[1] of discerning whether numbers are prime and incomposite or secondary and composite, relatively to each other since of the former unity is the common measure but of the latter some other number also besides unity and what this number is

[11] Suppose there be given us two odd numbers and some one sets the problem and directs us to determine whether they are prime and incomposite relatively to each other or secondary and composite and if they are secondary and composite what number is their common measure We must compare the given numbers and subtract the smaller from the larger as many times as possible then after this subtraction subtract in turn from the other as many times as possible for this changing about and subtraction from one and the other in turn will necessarily end either in unity or in some one and the same number which will necessarily be odd [12] Now when the subtractions terminate in unity they show that the numbers are prime and incomposite relatively to each other and when they end in some other number odd in quantity and twice produced then say that they are secondary and composite relatively to each other and that their common measure is that very number which twice appears

For example if the given numbers were 23 and 45 subtract 23 from 45 and 22 will be the remainder subtracting this from 23 the remainder is 1 subtracting this from 22 as many times as possible you will end with unity Hence they are prime and incomposite to one another and unity which is the remainder is their common measure

[13] But if one should propose other numbers 21 and 49 I subtract the smaller from the larger and 28 is the remainder Then again I subtract the same 21 from this for it can be done and the remainder is 7 This I subtract in turn from 21 and 14 remains from which I subtract 7 again for it is possible and 7 will remain But it is not possible to subtract 7 from 7 hence the termination of the process with a repeated 7 has been brought about and you may declare the original numbers 21 and 49 secondary and composite relatively to each

[1] This mode of determining common factors is found in Euclid (VII 1 X 2) and is commonly termed the Euclidean method of finding the greatest common divisor of numbers

other and 7 their common measure in addition to the universal unit

## CHAPTER XIV

*[1]* To make again a fresh start of the simple even numbers some are superabundant some deficient like extremes set over against each other, and some are intermediary between them and are called perfect *[2]* Those which are said to be opposites to one another the superabundant and deficient are distinguished from one another in the relation of inequality[1] in the directions of the greater and the less for apart from these no other form of inequality could be conceived nor could evil[2] disease disproportion unseemliness nor any such thing save in terms of excess or deficiency For in the realm of the greater[3] there arise excesses overreaching and superabundance and in the less need deficiency privation and lack but in that which lies between the greater and the less, namely the equal are virtues wealth moderation propriety beauty and the like to which the aforesaid form of number the perfect is most akin

*[3]* Now the superabundant number is one which has over and above the factors which belong to it and fall to its share others in addition just as if an animal should be created with too many parts or limbs with ten tongues as the poet says [4] and ten mouths or with nine lips or three rows of teeth or a hundred hands or too many fingers on one hand Similarly if when all the factors in a number are examined and added together in one sum it proves upon investigation that the number s own factors exceed the number itself this is called a superabundant number for it oversteps the symmetry which exists between the perfect and its own parts Such are 12 24 and certain others for 12 has a half 6 a third 4 a fourth 3 a sixth 2 and a twelfth 1 which added together make 16 which is more than the original 12 its parts therefore are greater than the whole itself *[4]* And 24 has a half a third fourth sixth eighth twelfth and twenty fourth, which are 12 8 6 4 3 2 1 Added together they make 36 which compared to the original number 24 is found to be greater than it, although made up solely of its factors Hence in this case also the parts are in excess of the whole

[1]Cf I 17 2 4 6 also I 23 4
[2]Cf Arist *Ethics* II 6 1106b 33
[3]Cf Arist *Ethics* II 6 1106b 24 33
[4]*Homer Odyssey* XII 85 ff

## CHAPTER XV

*[1]* The deficient number is one which has qualities the opposite of those pointed out and whose factors added together are less in comparison than the number itself It is as if some animal should fall short of the natural number of limbs or parts or as if a man should have but one eye as in the poem "And one round orb was fixed in his brow , or as though one should be one-handed or have fewer than five fingers on one hand or lack a tongue or some such member Such a one would be called deficient and so to speak maimed after the peculiar fashion of the number whose factors are less than itself such as 8 or 14 For 8 has the factors half fourth and eighth which are 4, 2, and 1 and added together they make 7 and less than the original number The parts therefore fall short of making up the whole *[2]* Again 14 has a half a seventh a fourteenth, 7 2 and 1 respectively and all together they make 10, less than the original number So this number also is deficient in its parts with respect to making up the whole out of them

## CHAPTER XVI

*[1]* While these two varieties are opposed after the manner of extremes the so-called perfect number[5] appears as a mean, which is discovered to be in the realm of equality and neither makes its parts greater than itself, added together nor shows itself greater than its parts but is always equal to its own parts For the equal is always conceived of as in the mid ground between greater and less and is as it were moderation between excess and deficiency and that which is in tune between pitches too high and too low

*[2]* Now when a number comparing with itself the sum and combination of all the factors whose presence it will admit neither exceeds them in multitude nor is exceeded by them then such a number is properly said to be perfect as one which is equal to its own parts Such numbers are 6 and 28 for 6 has the factors half third and sixth 3 2 and 1, respectively and these added together make 6 and are equal to the original number and neither more nor less Twenty-eight has the factors half fourth seventh fourteenth and twenty-eighth which are 14 7 4 2 and 1 these added together make 28 and so neither are the

[5]Euclid s definition *Elem* VII 22 is "A perfect number is one that is equal to its own parts."

parts greater than the whole nor the whole greater than the parts but their comparison is in equality which is the peculiar quality of the perfect number

[3] It comes about that even as fair and excellent things are few and easily enumerated while ugly and evil ones are widespread so also the superabundant and deficient numbers are found in great multitude and irregularly placed —for the method of their discovery is irregular—but the perfect numbers are easily enumerated and arranged with suitable order for only one is found among the units 6 only one other among the tens 28 and a third in the rank of the hundreds 496 alone and a fourth within the limits of the thousands that is below ten thousand 8 128 And it is their accompanying characteristic to end alternately in 6 or 8 and always to be even

[4] There is a method of producing them[1] neat and unfailing which neither passes by any of the perfect numbers nor fails to differentiate any of those that are not such which is carried out in the following way

You must set forth the even times even numbers from unity advancing in order in one line as far as you please 1 2 4 8 16 32 64 128 256 512 1 024 2 048 4 096 Then you must add them together one at a time and each time you make a summation observe the result to see what it is If you find that it is a prime incomposite number multiply it by the quantity of the last number added and the result will always be a perfect number If however the result is secondary and composite do not multiply but add the next and observe again what the resulting number is if it is secondary and composite again pass it by and do not multiply but add the next but if it is prime and incomposite multiply it by the last term added and the result will be a perfect number and so on to infinity In similar fashion you will produce all the perfect numbers in succession overlooking none

For example to 1 I add 2 and observe the sum and find that it is 3 a prime and incomposite number in accordance with our previous demonstrations for it has no factor with denominator different from the number itself but only that with denominator agreeing Therefore I multiply it by the last number to be taken into the sum that is 2 I get 6 and this I declare to be the first perfect number in actuality and to have those parts which are beheld in the numbers of which it is composed For it will have unity as the factor with denominator the same as itself, that is its sixth part and 3 as the half which is seen in 2 and conversely 2 as its third part

[5] Twenty-eight likewise is produced by the same method when another number 4 is added to the previous ones For the sum of the three 1 2 and 4 is 7, and is found to be prime and incomposite for it admits only the factor with denominator like itself the seventh part Therefore I multiply it by the quantity of the term last taken into the summation and my result is 28, equal to its own parts and having its factors derived from the numbers already adduced a half corresponding to 2 a fourth to 7 a seventh to 4 a fourteenth to offset the half and a twenty-eighth in accordance with its own nomenclature which is 1 in all numbers

[6] When these have been discovered, 6 among the units and 28 in the tens you must do the same to fashion the next [7] Again add the next number 8 and the sum is 15 Observing this I find that we no longer have a prime and incomposite number but in addition to the factor with denominator like the number itself it has also a fifth and a third with unlike denominators Hence I do not multiply it by 8 but add the next number 16 and 31 results As this is a prime incomposite number of necessity it will be multiplied in accordance with the general rule of the process by the last number added 16 and the result is 496 in the hundred and then comes 8 128 in the thousands and so on as far as it is convenient for one to follow

[8] Now unity is potentially a perfect number but not actually for taking it from the series as the very first I observe what sort it is according to the rule and find it prime and incomposite for it is so in very truth not by participation like the rest but it is the primary number of all and alone incomposite [9] I multiply it therefore by the last term taken into the summation that is by itself and my result is 1 for 1 times 1 equals 1 [10] Thus unity is perfect potentially for it is potentially equal to its own parts the others actually

## CHAPTER XVII

[1] Now that we have given a preliminary systematic account of absolute quantity we come in turn to relative quantity

[2] Of relative quantity then the highest generic divisions are two equality and inequal

[1]Cf Euclid (IX 36)

ity for everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these

*[3]* Now the equal is seen when of the things compared one neither exceeds nor falls short in comparison with the other for example 100 compared with 100 10 with 10 2 with 2, a mina with a mina a talent with a talent, a cu bit with a cubit and the like, either in bulk length weight or any kind of quantity *[4]* And as a peculiar characteristic also this relation is of itself not to be divided or separated, as being most elementary, for it admits of no difference For there is no such thing as this kind of equality and that kind, but the equal ex ists in one and the same manner *[5]* And that which corresponds to an equal thing to be sure, does not have a different name from it but the same like friend neighbor comrade" so also equal for it is equal to an equal

*[6]* The unequal on the other hand is split up by subdivisions and one part of it is the greater the other the less which have opposite names and are antithetical to one another in their quantity and relation For the greater is greater than some other thing and the less again is less than another thing in comparison, and their names are not the same but they each have different ones for example father and son striker and struck, teacher and pupil and the like

*[7]* Moreover of the greater separated by a second subdivision into five species one kind is the multiple another the superparticular an other the superpartient another the multiple superparticular and another the multiple su perpartient *[8]* And of its opposite the less there arise similarly by subdivision five species opposed to the foregoing five varieties of the greater the submultiple subsuperparticular subsuperpartient submultiple superparticu lar and submultiple-superpartient for as whole answers to whole smaller to greater so also the varieties correspond each to each in the aforesaid order with the prefix sub-

## CHAPTER XVIII

*[1]* Once more then the multiple is the species of the greater first and most original by nature as straightway we shall see and it is a number which when it is observed in compari son with another contains the whole of that number more than once For example com pared with unity all the successive numbers beginning with 2 generate in their proper order the regular forms of the multiple for 2 in the first place is and is called the double 3 triple, 4 quadruple and so on for "more than once" means twice, or three times, and so on in succession as far as you like

*[2]* Answering to this is the submultiple which is itself primary in the smaller division of inequality It is the number which when it is compared with a larger is able to measure it completely more than once and more than once starts with twice and goes on to infinity *[3]* If then it measures the larger number that is being compared twice only, it is properly called the subdouble as 1 is of 2 if thrice subtriple, as 1 of 3 if four times subquadruple as 1 of 4 and so on in succession

*[4]* While each of these the multiple and the submultiple is generically infinite the va rieties by subdivision and the species also are observed naturally to make an infinite series For the double beginning with 2 goes on through all the even numbers as we select al ternate numbers out of the natural series and these will be called doubles in comparison with the even and odd numbers successively placed beginning with unity *[5]* All the numbers from the beginning two places apart and third in order are triples for example 3 6 9 12 15, 18 21 24 It is their property to be alternately odd and even and they themselves in the regu lar series from unity are triples of all the num bers in succession as far as one wishes to go on with the process

*[6]* The quadruples are those in the fourth places three apart for instance 4 8 12 16, 20, 24 28 32 and so on These are the quad ruples of the regular series of numbers from unity going on as far as one finds it convenient to follow It belongs to them all to be even, for one needs only to take the alternate terms out of the even numbers already selected Thus necessarily it is true that the even numbers with no further designation are all doubles the alternate ones quadruples those two places apart sextuples and those three places apart octuples and this series will go on on this same analogy indefinitely

*[7]* The quintuples will be seen to be those four places apart placed fifth from one an other and them elves the quintuples of the successive numbers beginning with unity Al ternately they are odd and even like the triples

## CHAPTER XIX

*[1]* The superparticular the second species of the greater both naturally and in order, is a

number that contains within itself the whole of the number compared with it and some one factor of it besides

[9] If this factor is a half the greater of the terms compared is called specifically sesquialter and the smaller subsesquialter if it is a third sesquitertian and sub esquitertian and as you go on throughout it will always thus agree so that these species also will progress to infinity even though they are species of an unlimited genus

For it comes about that the first species the sesquialter ratio has as its consequents the even numbers in succession from 2 and no other at all and as antecedents the triples in succession from 3 and no other [3] These must be joined together regularly first to first, second to second third to third—3 2 6 4 9 6 12 8—and the analogous numbers to the ones corresponding to them in position

[4] If we care to investigate the second species of the superparticular the sesquitertian (for the fraction naturally following after the half is the third) we shall have this definition of it—a number which *contains* the whole of the number compared and a third of it in addition to the whole We may have examples of it in the proper order in the successive quadruples beginning with 4 joined to the triples from 3 each term with the one in the corresponding position in the series for example, 4 3 8 6 12 9 and so on to infinity [5] It is plain that that which corresponds to the sesquitertian but is called with the prefix sub- subsesquitertian is the number the whole of which is contained and a third part in addition for example 3 4 6 8 9 12 and the similar pairs of numbers in the same position in the series

[6] And we must observe the never failing corollary of all this that the first forms in each series the so-called root numbers are next to one another in the natural series the next after the root forms show an interval of only one number the third two the fourth three the fifth four and so on as far as you like [7] Furthermore that the fraction after which each of the superparticulars is named is seen in the lesser of the root numbers never in the greater

[8] That by nature and by no disposition of ours the multiple is a more elementary and an older form than the superparticular we shall shortly learn through a somewhat intricate process And here for a simple demonstration we must prepare in regular and parallel lines the multiples specified above according to their varieties first the double in one line then in a second the triple then the quadruple in a third and so on as far as the tenfold multiples so that we may detect their order and variety, their regulated progress and which of them is naturally prior and indeed other corollaries delightful in their exactness Let the diagram be as follows [9]

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 8 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

[10] Let there be set forth in the first row the natural series from unity and then in order those species of the multiple which we were bidden to insert

[11] Now then in comparison with the first rows beginning with unity if we read both across and up and down in the form of the letter gamma the next rows both ways themselves in the form of a gamma beginning with 4 are multiples according to the first form of the multiple for they are doubles The first differs by unity from the first the second from the second by 2 the third from the third by 3 the next by 4 those following by 5 and you will find that this follows throughout

The third rows in both directions from 9, their common origin will be the triples of the terms in that same first row according to the second form of the multiple the cross lines like the letter chi ending in the term 3 in either direction are to be taken into consideration [12] The difference for these numbers will progress after the series of the even numbers being 2 for the first 4 for the next 6 for the third and this difference Nature has of her own accord interpolated for us between these rows that are being examined as is evident in the diagram

[13] The fourth row whose common origin in both directions is 16 and whose cross-lines end with the terms 4 exhibits the third species of multiple the quadruple when it is compared with that same first row according to corresponding positions first term with first second with second third with third and so on Again the differences of these numbers are 3 6 then 9 then 12 and the quantities that progress by steps of 3 These numbers are detected in the

structure of the diagram in places just above the quadruples and in the subsequent forms of the multiple the analogy will hold throughout

*[14]* In comparison with the second line reading either way which begins with the common origin 4 and runs over in cross lines to the term 2 in each row, the lines which are next in order beneath display the first species of the superparticular that is, the sesquialter between terms occupying corresponding places Thus by divine nature not by our convention or agreement the superparticulars are of later origin than the multiples For illustration 3 is the sesquialter of 2 6 of 4 9 of 6 12 of 8 15 of 10 and throughout thus They have as a difference the successive numbers from unity like those before them

*[15]* The sesquitertians the second species of superparticular proceed with a regular even advance from 4 3 8 6 12 9 16 12 and so on having also a regular increase of their differences *[16]* And in the other multiple and superparticular relations you will see that the results are in harmony and not by any means inconsistent as you go on to infinity

*[17]* The following feature of the diagram moreover is of no less exactness The terms at the corners are units the one at the beginning a simple unit that at the end the unit of the third course and the other two units of the second course appearing twice so that the *product (of the first two) is equal to the square* (of the last) *[18]* Furthermore in reading either way there is an even progress from unity to the tens and again on the opposite sides two other progressions from 10 to 100

*[19]* The terms on the diagonal from 1 to 100 are all square numbers the products of equals by equals and those flanking them on either side are all heteromecic unequal and the products of sides of which one is greater than the other by unity and so the sum of two successive squares and twice the heteromecic numbers between them is always a square and conversely a square is always produced from the two heteromecic numbers on the sides and twice the square between them

*[20]* An ambitious person might find many other pleasing things displayed in this diagram upon which it is not now the time to dwell for we have not yet gained recognition of them from our *Introduction* and so we must turn to the next subject For after these two generic relations of the multiple and the superparticular and the *other two opposite* to them with the prefix sub- the submultiple and the subsuperparticular there are in the greater division of inequality the superpartient and in the less its opposite the subsuperpartient

## CHAPTER XX

*[1]* It is the superpartient relation when a number contains within itself the whole of the number compared and in addition more than one part of it and "more than one starts with 2 and goes on to all the numbers in succession Thus the root-form of the superpartient is naturally the one which has in addition to the whole two parts of the number compared and as a species will be called superbipartient after this the one with three parts besides the whole will be called supertripartient as a species then comes the superquadripartient the superquintipartient, and so forth

*[2]* The parts have their root and origin with the third for it is impossible in this case to begin with the half For if we assume that any number contains two halves of the compared number besides the whole of it we shall inadvertently be setting up a multiple instead of a superpartient because each whole plus two halves of it, added together makes double the original number Thus it is most necessary to start with two thirds then two fifths two sevenths and after these two ninths, following the advance of the odd numbers for two quarters, for example again are a half, two sixths a *third and thus again superparticulars* will be produced instead of superpartients which is not the problem laid before us nor in accord with the systematic construction of our science

*[3]* After the superpartient the subsuperpartient immediately is produced whenever a number is completely contained in the one compared with it, and in addition several parts of it 2 3 4 or 5 and so on

## CHAPTER XXI

*[1]* The regular arrangement and orderly production of both species are discovered when we set forth the successive even and odd numbers beginning with 3 and compare with them simple series of odd numbers only from 5 in succession first to first—that is 5 to 3—second to second—that is 7 to 4—third to third—that is 9 to 5—fourth to fourth—that is 11 to 6— and so on in the same order as far as you like In this way the forms of the superpartient and the subsuperpartient in due order will be disclosed through the root forms of each species the superbipartient first then the supertripartient superquadripartient and super

quintipartient and further in succession in similar manner, for after the root forms of each species the ones which follow them will be produced by doubling or tripling both the terms and in general by multiplying after the regular forms of the multiple

nerically, it will have in its subdivisions according to species a sort of diversification and change of names proper both to the first part of the name and to the second For instance in the first part that is the multiple it will have double triple, quadruple quintuple, and

TABLE OF THE SUPERPARTIENTS

| Root-forms | 5 | 3 | 7 | 4 | 9 | 5 | 11 | 6 | 13 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 10 | 6 | 14 | 8 | 18 | 10 | 22 | 12 | 26 | 14 |
| | 15 | 9 | 21 | 12 | 27 | 15 | 33 | 18 | 39 | 21 |
| | 20 | 12 | 28 | 16 | 36 | 20 | 44 | 24 | 52 | 28 |
| | 25 | 15 | 35 | 20 | 45 | 25 | 55 | 30 | 65 | 35 |
| | 30 | 18 | 42 | 24 | 54 | 30 | 66 | 36 | 78 | 42 |
| | 35 | 21 | 49 | 28 | 63 | 35 | 77 | 42 | 91 | 49 |
| | 40 | 24 | 56 | 32 | 72 | 40 | 88 | 48 | 104 | 56 |
| | 45 | 27 | 63 | 36 | 81 | 45 | 99 | 54 | 117 | 63 |

[7] It must be observed that from the two parts in addition to the whole which are contained in the greater term we are to understand third in the case of three parts, fourth with four parts fifth' with five sixth and so on so that the order of nomenclature is something like this superbipartient supertripartient superquadripartient then superquintipartient and similarly with the rest

[8] Now the simple uncompounded relations of relative quantity are these which have been enumerated Those which are compounded of them and as it were woven out of two into one are the following of which the antecedents are the multiple superparticular and multiple superpartient and the consequents the ones that immediately arise in connection with each of the former named with the prefix sub together with the multiple superparticular the submultiple superparticular and with the multiple superpartient the submultiple superpartient In the subdivision of the genera the species of the one will correspond to those of the other these also having names with the prefix sub

## CHAPTER XXII

[1] Now the multiple superparticular is a relation in which the greater of the compared terms contains within itself the lesser term more than once and in addition some one part of it whatever this may be

[2] As a compound such a number is doubly diversified after the peculiarities of nomenclature of its components on either side for inasmuch as the multiple superparticular is composed of the multiple and superparticular ge-

so forth and in the second part generically from the superparticular, its specific forms in due order the sesquialter sesquitertian sesquiquartan sesquiquintan and so on so that the combination will proceed in somewhat this order

Double sesquialter double sesquitertian, double sesquiquartan double sesquiquintan, double sesquisextan and analogously

Beginning once more triple sesquialter triple sesquitertian triple sesquiquartan triple sesquiquintan

Again quadruple sesquialter quadruple sesquitertian quadruple sesquiquartan quadruple sesquiquintan

Again quintuple sesquialter quintuple sesquitertian quintuple sesquiquartan quintuple sesquiquintan and the forms analogous to these *ad infinitum* Whatever number of times the greater contains the whole of the smaller by this quantity the first part of the ratio of the terms joined together in the multiple superparticular is named and whatever may be the factor in addition to the whole several times contained that is in the greater term from this is named the second kind of ratio of which the multiple superparticular is compounded

[3] Examples of it are these 5 is the double sesquialter of 2 7 the double sesquitertian of 3 9 the double sesquiquartan of 4 11 the double sesquiquintan of 5 You will furthermore always produce them in regular order in this fashion by comparing with the successive even and odd numbers from 2 the odd numbers, inclusively from 5 first with first second third with third and the with the one in the same position

The successive terms beginning with 5 and differing by 5 will be without exception double sesquialters of all the successive even numbers from 2 on, when terms in the same position in the series are compared and beginning with 3, if all those with a difference of 3 be set forth as 3 6 9 12 15 18 21 and in another series there be set forth those that differ by 7 to infinity as 7 14 21 28 35 42 49 and the greater be compared with the smaller first to first, second to second third to third fourth to fourth and so on the second species will appear the double sesquitertian disposed in its proper order

[4] Then again, to take a fresh start if the simple series of quadruples be set forth, 4 8 12 16 20 24 28 32 and then there be placed beside it in another series the successive numbers beginning with 9 and increasing by 9 as 9 18 27 36 45 54 we shall have revealed once more the multiple superparticular in a specific form that is the double sesquiquartan in its proper order and any one who desires can contrive this to an unlimited extent

[5] The second kind begins with the triple sesquialter such as 7 2 14 4 and in general the numbers that advance by steps of 7 compared with the even numbers in order from 2

[6] Then once more 10 3 is the first triple sesquitertian 20 6 the second and in a word the multiples of 10 in succession compared with the successive triples This indeed we can observe with greater exactitude and clearness in the table studied above for in comparison with the first row the succeeding rows in order[1] compared as whole rows display the forms of the multiple in regular order up to infinity when they are all compared in each case to the same first row and when each row is compared to all those above it in succession the second row being taken as our starting point all the forms of the superparticular are produced in their proper order and if we start with the third row all of those beginning with the fifth that are odd in the series when they are compared with this same third row and those following it will show all the forms of the superpartient in proper order In the case of the multiple superparticular the comparisons will have a natural order of their own if we start with the second row and compare the terms from the fifth first to first, second to second third to third and so on and then the terms of the seventh row to the third, those of the ninth to the fourth and follow the corresponding order as far as we are able to go [7] It is plain that here too the smaller terms have names corresponding to the larger ones with the prefix sub- according to the nomenclature given them all

## CHAPTER XXIII

[1] The multiple superpartient is the remaining relation of number This and the relation called by a corresponding name with the prefix sub-, exist when a number contains the whole of the number compared more than once (that is twice, thrice or any number of times) and certain parts of it more than one either two three or four and so on, besides [2] These parts[1] are not halves for the reasons mentioned above but either thirds fourths or fifths and so on

[3] From what has already been said it is not hard to conceive of the varieties of this relation for they are differentiated in the same way as and consistently with those that precede double superbipartient, double supertripartient double superquadripartient and so on For example 8 is the double superbipartient of 3 16 of 6 and in general the numbers beginning with 8 and differing by 8 are double superbipartients of those beginning with 3 and differing by 3 when those in corresponding places in the series are compared and in the case of the other varieties one could ascertain their proper sequence by following out what has already been said In this case too we must conceive that the nomenclature of the number compared goes along and suffers corresponding changes with the addition of the prefix sub-

[4] Thus we come to the end of our speculation upon the ten arithmetical relations for a first *Introduction* There is however a method very exact and necessary for all discussion of the nature of the universe which very clearly and indisputably presents to us the fact that that which is fair and limited and which subjects itself to knowledge[2] is naturally prior to the unlimited incomprehensible and ugly and furthermore that the parts and varieties of the infinite and unlimited are given shape and boundaries by the former and through it attain to their fitting order and sequence and like objects brought beneath some seal or measure all gain a share of likeness to it and similarity of name when they fall under its influence For thus it is reasonable that the rational

[1]Referring to the table in Chapter 19

[1]See 20 2 above
[2]Cf I 2 5

part of the soul will be the agent which puts in order the irrational part and passion and appetite which find their places in the two forms of inequality will be regulated by the reason ing faculty as though by a kind of equality and sameness [5] And from this equalizing process there will properly result for us the so-called ethical virtues sobriety,[1] courage gentleness self control fortitude and the like

[6] Let us then consider the nature of the principle that pertains to these universal mat ters It is capable of proving that all the com plex species of inequality and the varieties of these species are produced out of equality, first and alone as from a mother and root

[7] Let there be given us equal numbers in three terms first units then two s in another group of three then three s next four s five s and so on as far as you like For them as the setting forth of these terms has come about by a divine and not human contrivance nay by Nature herself, multiples will first be produced and among these the double will lead the way the triple after the double the quadruple next and then the quintuple and following the or der we have previously recognized *ad infini tum* second the superparticular and here again the first form, the sesquialter will lead and the next after it the sesquitertian will fol low and after them the next in order the ses quiquartan *the sesquiquintan the sesquisex* tan and so on *ad infinitum* third the super partient which once more the superbipartient will lead the supertripartient will follow im mediately upon it and then will come the superquadripartient the superquintipartient and according to the foregoing as far as one may proceed

[8] Now you must have certain rules like invariable and inviolable natural laws follow ing which the whole aforesaid advance and progress from equality may go on without fail ure These are the directions Make the first equal to the first the second equal to the sum of the first and second and the third to the sum of the first twice the second and the third For if you fashion according to these rules you would get first all the forms of the multiple in order out of the three given terms of the equal ity as it were sprouting and growing without your paying any heed or offering any aid From equality you will first get the double from the double the triple from the triple successively the quadruple and from this the quintuple in due order and so on [9] From the e same mul *tiples in their regular order reversed there are* immediately produced by a sort of natural ne cessity through the agency of the ame three rules the superparticulars and these not as it chances and irregularly but in their proper se quence for from the first the double reversed comes the first the sesquialter and from the second the triple the second in this class the se quitertian then the sesquiquartan from the quadruple and in general each one from the one of similar name [10] And with a fresh start if the superparticulars are set forth in the order of their production but with terms reversed the superpartients which naturally follow them are brought to light the superbi partient from the sesquialter, the supertripar tient from the sesquitertian the superquadri partient from the sesquiquartan and so on *ad infinitum* [11] If, however the superparticu lars are set forth with terms not in reverse but in direct order there are produced through the three rules the multiple superparticulars, the double sesquialter out of the first the sesqui alter the double sesquitertian from the second the sesquitertian, the double sesquiquartan from the third the sesquiquartan and so on [12] From those produced by the reversal of the superparticular that is the superpartients *and from tho e produced without such revers* al the multiple superparticulars there are once more produced in the same way and by the same rules both when the terms are in direct or reverse order the numbers that show the remaining numerical relations

[13] The following must suffice as illustrations of all that has been said hitherto the pro duction of these numbers and their sequence and the use of direct and of reverse order [14] From the relation and proportion in terms of the sesquialter reversed so as to begin with the largest term there arises a relation in super partient ratios the superbipartient and from it in direct order beginning with the smallest term a multiple superparticular relation the double se quialter For example from 9 6 4 we get either 9 15 25 or 4 10 25 From the relation in terms of sesquitertians beginning with the greatest term is derived a superpar tient the supertripartient beginning with the smallest term a double sesquitertian For ex ample from 16 12 9 comes either 16 28 49 or 9 21 49 And from the relation in terms sesquiquartans when it is arranged to with the largest term is derived a s

[1] Cf Aristotle *Ethics* 1107[b] 4 ff See *ibid* 1108 4 ff 1145[b] 8 ff

tient the superquadripartient when it starts with the smallest term a multiple superparticular, the double sesquiquintan for instance from 25 20, 16 comes either 25, 45, 81 or 16 36 81

*[16]* In the case of all the e relations that are thus differentiated and of the one from which both of the differentiated ones are derived the last term is always the same and a square the first term becomes the smallest and invariably the extremes are squares

*[16]* Moreover the multiple superpartients and superpartients of other kinds are made to appear in yet another way out of the superpartients for example from the superbipartient relation arranged so as to begin with the smallest term comes the double superbipartient but arranged so as to start with the greate t, the superpartient ratio of 8 5 Thus from 9 15 25 comes either 9 24 64 or 25 40, 64 From the supertripartient beginning with the smallest term we have the double supertripartient, and beginning with the largest the ratio of 11 7 Thus from 16 28 49 comes either 16, 44, 121 or 49 77, 121 *[17]* Again from the superquintipartient as for example, 25, 45 81 beginning with the lesser term we derive the double superquintipartient in the terms 25 70 196 but beginning with the greater a superpartient again, the ratio of 14 9 in the terms 81 126 196 And you will find the results analogous and in agreement with the foregoing in all successive cases to infinity

# BOOK TWO

## CHAPTER I

*[1]* An element is said to be and is the smallest thing which enters into the composition of an object and the least thing into which it can be analyzed Letters for example are called the elements of literate speech for out of them all articulate speech is composed and into them finally it is resolved Sounds are the elements of all melody for they are the beginning of its composition and into them it is resolved The so-called four elements of the universe in general are simple bodies fire water air and earth for out of them in the first instance we account for the constitution of the universe and into them finally we conceive of it as being resolved

We wish also to prove that equality is the elementary principle[1] of relative number for of absolute number number per se unity and the dyad are the most primitive elements the least things out of which it is constructed even to infinity by which it has its growth and with which its analysis into smaller terms comes to an end *[2]* We have however demonstrated that in the realm of inequality advance and increase have their origin in equality and go on to absolutely all the relations with a certain regularity through the operation of the three rules[2] It remains then in order to make it an element in very truth to prove that analyses also finally come to an end in equality Let this then be considered our procedure

## CHAPTER II

*[1]* Suppose then you are given three terms in any relation whatsoever and in any ratio whether multiple superparticular superpartient or a compound of these multiple superparticular or multiple superpartient provided only that the mean term is seen to be in the same ratio to the lesser as the greater to the mean and vice versa Subtract always from the mean the lesser term whether it be first or last in order, and set down the lesser term itself as the first term of your new series then put as your second term what remains from the second after the subtraction then after having subtracted the sum of the new first term and twice the new second term from the remaining number—that is the greater of the numbers originally given you—make the remainder your third term and the resulting numbers will be in some other ratio naturally more primitive *[2]* And if again in the same way you subtract the remainder from these same terms it will be found that your three terms have passed back into three others more primitive and you will find that this always takes place as a consequence until they are reduced to equality whence by every necessity it appears evident that equality is the elementary principle of relative quantity

*[3]* There follows upon this speculation a most elegant principle extremely useful in its application to the Platonic psychogony[3] and the problem of all harmonic intervals for in the Platonic passage we are frequently bidden, for the sake of the argument to set up series of intervals of two three four five or an infinite number of sesquialter ratios or two sesquitertians sesquiquartans sesquioctaves or superparticulars of any kind whatsoever, and in each case three four or five of them or as many as may be directed *[4]* It is reasonable that we should do this not in an unscientific unintelligent fashion it may be even blunderingly but artistically surely and quickly by the following procedure

## CHAPTER III

*[1]* Every multiple will stand at the head[4] of as many superparticular ratios corresponding in name with itself as it itself chances to be removed from unity and no more nor less under any circumstances

[1]See on I 23 4
[2]See on I 23 8.
[3]See Plato *Timaeus* 35 ff
[4]Refers to the table in c

*[2]* The doubles then, will produce[1] sesquialters the first one the second two the third three the fourth four, the fifth five, the sixth six and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number agrees with the multiples that have generated them at that point by a divine device as it were there is found the

Sixteen the fourth double will stand at the head of four sesquialters 24, 36 54 and finally 81, so that they may of necessity be equal in number to what generated them for 81 by its nature is not divisible by 2 And this as you go on you will find holds true in similar fashion to infinity

*[4]* For the sake of illustration let there be set down the table of the doubles thus

| The double ratio in the breadth of the table | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 4 | 8 | 16 | 32 | 64 | |
| | | 3 | 6 | 12 | 24 | 48 | 96 | |
| The triple ratio along | | | 9 | 18 | 36 | 72 | 144 | The sesquialter ra |
| the hypotenuse | | | | 27 | 54 | 108 | 216 | tio in the depth |
| | | | | | 81 | 162 | 324 | |
| | | | | | | 243 | 486 | |
| | | | | | | | 729 | |

number which terminates them all because it naturally is not divisible by that factor whereby the progression of the superparticular ratios went on

## CHAPTER IV

*[1]* We must make a similar table in illustration of the triple

| The triple ratio in the breadth | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 3 | 9 | 27 | 81 | 243 | 729 | |
| | | 4 | 12 | 36 | 108 | 324 | 972 | |
| The quadruple ratio | | | 16 | 48 | 144 | 432 | 1296 | The sesquitertian |
| on the hypotenuse | | | | 64 | 192 | 576 | 1728 | ratio in the depth |
| | | | | | 256 | 768 | 2304 | |
| | | | | | | 1024 | 3072 | |
| | | | | | | | 4096 | |

From the triples all the sesquitertians will proceed likewise equal in number to the number of the generating terms and coming to an end after the independence of their advance is lost in numbers not divisible by 3 Similarly the sesquiquartans come from the quadruples reaching a culmination after their independent progression in a number that is not divisible by 4

*[3]* As an example since doubles generate sesquialters corresponding to them in number the first row of multiples will be 1 2 4 8 16 32 64 Now since 2 is the first after unity this will be the origin of one sesquialter only 3 which number is not divisible by 2 so that another sesquialter might arise out of it The first double therefore is productive of but one sesquialter, and the second 4 of two For it produces its own sesquialter 6 and that of 6 9 but there is none for 9 because it has no half Eight which is the third double is father to three sesquialters one its own 12 the second 18 the sesquialter of 12 and third 27 that of 18 there is no fourth one however because of the general rule for 27 is not divisible by 2

In the foregoing table we shall observe that in the same way the first triple 3 stands at the head of but one sesquitertian ratio 4 its own sesquitertian which immediately shuts off the development of another like it for 4 is not divisible by 3, and hence will not have a sesquitertian The second triple is 9 and hence will begin a series of only two sesquitertian ratios 12 its own and 16 that of 12 but 16 cuts off further progress for it is not divisible by 3 and hence will not have a sesquitertian *[2]* Next in order is the triple 27 three times removed from 1 for the triples progress thus 1 3, 9 27 Therefore this number will stand at the head of three sesquitertian ratios and no more The first is its own 36 the second the sesquitertian of 36 48 the third that of the last 64 and this no longer has a third part and therefore will not admit of a sesquitertian The fourth leads a series of four sesquitertians and the fifth of course five

*[3]* Such then is the illustration and for the other multiples let the manner of your tables be the same Observe that likewise here as we found to be true in our previous discus

[1] See I 19 2

sion Nature shows us that the doubles are more nearly original than the triples the triples than the quadruples the e latter than the quintu ples and so on throughout For the highest rows of figures across the breadth of the tables if they are doubles will have doubles lying par allel to them and the numbers lying diagonal ly on the hypotenuse will be of the next suc ceeding variety greater by 1 that is triples seen also in a series of parallel lines If how ever there are triples across the breadth the diagonals will by all means be quadruples if the former are quadruples then the latter are quintuples and so forth

## CHAPTER V

*[1]* It remains after we have explained what other ratios are produced by combination of ratios to pass on to the succeeding topics of the *Introduction*

*[2]* Now the first two ratios of the super particular combined produce the first ratio of the multiple namely the double for every double is a combination of sesquialter and ses-quitertian and every sesquialter and sesqui tertian combined will invariably produce a double

For example since 3 is the sesquialter of 2 and 4 the sesquitertian of 3 4 will be the dou ble of 2 and is a combination of sesquialter and sesquitertian Again as 6 is the double of 3 we shall find between them some number that will of necessity preserve the sesquitertian ratio to the one and the sesquialter to the other and indeed 4 lying between 6 and 3 gives the ses quitertian ratio to 3 and the sesquialter to 6

*[3]* It was rightly said then that the double when resolved is resolved into the sesquialter and the sesquitertian and that when sesqui alter and sesquitertian are combined there arises the double and that the first two forms of the superparticular combined make the first form of the multiple

*[4]* But again to take another start this first form of the multiple which has thus been produced together with the first form of the superparticular will produce the next form of the same class that is the second multiple the triple for from every multiple and sesquialter combined a triple of necessity arises For ex ample as the double of 6 is 12 and the sesqui alter of this is 18 then immediately 18 is the *triple of 6 and to take another method if I do* not care to make 12 the mean term but rather 9 the sesquialter of 6 the same result will come about without deviation and harmoniously for while 18 is the double of 9 it will preserve the triple ratio to 6 Hence from the sesquialter and the double, the first forms of the superpar ticular and the multiple there arises by com bination the second form of the multiple, the triple and into them it is always resolved *[5]* For look you 6 which is the triple of 2 will have a mean term 3 which will exhibit two ratios the sesquialter with regard to 2 and the double ratio of 6 to itself

But if this triple ratio likewise the second form of the multiple is combined with the ses-quitertian which is the second form of the su-perparticular there would be produced from them the next form of the multiple namely the quadruple and this also will of necessity be resolved into them after the same fashion as the cases previously set forth and the quadru-ple taking into combination the sesquiquartan will make the quintuple and once more the latter with the sesquiquintan will make the sex tuple and so on to the end Thus the multiples in regular order from the beginning with the superparticulars in regular order from the be-ginning will be found to produce the next larger multiples For the double with the sesquialter makes the triple the triple with the sesquiter tian the quadruple the quadruple with the ses quiquartan the quintuple and as far as you wish to proceed no contrary result will appear

## CHAPTER VI

*[1]* Up to this point then we have sufficiently discussed relative number by a process of se-lection measuring out what is easily compre-hended and appropriate to the nature of the matters thus far introduced Whatever remains to be said on this topic will be filled in after we have put it aside and have first discussed cer tain subjects which involve a more serviceable inquiry having to do with the properties of ab-solute number not relative For mathematical speculations are always to be interlocked and to be explained one by means of another The subjects which we must first survey and ob-serve are concerned with linear plane and solid numbers cubical and spherical equilateral and scalene bricks beams 'wedges and the like the tradition concerning which to be sure since they are more closely related to magni tude is properly given in the *Geometrical Intro-duction* Yet the germs of these ideas are taken *over into arithmetic, as the science which is the* mother of geometry and more elementary than it For we recall that a short time ago we saw that arithmetic abolishes the other science

with itself but is not abolished by them, and conversely is of necessity implied by them but does not itself imply them

*[2]* First however, we must recognize that each letter by which we indicate a number, such as iota the sign for 10 kappa for 20 and omega for 800 designates that number by man s convention and agreement not by nature On the other hand the natural unartificial and therefore simplest indication of numbers would be the setting forth one beside the other of the units contained in each For example the writing of one unit by means of one alpha will be the sign for 1 two units side by side that is, a series of two alphas will be the sign for 2 when three are put in a line it will be the character for 3 four in a line for 4 five for 5 and so on For by means of such a notation and indication alone could the schematic arrangement of the plane and solid numbers mentioned be made clear and evident thus

The number 1 α
The number 2 α α
The number 3 α α α
The number 4 α α α α
The number 5 α α α α α

and further in similar fashion

*[3]* Unity then occupying the place and character of a point will be the beginning of intervals and of numbers but not itself an interval or a number, just as the point is the beginning of a line or an interval but is not itself line or interval Indeed when a point is added to a point it makes no increase for when an non-dimensional thing is added to another non-dimensional thing it will not thereby have dimension just as if one should examine the sum of nothing added to nothing which makes nothing We saw[1] a similar thing also in the case of equality among the relatives for a proportion is preserved—as the first is to the second so the second is to the third—but no interval is generated in the relation of the extremes to each other as there is in all the other relations with the exception of equality In exactly the same way unity alone out of all number when it multiplies itself produces nothing greater than itself

Unity, therefore, is non-dimensional and elementary and dimension first is found and seen in 2 then in 3 then in 4 and in succession in the following numbers for dimension is that which is conceived of as between two limits

*[4]* The first dimension is called line " for ' line is that which is extended in one direction Two dimensions are called surface ' for a surface is that which is extended in two directions Three dimensions are called 'solid ' for a solid ' is that which is extended in three directions and it is by no means possible to conceive of a solid which has more than three dimensions depth breadth and length By these are defined the six directions which are said to exist in connection with every body and by which motions in space are distinguished forward backward[2] up down right and left for of necessity two directions opposite to each other follow upon each dimension up and down upon one, forward and backward upon the second and right and left upon the third

*[5]* The statement also as it happens can be made conversely thus If a thing is solid it has by all means three dimensions length depth and breadth and conversely if it has the three dimensions it is always a solid, and nothing else

*[6]* That which has but two dimensions therefore will not be a solid but a surface for the latter admits of but two dimensions Here too it is possible similarly to reverse the statement directly stated a surface is that which has two dimensions and conversely that which has two dimensions is always a surface

*[7]* The surface then, is exceeded by the solid by one dimension and the line is exceeded by the surface by one for the line is that which is extended in but one direction and has only one dimension and it falls short of the solid by two dimensions The point falls short of the latter by one dimension and hence it has already been stated that it is non-dimensional since it falls short of the solid by three dimensions of the surface by two and of the line by one

## CHAPTER VII

*[1]* The point then is the beginning of dimension but not itself a dimension and likewise the beginning of a line but not itself a line the line is the beginning of surface but not surface and the beginning of the two-dimensional but not itself extended in two directions *[2]* Naturally too surface is the beginning of body, but not itself body and likewise the beginning of the three-dimensional but not itself extended in three directions

[1] I 23 7 ff

[2] Cf Plato *Timaeus* 43

*[3]* Exactly the same in numbers unity is the beginning of all number that advances unit by unit in one direction linear number is the beginning of plane number which spreads out like a plane in one more dimension and plane number is the beginning of solid number which possesses a depth[1] in the third dimension besides the original ones To illustrate and classify, linear numbers are all those which begin with 2 and advance by the addition of 1 in one and the same dimension and plane numbers are those[2] that begin with 3 as their most elementary root and proceed through the next succeeding numbers They receive their names also in the same order for there are first the triangles then the squares the pentagons after these then the hexagons the heptagons and so on indefinitely and as we said they are named after the successive numbers beginning with 3

*[4]* The triangle therefore is found to be the most original and elementary form of the plane number This we can see from the fact that, among plane figures[3] graphically represented if lines are drawn from the angles to the centers each rectilinear figure will by all means be resolved into as many triangles as it has sides but the triangle itself if treated like the rest will not change into anything else but itself Hence the triangle is elementary among these figures for everything else is resolved into it but it into nothing else From it the others likewise would be constituted but it from no other It is therefore the element of the others and has itself no element *[5]* Likewise as the argument proceeds in the realm of numerical forms it will confirm this statement

## CHAPTER VIII

*[1]* Now a triangular number is one which when it is analyzed into units shapes into triangular form the equilateral placement of its parts in a plane 3 6 10 15 21 28 and so on are examples of it for their regular formations expressed graphically will be at once triangular and equilateral As you advance you will find that such a numerical series as far as you like takes the triangular form if you put as the most elementary form the one that arises from unity so that unity may appear to be potentially a triangle and 3 the first actually

*[2]* Their sides will increase by the successive numbers for the side of the one potentially first is unity that of the one actually first that is 3 is 2 that of 6 which is actually second 3 that of the third 4 the fourth, 5 the fifth 6, and so on

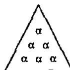
*[3]* The triangular number is produced from the natural series of number set forth in a line and by the continued addition of successive terms one by one, from the beginning for by the successive combinations and additions of another term to the sum the triangular numbers in regular order are completed For example from this natural series 1 2 3 4 5 6 7 8 9 10 11 12 13, 14 15 I take the first term and have the triangular number which is potentially first, 1,



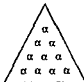

then adding the next term I get the triangle actually first for 2 plus 1 equals 3 In its graphic representation it is thus made up Two units side by side are set beneath one unit and the number three is made a triangle




Then when next after these the following number 3 is added simplified into units and joined to the former it gives 6 the second triangle in actuality and furthermore it graphically represents this number



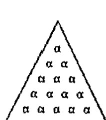

Again the number that naturally follows 4 added in and set down below the former reduced to units gives the one in order next after the aforesaid 10 and takes a triangular form

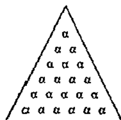


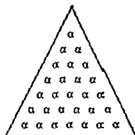

5 after this then 6 then 7 and all the numbers in order are added so that regularly the sides of each triangle will consist of as many

[1]Cf Plato *Timaeus* 53
[2]But cf Euclid in *Elements* VII Def 17
[3]Cf Plato *Timaeus* 53 ff

numbers as have been added from the natural series to produce it

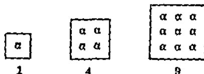

Side 5

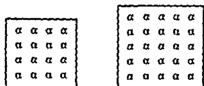

Side 6


Side 7

## CHAPTER IX

*[1]* The square is the next number after this which shows us no longer 3 like the former but 4 angles in its graphic representation but is none the less equilateral Take for example 1 4 9 16 25 36 49 64 81, 100 for the representations of these numbers are equilateral, square figures as here shown, and it will be similar as far as you wish to go


1 4 9


16 25

*[2]* It is true of these numbers as it was also of the preceding that the advance in their sides progresses with the natural series The side of the square potentially first 1 is 1 that of 4 the first in actuality 2 that of 9 actually the second 3 that of 16 the next actually the third 4 that of the fourth 5 of the fifth, 6, and so on in general with all that follow

*[3]* This number also is produced if the natural series is extended in a line increasing by 1, and no longer the successive numbers are added to the numbers in order, as was shown before but rather all those in alternate places that is, the odd numbers For the first 1 is potentially the first square the second 1 plus 3 is the first in actuality, the third 1 plus 3 plus 5 is the second in actuality the fourth 1 plus 3 plus 5 plus 7 is the third in actuality the next is produced by adding 9 to the former numbers the next by the addition of 11 and so on

*[4]* In these cases also it is a fact that the side of each consists of as many units as there are numbers taken into the sum to produce it

## CHAPTER X

*[1]* The pentagonal number is one which likewise upon its resolution into units and depiction as a plane figure assumes the form of an equilateral pentagon 1, 5 12 22 35 51 70 and analogous numbers are examples *[2]* Each side of the first actual pentagon 5 is 2 for 1 is the side of the pentagon potentially first 1 3 is the side of 12 the second of those listed 4 that of the next 22 5 that of the next in order 35 and 6 of the succeeding one, 51 and so on In general the side contains as many units as are the numbers that have been added together to produce the pentagon chosen out of the natural arithmetical series set forth in a row For in a like and similar manner there are added together to produce the pentagonal numbers the terms beginning with 1 to any extent whatever that are two places apart that is those that have a difference of 3

Unity is the first pentagon potentially and is thus depicted

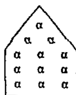


5 made up of 1 plus 4 is the second similarly represented



12 the third is made up out of the two former numbers with 7 added to them so that it may have 3 as a side as three numbers have been

added to make it Similarly the preceding pentagon 5 was the combination of two numbers and had 2 as its side The graphic representation of 12 is this




The other pentagonal numbers will be produced by adding together one after another in due order the terms after 7 that have the difference 3, as for example 10 13 16 19 22 25 and so on The pentagons will be 22 35 51 70 92, 117 and so forth

## CHAPTER XI

*[1]* The hexagonal heptagonal and succeeding numbers will be set forth in their series by following the same process if from the natural series of number there be set forth series with their differences increasing by 1 For as the triangular number was produced by admitting into the summation the terms that differ by 1 and do not pass over any in the series as the square was made by adding the terms that differ by 2 and are one place apart and the pentagon similarly by adding terms with a difference of 3 and two places apart (and we have demonstrated these by setting forth examples both of them and of the polygonal numbers made from them) so likewise the hexagons will have as their root numbers those which differ by 4 and are three places apart in the series which added together in succession will produce the hexagons For example 1 5 9 13 17 21 and so on so that the hexagonal numbers produced will be 1 6 15 28 45 66 and so on as far as one wishes to go

*[2]* The heptagonals which follow these have as their root numbers terms differing by 5 and four places apart in the series like 1 6 11 16 21 26 31 36 and so on The heptagons that thus arise are 1 7 18 34 55 81 112 148 and so forth

*[3]* The octagonals increase after the same fashion with a difference of 6 in their root-numbers and corresponding variation in their total constitution

*[4]* In order that as you survey all cases you may have a rule generally applicable note that the root-numbers of any polygonal differ by 2 less than the number of the angles shown by the name of the polygonal—that is by 1 in the triangle 2 in the square 3 in the pentagon 4 in the hexagon 5 in the heptagon and so on with similar increase

## CHAPTER XII

*[1]* Concerning the nature of plane polygonals this is sufficient for a first *Introduction* That however the doctrine of these numbers is to the highest degree in accord with their geometrical representation and not out of harmony with it would be evident not only from the graphic representation in each case but also from the following Every square figure diagonally divided is resolved into two triangles and every square number is resolved into two consecutive triangular numbers and hence is made up of two successive triangular numbers For example 1 3 6 10 15 21 28 36 45 55 and so on are triangular numbers and 1 4 9 16 25 36 49 64 81 100 squares *[2]* If you add any two consecutive triangles that you please you will always make a square, and hence whatever square you resolve you will be able to make two triangles of it

Again any triangle joined to any square figure makes a pentagon for example the triangle 1 joined with the square 4 makes the pentagon 5 the next triangle 3 of course with 9 the next square makes the pentagon 12 the next 6 with the next square 16 gives the next pentagon 22 10 and 25 give 35 and so on

*[3]* Similarly if the triangles are added to the pentagons following the same order they will produce the hexagonals in due order and again the same triangles with the latter will make the heptagonals in order the octagonals after the heptagonals and so on to infinity

*[4]* To remind us let us set forth rows of the polygonals written in parallel lines as follows The first row triangles the next squares after them pentagonals then hexagonals then heptagonals then if one wishes the succeeding polygonals

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Triangles | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 |
| Squares | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
| Pentagonals | 1 | 5 | 12 | 22 | 35 | 51 | 70 | 92 | 117 | 145 |
| Hexagonals | 1 | 6 | 15 | 28 | 45 | 66 | 91 | 120 | 153 | 190 |
| Heptagonals | 1 | 7 | 18 | 34 | 55 | 81 | 112 | 148 | 189 | [illegible] |

You can also set forth the succeeding polygon als in similar parallel lines

*[5]* In general you will find that the squares are the sum of the triangles above those that occupy the same place in the series, plus the numbers of that same class in the next place back for example 4 equals 3 plus 1 9 equals 6 plus 3 16 equals 10 plus 6 25 equals 15 plus 10, 36 equals 21 plus 15 and so on

The pentagons are the sum of the squares above them in the same place in the series, plus the elementary triangles that are one place further back in the series for example 5 equals 4 plus 1 12 equals 9 plus 3 22 equals 16 plus 6, 35 equals 25 plus 10 and so on

*[6]* Again the hexagonals are similarly the sums of the pentagons above them in the same place in the series plus the triangles one place back for instance, 6 equals 5 plus 1 15 equals 12 plus 3, 28 equals 22 plus 6 45 equals 35 plus 10 and as far as you like

*[7]* The same applies to the heptagonals for 7 is the sum of 6 and 1 18 equals 15 plus 3 34 equals 28 plus 6 and so on Thus each polygonal number is the sum of the polygonal in the same place in the series with one less angle plus the triangle in the highest row one place back in the series

*[8]* Naturally then the triangle is the element of the polygon both in figures and in numbers and we say this because in the table reading either up and down or across the successive numbers in the rows are discovered to have as differences the triangles in regular order

## CHAPTER XIII

*[1]* From this it is easy to see what the solid number is and how its series advances with equal sides for the number which in addition to the two dimensions contemplated in graphic representation in a plane length and breadth has a third dimension which some call depth others thickness and some height that number would be a solid number extended in three directions and having length depth and breadth

*[2]* This first makes its appearance in the so-called pyramids These are produced from rather wide bases narrowing to a sharp apex first after the triangular form from a triangular base second after the form of the square from a square base, and succeeding these after the pentagonal form from a pentagonal base then similarly from the hexagon heptagon octagon and so on indefinitely

*[3]* Exactly so among the geometrical solid figures if one imagines three lines from the three angles of an equilateral triangle equal in length to the sides of the triangle converging in the dimension height to one and the same point a pyramid would be produced, bounded by four triangles equilateral and equal one to the other one the original triangle and the other three bounded by the aforesaid three line *[4]* And again if one conceives of four lines starting from a square equal in length to the sides of the square each to each and again converging in the dimension height to one and the same point a pyramid would be completed with a square base and diminishing in square form, bounded by four equilateral triangles and one square the original one *[5]* And starting from a pentagon hexagon heptagon and however far you care to go lines equal in number to the angles erected in the same fashion from the angles and converging to one and the same point will complete a pyramid named from its pentagonal hexagonal, or heptagonal base, or similarly

*[6]* So likewise among numbers each linear number increases from unity as from a point as for example 1, 2 3, 4 5 and successive numbers to infinity and from these same numbers which are linear and extended in one direction combined in no random manner the polygonal and plane numbers are fashioned—the triangles by the combination of root numbers immediately adjacent the square by adding every other term the pentagons every third term and so on *[7]* In exactly the same way, *if the plane polygonal numbers are piled one upon* the other and as it were built up the pyramids that are akin to each of them are produced the triangular pyramid from the triangles the square pyramid from the squares the pentagonal from the pentagons the hexagonal from the hexagons and so on throughout

*[8]* The pyramids with a triangular base then in their proper order are these 1, 4 *10 20 35 56 84 and so on and their origin is the* piling up of the triangular numbers one upon the other first 1, then 1 3 then 1 3 6 then 10 in addition to these and next 15 together with the foregoing then 21 besides these next 28 and so on to infinity

*[9]* It is clear that the greatest number is conceived of as being lowest for it is discovered to be the base the next succeeding one is on top of it and the next on top of that until unity appears at the apex and so to speak tapers off the completed pyramid into a point

## CHAPTER XIV

*[1]* The next pyramids in order are those with a square base which rise in this shape to one and the same point These are formed in the same way as *the triangular pyramids of* which we have just spoken For if I extend in series the square numbers in order beginning with unity thus 1 4, 9 16 25 36 49 64 81 100 and again set the successive terms as in a pile one upon the other in the dimension height when I put 1 on top of 4 the first actual pyramid with square base 5 is produced for here again unity is potentially the first *[2]* Once more I put this same pyramid entire compo ed of 5 units just as it is upon the square 9 and there is made up for me the pyramid 14 with square base and side 3—for the former pyramid had the side 2 and the one potentially first 1 as a side I or here too each side of any pyramid whatsoever must consist of as many units as there are polygonal numbers piled together to create it

*[3]* Again I place the whole pyramid 14 with the square 9 as its base upon the square 16 and I have 30 the third actual pyramid of those that have a square base and by the same order and procedure from a pentagonal hexagonal or heptagonal base, and even going on farther we shall produce pyramids by piling upon one another the corresponding polygonal numbers tarting with unity as the smallest and going on to infinity in each case

*[4]* From this too it becomes evident that triangles are the most elementary for absolutely all of the pyramids that are exhibited and shown, with the various polygonal bases are bounded by triangles up to the apex

*[5]* But lest we be heedless of truncated bi truncated and tri truncated pyramids the names of which we are sure to encounter in scientific writings you may know that if a pyramid with any sort of polygon as its base triangle square pentagon or any of the succeeding polygons of the kind when it increases by this process of piling up does not taper off into unity it is called simply truncated when it is left without the natural apex that belongs to all pyramids for it does not terminate in the potential polygon unity as in some one point but in another polygon and an actual one and unity is not its apex but its upper boundary becomes a plane figure with the same number of angles as the base If however in addition to the failure to terminate in unity it does not even terminate in the polygon next to unity and the first in actuality, such a pyramid is called bi truncated and if, still further it does not have the second actual polygon at its upper limit but only the one next beneath it will be called tri truncated yes even four times truncated if it does not have the next one as its limit, or five times truncated at the next step and so on as far as you care to carry the nomenclature

## CHAPTER XV

*[1]* While the origin advance increase and nature of the equilateral solid numbers of pyramidal appearance is the foregoing, with its seed and root in the polygonal numbers and the piling up of them in their regular order there is another series of solid numbers of a different kind consisting of the so-called cubes, beams,' 'bricks' 'wedges spheres and parallelepipedons which has the order of its progress somewhat as follows

*[2]* The foregoing squares 1 4 9 16 25 36 49 64 and so on which are extended in two directions and in their graphic representation in a plane have only length and breadth will take on yet a third dimension and be solids and extended in three directions if each is multiplied by its own side 4 which is 2 times 2 is again multiplied by 2 to make 8 9 which is 3 times 3 is again increased by 3 in another dimension and gives 27 16 which is 4 times 4 is multiplied by its own side 4 and 64 results and so on with the succeeding squares throughou

*[3]* Here too the sides will be composed of as many units as were in the sides of the squares from which they arose in each case the sides of 8 will be 2 like those of 4 those of 27 3, like those of 9 those of 64 4 like those of 16 and so on so that likewise the side of unity the potential cube will be 1 which is the side of the potential square 1

In general each square is a single plane and has four angles and four sides while each several cube having increased out of some one square multiplied by its own side will have always six plane surfaces each equal to the original square and twelve edges each equal to and containing exactly the same number of units as each side of the original square and eight solid angles each of which is bounded by three edges like in each case to the sides of the original square

## CHAPTER XVI

*[1]* Now since the cube is a solid figure with equal sides in all dimensions in length depth,

and breadth and is equally extended in all the six so called directions [1] it follows that there is opposed to it that which has its dimensions in no case equal to one another but its depth un equal to its breadth and its length unequal to either of these for example 2 times 3 times 4 or 2 times 4 times 8, or 3 times 5 times 12 or a figure which follows some other scheme of in equality

*[2]* Such solid figures in which the dimen sions are everywhere unequal one to another, are called scalene in general Some however, using other names call them ' wedges for carpenters house builders and blacksmiths wedges and those used in other crafts having unequal sides in every direction are fashioned so as to penetrate they begin with a sharp end and continually broaden out unequally in all the dimensions Some also call them *sphekiskoi* wasps because wasps bodies also are very like them compressed in the middle and show ing the resemblance mentioned From this also the *sphekoma* point of the helmet must derive its name for where it is compressed it imi tates the waist of the wasp Others call the same numbers altars using their own metaphor for the altars of ancient style particularly the Ionic do not have the breadth equal to the depth nor either of these equal to the length nor the base equal to the top but are of varied dimensions everywhere

*[3]* Now whereas the two kinds of numbers cube and scalene are extremes the one equally extended in every dimension the other un equally the so-called parallelepipedons are sol id numbers like means between them The plane surfaces of these are heteromecic numbers [2] just as in the case of the cubes the faces were squares as has been shown

## CHAPTER XVII

*[1]* Again then to take a fresh start a num ber is called heteromecic if its representation when graphically described in a plane is quad rilateral and quadrangular to be sure but the sides are not equal one to another nor is the length equal to the breadth but they differ by 1 Examples are 2 6 12 20 30 42 and so on for if one represents them graphically he will always construct them thus 1 times 2 equals 2 2 times 3 equals 6 3 times 4 equals 12 and the succeeding ones similarly 4 times 5 5 times 6, 6 times 7, 7 times 8 and thus indefinitely

provided only that one side is greater than the other by 1 and by no other number If how ever the sides differ otherwise than by 1 for instance by 2 3 4 or succeeding numbers as in 2 times 4 3 times 6 4 times 8 or however else they may differ then no longer will such a number be properly called a heteromecic but an oblong number For the ancients of the school of Pythagoras and his successors saw the other' [3] and otherness primarily in 2 and ' the same and sameness in 1 as the two beginnings of all things and these two are found to differ from each other only by 1 Thus the other ' is fundamentally other by 1, and by no other number, and for this reason cus tomarily other is used among those who speak correctly of two things and not of more than two

*[2]* Moreover, it was shown that all odd num ber is given its specific form[4] by unity and all even number by 2 Hence we shall naturally say that the odd partakes of the nature of the same and the even of that of the other for indeed there are produced by the succes sive additions of each of these—naturally and not by our decree—by the addition of the odd numbers from 1 to infinity the class of the squares and by the addition of the evens from 2 to infinity that of the heteromecic numbers [5]

*[3]* There is accordingly every reason to think that the square once more shares in the nature of the same for its sides display the same ratio alike unchanging and firmly fixed in equality to themselves while the hetero mecic number partakes of the nature of the other for just as 1 is differentiated from 2 dif fering by 1 alone thus also the sides of every heteromecic number differ from one another, one differing from the other by 1 alone

To illustrate if I have set out before me the successive numbers in series beginning with 1 and select and arrange by themselves the odd numbers in the line and the even by themselves in another there are obtained these two series

1 3 5 7 9 11 13 15 17 19 21 23 25 27
2 4 6 8 10 12 14 16 18 20 22 24 26 28

*[4]* Now then the beginning of the odd series is unity which is of the same class as the series and possesses the nature of the same and so whether it multiplies itself in two dimensions

[1] Cf II 6 4
[2] See the following chapter

[3] Cf Plato *Timaeus* 35 ff
[4] Cf I 7 2
[5] Cf II 18 2 and 20 3

or in three it is not made different nor yet does it make any other number depart from what it was originally [1] but keeps it just as it was Such a property it is impossible to find in any other number *[5]* Of the other series the beginning is 2 which is similar in kind to this series and imitates otherness for whether it multiplies itself or another number it causes a change [2] for example 2 times 2 2 times 3

*[6]* But in cases like 8 times 8 times 2 or 8 times 8 times 3 such solid forms are called bricks, ' the product of a number by itself and then by a smaller number if however a greater height is joined to the square as in 3 times 3 times 7, 3 times 3 times 8 or 3 times 3 times 9 or however many times the square be taken provided only it be a greater number of times than the square it elf then the number is a beam the product of a number by itself and then by a larger number The wedges to be sure were the products of three unequal num bers and cubes of three equal ones

*[7]* Among the cubes some of them in addi tion to being the product of three equal num bers have the further property of ending at every multiplication in the same number as that from which they began these are called spherical and also recurrent Such indeed are those with sides 5 or 6 for however many times I increase each one of these it will by all means end each time in the same figure the derivative of 6 in 6 and that of 5 in 5 For example the product of 5 times 5 will end in 5 and so will 5 times this product and if necessary 5 times this again and to infinity no other concluding term will be found except 5 From 6 too in the same fashion 6 and no other will be the con cluding term and so 1 likewise is potentially spherical and recurrent for as is reasonable it has the same property as the spheres and circles For each one of them circling and turning around ends where it begins And so these numbers afore aid are the only ones of the products of equal factors to return to the same starting point from which they began in the course of all their increases If they increase in the manner of planes in two dimensions they are called circular like 1 25 and 36 derived from 1 times 1 5 times 5 and 6 times 6 but if they have three dimensions or are multiplied still further than this they are called spherical solid numbers for example 1 125 216 or again 1 625 1 296

## CHAPTER XVIII

*[1]* Regarding the solid numbers this is for the present sufficient The physical philosophers however and those that take their start with mathematics call the same and the other [3] the principles of the universe and it has been shown that "the same inheres in unity and the odd numbers to which unity gives specific form and to an even greater degree in the squares made by the continued addition of odd numbers because in their sides they share in equality while "the other' inheres in 2 and the whole even series which is given specific form by 2 and particularly in the heteromecic numbers which are made by the continued ad dition of the even numbers because of the share of the original inequality[3] and otherness which they have in the difference between their sides Therefore it is most necessary further to demonstrate how in these two as in origins and seeds there are potentially existent all the peculiar properties of number of its forms and subdivisions of all its relations of polygonals and the like

*[2]* First however we must make the dis tinction whereby the oblong (promecic) number differs from the heteromecic The heteromecic is as was stated above [4] the product of a number multiplied by another larger than the first by 1 for example 6 which is 2 times 3 or 12 which is 3 times 4 But the oblong is simi larly the product of two differing numbers dif fering however not by 1 but by some larger number as 2 times 4 3 times 6 4 times 8 and similar numbers which in a way exceed in length and overstep the difference of 1

*[3]* Therefore since squares are produced from the multiplication of numbers by their own length and have their length the same as their breadth properly speaking they would be called idiomecic or tautomecic for exam ple 2 times 2 3 times 3 4 times 4 and the rest And if this is true they will admit in every way of sameness and equality and for this reason are limited and come to an end for the equal and the same are so in one definite way But since the heteromecic numbers are produced by the multiplication of a number by not its own but another number s length they are therefore called heteromecic and admit of infinity and boundlessness

*[4]* In this way then all numbers and the

[1] Cf II 6 3
[2] Cf Aristotle, *On the Soul* 406^b 13
[3] Cf II 17 2
[4] Cf II 17 1

objects in the universe which have been created with reference to them are divided and classified and are seen to be opposite one to another and well do the ancients at the very beginning of their account of Nature make the first subdivision in their cosmogony on this principle Thus Plato[1] mentions the distinction between the natures of the same and "the other and again that between the essence which is indivisible and always the same and the one which is divided and Philolaus says that existent things must all be either limitless or limited or limited and limitless at the same time, by which it is generally agreed that he means that the universe is made up out of limited and limitless things at the same time obviously after the image of number for all number is composed of unity and the dyad even and odd and these in truth display equality and inequality sameness and otherness the bounded and the boundless the defined and the undefined

## CHAPTER XIX

*[1]* That we may be clearly persuaded of what is being said namely that things are made up of warring and opposite elements[2] and have in all likelihood taken on harmony—and harmony always arises from opposites for harmony is the unification of the diverse and the reconciliation of the contrary minded—let us set forth in two parallel lines no longer as just previously the even numbers from 2 by themselves and the odd numbers from 1 but the numbers that are produced from these by adding them successively together the squares from the odd numbers and the heteromecic from the even For if we give careful attention *to their setting forth we shall admire their mu*tual friendship and their cooperation to produce and perfect the remaining forms to the end that we may with probability conceive that also in the nature of the universe from some such source as this a similar thing was brought about by universal providence

*[2]* Let the two series then be as follows That of the squares from unity 1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 and that of the heteromecic numbers beginning with 2 and proceeding thus 2 6 12 20 30 42 56 72 90 110 132 156 182 210 240

*[3]* In the first place then the first square is the fundamental multiple of the first heteromecic number the second compared to the second is its sesquialter the third, sesquitertian of the third the fourth sesquiquartan of the fourth then sesquiquintan sesquisextan and so on similarly *ad infinitum* Their differences too will increase according to the successive numbers from 1 the difference of the first terms is 1 of the second 2 of the third 3, and so on Next if first the second term of the squares be compared with the first heteromecic number, the third with the second the fourth with the third and the rest similarly they will keep unchanged the same ratios as before but their differences will begin to progress no longer from 1 but from 2 remaining the same as before and according to the advance observed in the former comparison the first to the first will be the first or root form multiple the second to the second the second sesquialter from the root-form the third to the third the third sesquitertian from the root-form and the succeeding terms will go on in similar fashion

*[4]* Furthermore the squares among themselves will have only the odd numbers as differences the heteromecic even numbers And if we put the first heteromecic number as a mean term between the first two squares the second between the next two the third between the two following and the fourth between the two next succeeding therein will be seen still more regularly the numerical relations in groups of three terms For as 4 is to 2 so is 2 to 1 and as 9 is sesquialter to 6 so is 6 to 4 and as 16 to 12 so is 12 to 9 and so on with both numbers and ratios regularly advancing As the greater is to the mean so will the mean be to the lesser and not in the same ratio but always a different one by an increase In all the groupings *too the product of the extremes is equal* to the square of the mean and the extremes plus twice the mean by exchange will always give a square What is neatest of all from the addition of both there comes about the production of the triangles in due order showing that the nature of these is more ancient[1] than the origin of all things thus 1 plus 2 2 plus 4 4 plus 6 6 plus 9 9 plus 12 12 plus 16 16 plus 20 and by this process the triangles which give rise to the polygons come forth in order

## CHAPTER XX

*[1]* Still further every square plus its own side becomes heteromecic or by Zeus if its side is subtracted from it Thus the other' is con

[1] Cf Plato *Timaeus* 35

[2] Plato *Timaeus* 30

[1] Cf II 17 3 18 1 and II 7 4 cf 12 8

ceived of as being both greater and smaller than the same since it is produced both by addition and by subtraction in the same way that the two kinds of inequality[1] also the greater and the less have their origin from the application of addition or subtraction to equality [2] This also is sufficient evidence that the two forms partake of sameness and otherness of otherness in an indefinite fashion but of sameness definitely 1 and 2 generically but the odd of sameness after the manner of a subordinate species because it belongs to the same class as 1, and the even of otherness because it is homogeneous with 2

[3] There is also a still clearer reason why the square since it is the product of the addition of odd numbers is akin to sameness and the heteromecic numbers to otherness because it is made up by adding even numbers for as though they were friends of one another these two forms share in their two rows the same differences when they do not have the same ratios and conversely the same ratios when they do not have the same differences For the difference between 4 and 2 in the double ratio is found between 6 and 4 as a superparticular and again the difference between 9 and 6 as a sesquialter is found between 12 and 9 as a sesquitertian and so on What is the same in quality is different in quantity and just the opposite what is the same in quantity is different in quality [4] Again it is clear that in all their relations the same difference between two terms will necessarily be called fractions with names that differ by 1 and be the half of one and the third of the other or the third of one and the quarter of the other or the fourth of one and the fifth of the other and so on

[5] But what will most of all confirm the fact that the odd and never the even is preeminently the cause of sameness is to be demonstrated in every series beginning with 1 following some ratio for example the double ratio 1 2 4 8 16 32 64 128 256 or the triple 1 3 9 27 81 243 729 2 187 and as far as you like You will find that of necessity all the terms in the odd places in the series are squares and no others by any device whatsoever and that no square is to be found in an even place

But all the products of a number multiplied twice into itself that is the cubes which are extended in three dimensions and seen to share in sameness to an even greater extent are the product of the odd numbers not the even 1 8 27 64 125 and 216 and those that go on analogously, in a simple unvaried progression as well For when the successive odd numbers are set forth indefinitely beginning with 1 observe this The first one makes the potential cube the next two added together the second the next three the third the four next following the fourth the succeeding five the fifth the next six the sixth and so on

[1]Cf I 17 6

## CHAPTER XXI

[1] After this it would be the proper time to incorporate the nature of proportions a thing most essential for speculation about the nature of the universe and for the propositions of music astronomy, and geometry and not least for the study of the works of the ancients and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting

[2] A proportion then is in the proper sense the combination of two or more *ratios* but by the more general definition the combination of two or more *relations* even if they are not brought under the same ratio but rather a difference or something else

[3] Now a ratio[2] is the relation of two terms to one another and the combination of such is a proportion so that three is the smallest number of terms of which the latter is composed although it can be a series of more subject to the same ratio or the same difference For example 1 2 is one ratio where there are two terms but 2 4 is another similar ratio hence 1 2 4 is a proportion for it is a combination of ratios or of three terms which are observed to be in the same ratio to one another [4] The same thing may be observed also in greater numbers and longer series of terms for let a fourth term 8 be joined to the former after 4 *again in a similar relation the double and then* 16 after 8 and so on

[5] Now if the same term one and unchanging is compared to those on either side of it to the greater as consequent and to the lesser as antecedent such a proportion is called continued for example 1 2 4 is a continued proportion as regards quality[3] for 4 2 equals 2 1 and conversely 1 2 equals 2 4 In quantity 1 2 3 for example is a continued proportion for as 3 exceeds 2 so 2 exceeds 1 and conversely as 1 is less than 2 by so much 2 is less than 3

[6] If however one term answers to the lesser term and becomes its antecedent and a

[2]Cf Euclid *Elements* V init

[3]Cf II 22 2 23 4 below

greater term and another not the same takes the place of consequent and lesser term with reference to the greater such a mean and such a proportion is called *no longer continued* but disjunct *for example* as regards quality, 1 2 4 8 for 2 1 equals 8 4 and conversely 1 2 equals 4 8 and again 1 4 equals 2 8 or 4 1 equals 8 2 and in quantity, 1 2 3 4 for as 1 is exceeded by 2 by so much 3 is exceeded by 4 or as 4 exceeds 3 so 2 exceeds 1 and by interchange as 3 exceeds 1 so 4 exceeds 2 or as 1 *is exceeded* by 3 by so much 2 *is exceeded* by 4

## CHAPTER XXII

*[1]* The first three proportions then which are acknowledged by all the ancients Pythagoras Plato and Aristotle are the arithmetic geometric and harmonic and *there are three others subcontrary to them* which do not have names of their own but are called *in more general* terms the fourth fifth and sixth forms of mean after which the moderns discover four others as well making up the number ten which according to the Pythagorean view is the most perfect possible It was in accordance *with this* number indeed *that not long ago the* ten relations *were observed to* take their proper number the so-called ten categories the divisions and forms of the extremities of our hands and feet and countless other things which we shall notice in the proper place

*[2]* Now however we must treat *from the* beginning *first that form of* proportion which by *quantity reconciles* and binds *together the* comparison of the terms which is a quantitative equality as regards the difference of the several terms to one another This would be the arithmetic proportion for it was previously reported that quantity is *its* peculiar belonging *[3]* What then *is* the reason *that we shall* treat of this first *and not* another? Is it not clear that Nature shows it forth before the rest? For in the natural series of simple numbers beginning with 1 with no term passed over or omitted the definition of this proportion alone is preserved moreover in our previous statement [1] *we demonstrated* that the *Arithmetical Introduction* itself is antecedent to all the others because it abolishes them together with itself but is not abolished together with them and because it is implied by them but does not imply them Thus it is natural *that the mean* which shares the name of arithmetic will not unreasonably take precedence of the means which are named for the other sciences the geometric and harmonic for it is plain that all the more *will it take precedence* over the *subcontraries* over which the first three *hold* the leadership *[4]* As the first and original therefore since it is most deserving of the honor let the arithmetic proportion have its discussion at our hands before the others

## CHAPTER XXIII

*[1]* It is an arithmetic *proportion* then, whenever *three or more terms* are set forth in succession or are so conceived and the same quantitative difference is found to exist between the successive numbers but not the same ratio among the terms, one to another For example 1 2 3 4, 5 6 7 8 9 10 11, 12 13 *for* in this natural series of numbers examined consecutively *and without* any *omissions every* term whatsoever is *discovered to* be placed between two and to preserve the arithmetic proportion to them For its differences as compared with those ranged on either side of *it are* equal the same ratio however is not preserved among them

*[2]* And we understand *that in such a* series *there comes about both a continued* and a disjunct proportion for if the same middle term answers to those *on* either side as both antecedent and consequent it would be a continued proportion but *if* there is another mean *along* with it a disjunct proportion comes about

*[3] Now if we separate* out of this *series any* three consecutive terms whatsoever after the form of the continued *proportion* or four or more terms after the disjunct form and consider them the difference of them all would be 1 but their ratios would be different throughout *If however again we* select three *or more terms* not adjacent but *separated* separated nevertheless *by a constant* interval if one term was omitted in setting down each term the difference in every case will be 2 and once more with three terms it will be a continued proportion with more disjunct If two terms *are* omitted the difference will *always* be 3 in all *of them continued or disjunct* if three 4 if four 5 and so on

*[4]* Such *a* proportion therefore partakes in equal quantity in its differences but of unequal quality for this reason it is *arithmetic* If on the contrary it *partook of similar quality* *but not quantity* it would be geometric instead *of* arithmetic

*[5]* A thing is peculiar to this proportion

[1] Cf I 4 2 ff

that does not belong to any other namely, the mean is either half of or equal to the sum of the extremes whether the proportion be viewed as continuous or disjunct or by alternation for either the mean term with itself or the mean terms with one another are equal to the sum of the extremes

[6] It has still another peculiarity, what ratio each term has to itself, this the differences have to the differences that is they are equal

Again the thing which is most exact and which has escaped the notice of the majority the product of the extremes when compared to the square of the mean is found to be smaller than it by the product of the differences whether they be 1 2, 3 4 or any number whatever

In the fourth place a thing which all previous writers also have noted the ratios between the *smaller terms are larger as compared to those between the greater terms* It will be shown that in the harmonic proportion on the contrary the ratios between the greater terms are greater than those between the smaller for this reason the harmonic proportion is subcontrary to the arithmetic and the geometric is midway between them as it were between extremes for this proportion has the ratios between the greater terms and those between the smaller equal and we have seen that the equal is in the middle ground between the greater and the less So much then about the arithmetic proportion

## CHAPTER XXIV

[1] The next proportion[1] after this one the geometric is the only one in the strict sense of the word to be called a proportion because its terms are seen to be in the same ratio It exists whenever of three or more terms as the greatest is to the next greatest so the latter is to the one following and if there are more terms as this again is to the one following it but they do not however differ from one another by the same quantity but rather by the same quality of ratio the opposite of what was seen to be the case with the arithmetic proportion

[2] For an example set forth the numbers beginning with 1 that advance by the double ratio 1 2 4 8 16 32 64 and so on or by the triple ratio 1 3 9 27 81 243 and so on or by the quadruple or in some similar way In each one of these series three adjacent terms or four or any number whatever that may be taken will give the geometric proportion to one another as the first is to the next smaller so is that to the next smaller and again that to the next smaller and so on as far as you care to go and also by alternation For instance, 2 4 8 the ratio which 8 bears to 4 that 4 bears to 2, and conversely they do not, however have the same quantitative difference Again 2, 4 8, 16 for not only does 16 have the same ratio to 8 as before though not the same difference but also by alternation it preserves a similar relation—as 16 is to 4 so 8 is to 2 and conversely as 2 is to 8 so 4 is to 16 and disjunctly as 2 is to 4 so 8 is to 16 and conversely and in disjunct form as 16 is to 8 so 4 is to 2 for it has the double ratio

[3] The geometric proportion has a peculiar property shared by none of the rest that the differences of the terms are in the same ratio to *each other as the terms to those adjacent to them the greater to the less and vice versa* Still another property is that the greater terms have as a difference with respect to the lesser the lesser terms themselves and similarly difference differs from difference by the smaller difference itself if the terms are set forth in the double ratio in the triple ratio both terms and differences will have as a difference twice the next smaller in the quadruple ratio thrice in the quintuple four times and so on

[4] Geometric proportions come about not only among the multiples but also among all the superparticular, superpartient and mixed forms and the peculiar property of this proportion in all cases is preserved that in the continued proportions the product of the extremes is equal to the square of the mean but in disjunct proportions or those with a greater number of terms even if they are not continued but with an even *number of terms* that the product of the extremes equals that of the means

[5] As an illustration of the fact that in all the relations all kinds of multiple superparticulars superpartients and mixed ratios the peculiar property of this proportion is preserved let that suffice and be sufficient for us wherein we fashioned beginning with equality by the three rules all the kinds of inequality out of one another when they were in both direct and reverse order for each act of fashioning and each series set forth is a geometric proportion with all the aforesaid *properties* as well as a fourth namely that they keep the

[1] Cf Euclid *Elements* VII Def 21

[2] Cf I 23 7 ff

sponse for diagnostic purposes (1175). Dehydroisoandrosterone excretion may aid in the differential diagnosis of adrenal hyperfunction (370). Similarly, increased urinary pregnanediol and pregnanetriol, in cases of adrenal hyperplasia, has been reduced by cortisone therapy (100) and urinary estrogens may also be reduced (371, 771). The therapeutic use of cortisone to depress adrenal secretion in nonmalignant cases has been encouraging.

Adrenal tumors in the mouse have been discussed in a review by Woolley (1237), and the adrenal genital relationship reviewed by Zuckerman (1262).

## B. Ovarian Disturbances

The use of cortisone in various ovarian disturbances and its use in cases of sterility have been described (598). ACTH caused a two- to eight-fold increase in urinary output of gonadotropins in eight of nine patients treated (1042). From the studies on adrenalectomized patients it appears that the adrenals normally produce estrogens (203). In rats, cortisone caused a slight augmentation of the increase in ovarian weight after administration of chorionic gonadotrophin (330). Cortisone alone did not exert any sex hormone or gonadotrophic activity (787). Szego found that the acute stimulation of the uterus by estrogens could be antagonized by cortisone or hydrocortisone but not by DCA. Such action was independent of pituitary function (1128). Others observed a similar action using ovariectomized rats (1132). In castrated male guinea pigs cortisone in small doses exerted a stimulating estrogen-like action on nipple growth. Larger doses, however, inhibited nipple growth (1202). The correlation between ovarian and adrenal cyclic function in guinea pigs has been suggested as related to the estrogen stimulation of the anterior pituitary to release ACTH (1259). In the dog cortisone did not exert androgenic effects nor did it antagonize the action of testosterone propionate on the prostate (1132). A recent detailed review containing extensive experimental data on the influence of the adrenals on the reproductive system has been published by Moore (787).

## C. Maintenance of Lactation

The controversy over the effectiveness of DCA or cortisone and 11-dehydrocortisone in maintaining lactation in adrenalectomized rats has been reinvestigated by Cowie. Whereas no agent alone was completely effective, it was found that in the rat cortisone pellets (2 of 11 mg.) and DCA (1 of 50 mg.) allowed complete restoration of lactation (184). The relationship between the adrenal cortex and the mammary gland has been reviewed by Folley (333).

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sponse for diagnostic purposes (1175) Dehydroisoandrosterone excretion may aid in the differential diagnosis of adrenal hyperfunction (370) Similarly, increased urinary pregnanediol and pregnanetriol in cases of adrenal hyperplasia has been reduced by cortisone therapy (100) and urinary estrogens may also be reduced (371, 771) The therapeutic use of cortisone to depress adrenal secretion in nonmalignant cases has been encouraging

Adrenal tumors in the mouse have been discussed in a review by Woolley (1237), and the adrenal genital relationship reviewed by Zuckerman (1262)

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CHAPTER XV

# Clinical Endocrinology

By K. E. PASCHKIS AND A. E. RAKOFF

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## I Introduction

The task of reviewing, within the framework of this book recent progress in clinical endocrinology is not easy In view of the extensive literature we had to impose upon ourselves certain limitations It would hardly appear likely that the student or the practicing physician would turn for detailed practical guidance, to a book dealing quite predominantly with the chemistry and physiology of the hormones Rather the biochemist and physiologist would seek here information on the clinical applications of the scientific achievements in his field In fact, the line between physiological and clinical investigations is being reduced almost to the vanishing point We are therefore presenting advances in principles and approaches rather than details of procedures or of therapeutic results

## II Anterior Pituitary Gland

### A General Remarks

In the not so distant past the anterior pituitary was supposed to secrete some twenty different hormones For every and any effect of crude pituitary extracts a separate hormone entity was postulated and a name coined Chemical isolation procedures have yielded six hormones and a better knowledge of metabolic interrelations has led to an understanding of multiple effects of individual single hormones

The anterior pituitary then produces six hormones growth hormone (also called somatotropic hormone) thyrotropic hormone (TSH), adrenocorticotropic hormone (ACTH) and the three gonadotropic hormones follicle stimulating hormone (FSH) luteinizing or interstitial cell stimulating hormone (LH ICSH) and luteotropic hormone (LTH), identical with lactogenic hormone

At present the trend appears to reverse itself Observations have been recorded suggesting the existence of two adrenocorticotropic hormones (one responsible for adrenocortical growth another for secretory function) (8, 125) as well as of two thyrotropic hormones (73) Regarding growth hormone a separation of the diabetogenic factor from the protein anabolic growth hormone has been reported (169) These problems are under current investigation and cannot as yet be evaluated from the standpoint of the physiologist or clinician Even if the fractionations and chemical separations mentioned above should be borne out by further work the question will have to be answered whether the pituitary gland at any time secretes these factors separately or whether it secretes a complex hormone with different actions residing in different parts of the molecule, parts which could be separated *in vitro*

Regarding ACTH, various preparations have been made and have be

come available for clinical use The action of these preparations is qualitatively identical A highly purified preparation designated ACTX (Corticotropin A) is much less subject to destruction than is ACTH (the crude preparation) when administered intramuscularly Actide corticotropin (Corticotropin B) is a preparation obtained by peptic hydrolysis of ACTH It evidently represents the active peptides of the larger ACTH molecule It is readily inactivated or destroyed in the extravascular tissues and at present appears to offer little advantage for clinical use (239)

As far as the six hormone entities are concerned *it appears that their* secretion is to a large extent independent and that the gland does not function as a unit There is no panhyperpituitarism hyperfunction of the entire anterior pituitary, nor is there a concerted hypofunction panhypopituitarism except that due to destruction of the glandular tissue As a matter of fact, interrelations of secretion of individual pituitary hormones appear to be the result of retrograde action upon the pituitary from secretions of the secondary glands Examples of this type of interrelation may be the dependence of growth hormone secretion upon thyroid hormone secretion and the release of LH through changes in estrogen levels

The independence of secretion of individual pituitary hormones has become more intelligible through the discovery of individual cell types responsible for elaboration of the individual hormones Traditionally the hormone secreting cells of the anterior pituitary are classified on the basis of the staining properties of their granules and of the morphologic characteristics of the Golgi apparatus as acidophilics (eosinophilics) and basophilics The elaboration of FSH TSH and ACTH is tentatively ascribed to the basophilics and that of the other three hormones to the eosinophilics It has however become evident that basophilic staining is the common property of specialized cells, differentiated by histochemical reactions A basophilic FSH producing cell is different from the basophilic TSH producing cell (78 79 165 166) No similar recent differentiations have been reported within the eosinophilic cells but an older observation of carminophilic cells in the cat pituitary believed responsible for LH production points in the same direction (60)

## B Acromegaly and Gigantism

The clinical picture of gigantism and acromegaly was outlined in *The Hormones* Volume II More information has accumulated regarding the natural history of acromegaly The excessive secretion of growth hormone by the eosinophilic cells occurs periodically Periods of excessive secretion are followed by periods of relative quiescence Such cycles may be repeated several times in the course of years It is of more than academic interest to recognize the active phase of the disease earlier than in retro

spect by increased growth phenomena such as enlargement of feet or separation of teeth Bio assay of growth hormone in the blood has been reported in one case (115) but assay methods which would lend them elves readily to clinical work have yet to be developed Increased levels of serum inorganic phosphate have been found in the active phase of acromegaly, the levels return to normal under treatment with sex steroids as well as with radiation therapy (115)

Diabetes has long been known to be a frequent complication of acromegaly In many respects it appears to be the human equivalent to the growth hormone-induced diabetes of dogs Even the reversibility of early stages of growth hormone-induced diabetes in dogs has been paralleled in diabetes of acromegaly in an acromegalic, observed and studied repeatedly for several years a diabetic glucose tolerance made its appearance and reverted to normal following intensive roentgen irradiation of the pituitary (152) It has been pointed out that diabetes in acromegaly is more frequent in acromegalics in whose families non acromegalic members have diabetes This would tentatively suggest that an inherited 'vulnerability of the islets ( doglike' islets) is a prerequisite for the growth hormone to exert its diabetogenic effect The bearer of nonvulnerable' islets would respond much like those species (*e g*, the rat) in whom diabetes cannot readily be induced by growth hormone

It has long been known that goiter is a frequent occurrence in acromegalics Also in many cases a high BMR is recorded (39) In view of the well known stimulation of the thyroid by anterior pituitary TSH the obvious assumption was that in this type of hyperpituitari m an increased secretion of TSH was associated with the increased secretion of growth hormone and that the former caused goiter and increased function of the thyroid The situation is undoubtedly more complicated and the whole story has yet to be told Cushing and his collaborators pointed out early that the histologic appearance of goiters of acromegalics differed markedly from that in thyrotoxicosis also thyroidectomy caused regression of the elevated BMR and of other manifestations only in exceptional cases of acromegaly (39) More recently it has been found that the I 131 uptake of the thyroid gland of acromegalics was normal regardless as to whether there was a goiter or whether the BMR was elevated or normal (136) These observations have suggested as did the older clinical observations of Cushing and associates that the high BMR of many acromegalics may be due to factors other than increased thyroid function (136) TSH levels in the serum of acromegalics have been found elevated regardless of the presence or absence of goiter and of the levels of the BMR (42) Neither the significance of this finding in the pathophysiology of acromegaly nor

the cause of excessive secretion of TSH (believed to be secreted by the basophilic anterior pituitary cells) is at present understood

*Treatment* In the opinion of most clinicians indication for surgical treatment is not the demonstrable evidence for presence of an adenoma but rather a rapidly progressive encroachment of the optic nerves Other cases should be subjected to roentgen ray treatment its effectiveness has been greatly improved with the use of higher doses of radiation than were previously employed (112) If there is no evidence of increased hormonal activity nor evidence of expansive growth of an adenoma treatment should be withheld Treatment with androgens or with androgens and estrogens has been tried in an attempt to suppress growth hormone secretion through the suppressive action of the steroid hormones upon the anterior pituitary With this form of therapy return of the elevated serum inorganic phosphorus levels to normal as well as improvement of the diminished carbohydrate tolerance has been observed but it is as yet too early to judge the practical usefulness of steroid hormone therapy in acromegaly (115)

## C Hypopituitarism

Destruction of the anterior pituitary resulting from tumor (chromophobe adenoma parapituitary tumors *e g* craniopharyngioma) or post partum necrosis causes panhypopituitarism Depending on the amount of pituitary tissue destroyed the severity of the disorder may vary from complete loss of pituitary function to milder forms It is interesting that in the incomplete forms not all pituitary functions are diminished or abolished in the same proportions (151) It is our impression that the gonadotropin secretion is most sensitive and is usually lost first The various combinations of functional deficiencies seen in tumor cases in which the nontumorous part of the gland is exposed to compression from the tumor cannot be explained readily by topographic arrangement of the cells responsible for the respective hormone secretion but perhaps rather by differences of vulnerability of these cells to the pressure exerted on all

The essential findings are described in Volume II It should be added that sodium loss evidently due to adrenocortical salt hormone deficiency *is found in severe cases (151) this is of interest because it has been stated* that in rodents salt hormone function of adrenal cortex is not governed by the anterior pituitary The lack of response of the adrenal cortex to ACTH mentioned in Volume II pertains only to single intramuscular injections With repeated injections and especially with intravenous drip infusion the atrophic adrenal can be stimulated and evidence of function be obtained (rise of 17 ketosteroid and 11 oxysteroid excretion eosinopenia) We have been able to stimulate the adrenal cortex 18 years after

post partum necrosis (152) The therapeutic regime as outlined in Volume II now includes cortisone and hydrocortisone

1 *Single Hormone Hypopituitarism*

In contradistinction to panhypopituitarism due to destruction of the anterior pituitary the secretion of individual single hormones may be deficient, in the presence of normal secretion of the others

Hypogonadotropism occurs both in males and females and causes clinical pictures indistinguishable from that due to primary gonadal failure The distinction is made chiefly by assay of urinary levels of gonadotropin excretion which is high in primary gonadal failure and absent in hypogonadotropism This will be further discussed in the chapters on Testes (VII) and Ovaries (X)

Hypothyrotropism is indistinguishable clinically from hypothyroidism due to primary thyroid failure A distinction based on thyrotropin assay, analogous to the gonadotropin studies mentioned above, can be attempted only in laboratories equipped for TSH assays In view of the difficulties of this assay procedure the two types of hypothyroidism are usually differentiated by the response to injections of TSH (102, 159) An increase of I 131 uptake, protein bound iodine (PBI), or BMR can be expected only if the thyroid gland is intact, and can therefore be accepted as evidence of hypothyrotropism

There is at present no information as to whether these single hormone deficiencies of the anterior pituitary are the result of an anatomical derangement or whether the lack of secretion of the respective hormones is due only to a functional disturbance

The growth hormone deficiency causing pituitary dwarfism of the Lorain Levy type does not belong strictly in the category of single hormone deficiencies because in this disorder growth hormone deficiency is usually associated with gonadotropin deficiency and consequent hypogonadism In exceptional cases however gonadal structure and function are normal (88 180) and these cases legitimately can be considered under the heading of single hormone deficiencies As is discussed below under the heading of pituitary dwarfism there is evidence for morphological changes in the eosinophilic cells of the anterior pituitary

2 *Pituitary Dwarfism*

Dwarfism due to lack of growth hormone secretion is seen in children in whom the anterior pituitary is destroyed or encroached upon by tumors in most instances craniopharyngiomas Almost always the growth deficiency in these cases is associated with deficiencies of the tropic hormones, in other words the growth deficiency is part of panhypopituitarism In

cases of pituitary dwarfism without clinically or radiologically demonstrable pituitary lesion (Lorain Levy type) the diagnosis is difficult In most cases hypogonadism is present (but of course recognizable only after the age of puberty) but normal sexual development and even fertility have been recorded in exceptional cases (88 180) Delayed closure of the epiphyses appears to be typical Increased insulin sensitivity might be expected in view of the anti insulin action of growth hormone however few observations are available and it would appear far from certain whether a normal insulin sensitivity would rule out growth hormone deficiency in the presence of normal adrenocortical function The pituitary dwarfism of the Loram Levy type may be compared with the dwarfism of the dwarf Silver mouse In the latter dwarfism and hypogonadism occur as a recessive hereditary trait (188 189) The pituitary shows absence of eosinophilic cells A marked diminution of the number of eosinophilic cells and degenerative changes in those present have been reported in a case of pituitary dwarfism in a woman (88)

Whereas most growth hormone preparations have been found to be entirely ineffective in man inducing neither N retention nor growth in length (in spite of good potency in the rat assay) positive results have been obtained recently by several investigators (114 184) The active growth hormone preparations are not yet commercially available

## II Thyroid Gland

### A Diagnostic Procedures

A considerable amount of work during the last several years has been concerned with diagnostic methods based on the role of the thyroid gland in iodine metabolism The interest in these aspects has overshadowed that in the older procedures especially in the determination of the BMR Conditions other than those of abnormal thyroid function are known to effect the oxygen consumption In clinical diagnostic use one of the most common and most disturbing factors is that of emotional unrest causing high oxygen consumption probably by release of epinephrine Techniques have been introduced for performing the test in barbiturate induced sleep or anesthesia under such conditions the high oxygen consumption resulting from nervous factors is reduced to normal values whereas the high BMR due to hyperthyroidism is not appreciably affected (13 172)

The use of I 131 tracer studies and PBI determinations has added materially to the knowledge of the pathophysiology of thyroid disease and these procedures have proved invaluable as diagnostic tools Obviously a study of these parameters of thyroid function gives no information regarding the target response of the body tissues This is highlighted by the unex

plained syndrome of "hypometabolism" in which the basal oxygen consumption may be as low as in severe hypothyroidism, but in which the manufacture and release of thyroid hormone is entirely normal, as are all other body functions dependent on thyroid hormone.

## 1. *Tracer Studies*

The physiological basis for the diagnostic use of I 131 tracers has been discussed in Volume II. In many clinics the uptake is determined 24 hours after an oral tracer dose by measuring the radiation over the thyroid region. It has generally been confirmed that the uptake is high (more than 40%) in hyperthyroidism and low (less than 10%) in hypothyroidism. The tracer dose employed depends largely upon the sensitivity of the measuring instruments and of course on the particular parameters to be studied. With the use of scintillation counters the dose can be as low as 1 to 10 microcuries (3). Urinary excretion studies can supplement the measurements over the thyroid, being inversely related to the latter. Most of the iodine not taken up by the gland is rapidly excreted in the urine. It should be remembered that conditions other than thyrotoxicosis cause increased uptake of I 131: the high uptake of iodine-deficiency goiter, long postulated on the basis of animal experiments, has now been demonstrated in an endemic goiter population not exposed to iodine prophylaxis such as the use of iodized table salt (198). The high uptake in some goitrous cretins is discussed on page 837. Drug induced situations of internal iodine starvation also may cause a temporary rise of uptake considerably above normal. This is the case in individuals in whom the iodine trapping mechanism was blocked by thiocyanate with development of goiter. Following withdrawal of thiocyanate the uptake by the iodine starved goitrous gland is very high. A similar situation occurs after withdrawal of thiouracils in experimental animals but has not been definitely observed in man (41).

Conversely, low uptakes occur in conditions other than hypothyroidism. Antithyroid drugs (thiourea derivatives, imidazoles) as well as thiocyanate block the uptake, though in different ways. Thiocyanate truly blocks iodide uptake under ordinary conditions; only the presence of very large amounts of iodide would force the latter into the gland through the block. The antithyroid drugs do not materially interfere with the iodide concentrating mechanism but inhibit incorporation of molecular iodine into the amino acid. Since 24 hours after the tracer dose mainly the protein bound moiety remains, the result at this time interval is one of decreased uptake. Low uptake values are also obtained during and for several weeks after medication with thyroid hormone, evidently through suppression of TSH secretion from the anterior pituitary. Administration of large amounts of iodine either as iodide or as organically bound compound

(as used in roentgenographic studies of cholecystograms uterosalpingograms urograms bronchograms encephalograms) also causes low tracer uptakes Two factors may be involved The iodide may cause a real block perhaps by inactivating TSH Furthermore if the iodide pool (circulating and extracellular) is considerably increased the tracer is diluted much more than in an average pool the uptake of the tracer may thereby be smaller owing to dilution In renal disease the urinary excretion of iodide is diminished both because of impaired glomerular filtration and increased tubular reabsorption Tracer studies based on urinary excretion of I 131 would give erroneous results The increased I 127 pool also causes a greater dilution of the administered tracer and the accumulation of I 131 in the thyroid is delayed However, values obtained by direct measurement over the thyroid at 26 hours are within normal range (157)

The e physiological and pharmacological factors influencing I 131 uptake are of interest not only because of the light they throw on the pathophysiology of thyroid disease but also because they are pitfalls in diagnostic interpretation of the results of such studies Another interesting finding is the almost complete obliteration of I 131 uptake in acute thyroiditis without other evidence of hypothyroidism

Several procedures, using I 131 tracers have been introduced which add materially to the information obtained by determining the uptake after 24 hours By repeated measuring of activity over the neck following a tracer dose the concentration of I 131 in the thyroid is found to be a linear function of the square root of time The slope of this curve is called the accumulation gradient (9 74)

The mathematical relationship is simple but rather obscure in terms of biological significance The linear relationship just mentioned is found in euthyroid and hypothyroid individuals but not in hyperthyroidism The method is valuable inasmuch as it establishes the accumulation gradient within 1 to 2 hours after administration of the tracer and permits study of factors influencing the iodine accumulation of the thyroid (168) For example the activity of antithyroid drugs has been screened by this method (200)

Determination of thyroidal and renal plasma I 131 clearance rates has been recommended as a routine diagnostic test Whereas determinations of the 24 hour uptake fail to correlate with the clinical condition in 7% to 10% of cases the claim is made that the clearance rates show no overlapping of values in normal and hyperthyroid individuals (17) It will be interesting to ee whether such a sharp dividing line between euthyroid and hyperthyroid individuals will be confirmed on a larger material which will include mild cases of hyperthyroidism It may appear questionable to some whether there is biologically a sharp line of demarcation between a thyroid

gland with 'top normal' function and that functioning slightly in excess of normal.

Another parameter of thyroid function which can be studied by the use of I 131 tracers is that of release of thyroid hormone from the gland. By daily *direct determination of the radiation over the thyroid for several days* a measure for the secretion of thyroid hormone can be obtained (9). This is also referred to as 'physiological decay,' and the time during which the gland loses 50% of its activity (corrected for physical decay) is called the biological half life. For more precise determinations it is necessary to administer antithyroid drugs (thiouracil derivatives or imidazole derivatives) after the tracer dose in order to block uptake of that iodide moiety originating from the secreted thyroid hormone (9). Because of the necessary length of observation period this method is not generally used in clinical practice. Measurements of protein bound iodine I 131 following a tracer dose reflect the secretory activity of the thyroid (186). Under certain circumstances the values of $PBI_{131}$ do not correlate with the clinico-physiological status of the individual. In some cases of postoperative myxedema the $PBI_{131}$ values have been found high in the presence not only of clinical manifestations of hypothyroidism but also of low $PBI_{127}$ values. This has been thought to indicate a very small but very active thyroid remnant with rapid turnover rate but small total output (19). By determining total plasma I 131 and protein bound plasma I 131 a conversion ratio has been calculated by the formula conversion ratio = $\frac{PBI_{131}}{\text{total plasma I 131}} \times 100$. This ratio is believed to be related to the rate at which the gland utilizes iodine for hormone synthesis and release. There is some doubt as to the validity of such a simple formula on physiological grounds; correlation with the clinical condition has been found only in hyperthyroidism, whereas the conversion ratio of hypothyroid and euthyroid individuals did not differ materially (183).

*A further application of the tracer technique made possible through the* introduction of the highly sensitive scintillation counter is the scanning procedure (68, 69). One or several scintillation counters are moved by a motor driven device across the thyroid region; the impulses are transmitted to a stylus which records them on graph paper. Such 'scintigrams' record the outlines of the thyroid gland and have been used for determining thyroid weight as a basis for calculating therapeutic doses (69). Inactive and hyperactive nodules can be demonstrated inasmuch as the scintigram will show regions of diminished or absent density or regions of increased density (68). The procedure has furthermore been used as an aid in differential diagnosis of masses of the neck and by scanning body regions other than the neck for detection of thyroid cancer metastases (10, 195).

## 2 *Protein Bound Iodine (PBI)* (207)

Methods for determination of PBI have been improved a discussion of these methods is beyond the scope of this review In interpreting PBI values as indicative of abnormal thyroid function, sources of error other than those of a technical nature must be kept in mind The most important error results from the presence in the body fluids of large amounts of nonhormonal iodine This may be the result of ingestion of large amounts of iodide organic iodine compounds used as contrast media in various diagnostic procedures (cholecystography pyelography bronchography encephalomyelography, utero alpingography) cause even greater and more prolonged spuriously high PBI values It appears that in the presence of large amounts of iodide in the serum some iodine will come down with the protein precipitate causing high yields in this fraction High PBI levels are regularly found in pregnancy (120) This finding poses an intriguing question Evidently the normal pregnant woman exhibits no signs of hyperthyroidism Is this because the body needs more thyroid hormone in pregnancy? Spuriously low PBI levels have been obtained during the use of mercurial diuretics, during the technical procedures of digestion insoluble mercuric iodates or iodides are formed (139)

### B Thyrotoxicosis

No progress has been made regarding the etiology of thyrotoxicosis

Advances in diagnostic methods are discussed on page 827 Diagnosis is not difficult in fairly typical cases certain types of patients pose a more difficult diagnostic problem in which the use of the newer procedures outlined above are of particular value One is the thyrocardiac individual presenting cardiac failure frequently without a history typical of thyrotoxicosis nor are the clinical manifestations unequivocal Frequently a failure to respond to digitalis first arouses suspicion Signs and symptoms such as auricular fibrillation tachycardia and nervousness can be the result of the cardiac condition and therefore help little in establishing the diagnosis of thyrotoxicosis In the presence of cardiac failure and passive congestion a true basal metabolic rate cannot be determined PBI and tracer studies have been of great diagnostic value in these cases Control of the thyrotoxicosis will improve the cardiac condition significantly Whether thyrotoxicosis can cause failure and consequent passive congestion of an otherwise entirely normal heart is debatable The fact that the majority of 'thyrocardiac patients are in the older age group would lend support to the assumption that thyrotoxicosis causes failure only of a damaged heart The argument loses practical significance if one keeps in mind that the pre existing cardiac damage (most frequently arteriosclerotic in origin) may be subclinical and may revert to a symptomless stage after the thyrotoxic component is controlled

Another group of thyrotoxic patients in whom the diagnosis can be established only with the aid of the tests described is that of certain severely agitated psychoneurotics It is generally believed that psychic factors can precipitate or aggravate thyrotoxicosis, equally thyrotoxicosis is apt to aggravate or bring into the open psychoneurotic tendencies Regardless of what is cause or effect in an individual case control of thyrotoxicosis if present is of greatest importance Sleeping BMR, PBI, and tracer studies will help to establish the presence or absence of thyrotoxicosis

*Therapy* No basically new approach in the treatment of thyrotoxicosis has evolved However a great amount of experience has accumulated with the use of the methods outlined in Volume II

Numerous antithyroid drugs have been tested Those which have proved most useful and are generally used, are two thiouracil derivatives propylthiouracil and methylthiouracil and an imidazole derivative 1 methyl 2 mercaptoimidazole (trade name Tapazole) They can be used in preparation for surgery or for definitive medical treatment If given preoperatively the increased vascularity and friability of the hyperplastic gland may cause technical difficulties at operation Iodide (*e g* in the form of liquor iodi comp Lugol's solution) causes involution of the hyperplastic gland with decreased vascularity and friability Iodide is therefore administered together with the antithyroid drug for a few days prior to operation The mechanism of action of iodide under these conditions is but poorly understood It may enter the gland but is not oxidized and rapidly leaves the gland again It may possibly inactivate endogenous TSH If the antithyroid drugs are used for definitive medical treatment they should be administered for long periods of time It is advisable to continue maintenance doses for 8 to 12 months after a euthyroid condition has been attained Then treatment is discontinued Fifty to sixty per cent of the patients remain in remission the others relapse It has been possible in some instances to keep up maintenance treatment for several years if for any reason the other forms of therapy surgery or radiation were inadvisable

Toxic reactions are observed in a small number of cases both with propyl and methyl thiouracil and with Tapazole The total incidence is less than 2% in a few instances agranulocytosis has occurred

Experience with radioactive iodine has also been greatly extended There still is considerable difficulty in determining the dose which will reduce thyroid activity to normal but not cause hypothyroidism (see also p 830) In many cases there occurs an exacerbation of the thyrotoxicosis during the first two to three weeks following the administration of I 131 owing to the liberation of stored hormone incident to the destruction of thyroid tissue This exacerbation is rarely serious but thyroid crisis has

been observed We have seen the most gratifying results of I 131 therapy in the thyrocardiac patient but the temporary exacerbation of the thyrotoxicosis may cause the patient to go through a period of more severe cardiac failure

The possibility, that the radiation from therapeutic doses of I 131 may be carcinogenic is considered remote by most investigators However thyroid cancer has been observed in rats treated with I 131 (65) Further more the possibility of late development of radiation induced fibrosis has to be considered Many clinics therefore prefer not to treat thyrotoxicosis with I 131 in individuals younger than 35 to 40 years

1 *Ophthalmopathy*

Interest in the ophthalmopathic facet of Graves disease continues There appears to be no question that the severe ophthalmopathy is caused by a pituitary factor as outlined in Volume II In the older experiments in which exophthalmos had been induced in experimental animals with pituitary extracts this action had been ascribed to TSH or to a pituitary factor closely allied to TSH Dobyns and Steelman have succeeded in separating the exophthalmos producing principle from the thyroid stimulating hormone (49) This still leaves open the question whether the pituitary actually secretes the two factors separately, or whether the separation is a chemical one only The most severe forms of ophthalmopathy are usually observed following thyroidectomy for thyrotoxicosis occasionally they occur during treatment with antithyroid drugs As long as the thyrotoxicosis is uncontrolled the ophthalmic picture is usually less severe in exceptional cases the dissociation of thyrotoxicosis and ophthalmopathy is complete the latter occurring in euthyroid individuals or in spontaneous myxedema, in either case without any indication of a preceding hyperthyroid episode

The early changes in the retrobulbar tissue both in exophthalmos produced experimentally by injections of TSH and in human cases are those of edema the edema fluid is rich in mucopolysaccharides (127) The muscle changes are swelling edema and lymphocytic infiltration (pseudo hypertrophy) Whereas these muscular changes occur quite predominantly in the extrinsic muscles of the eye they are demonstrable to a minor extent also in skeletal muscles

Occasionally severe ophthalmopathy occurs simultaneously with pretibial localized myxedema (38 224) The cutaneous localized infiltration shows the same chemical characteristics as the retrobulbar tissue (227) Injection of hyaluronidase into such lesions has been reported to cause resolution evidently owing to the specific depolymerizing action of this enzyme on the mucopolysaccharides (77 142) in one case of our own

observation we were not able to show any changes following local hyaluronidase injection (152). Similarly hyaluronidase has been injected into the retrobulbar tissue in cases of severe ophthalmopathy but results have been unsatisfactory (77, 107). Because TSH was believed to be the active factor from the anterior pituitary several investigators have studied the TSH levels in blood or the urinary excretion of TSH in ophthalmopathic cases (5, 42, 161). In most instances high blood levels or high urinary excretion was found (5) but this appears not to be invariably the case (42, 161). If the *in vitro* separation of an exophthalmos inducing factor from the thyroid stimulating moiety is confirmed the TSH studies just described would suggest that TSH secretion usually but not invariably parallels that of exophthalmos inducing factor.

The question whether the ophthalmopathy associated with Graves' disease is always and in all cases due to a pituitary factor (so called "thyrotropic ophthalmopathy") can as yet not be definitely answered. Most investigators favor a unitarian etiology for both the severe ophthalmopathy and the ocular manifestations of "classical" Graves' disease. However, the occurrence of two etiologically different types of ophthalmopathy has been suggested, thyrotropic and thyrotoxic in nature (143). In the latter the ocular changes are supposed to be the direct effect of excessive thyroid hormone secretion. If this type actually exists, the eye manifestation would be expected to subside after the thyrotoxicosis is brought under control. In many instances improvement of the eyes following control of the thyrotoxicosis is more apparent than real: the lid retraction (a definitely thyrotoxic manifestation) subsides and with weight gain the cheeks fill out giving the eyes a more normal appearance, whereas the exophthalmos may not recede or actually increase even if slightly. However occasional observations do suggest a real regression of exophthalmos and of paresis of extrinsic ocular muscles.

It is most important to prevent the development of the most severe or malignant forms. Since they occur mostly postoperatively, thyroidectomy should be advised against in thyrotoxic patients in whom severe ophthalmopathy, including swelling of the lids, pain, extensive muscle paresis and edema of the conjunctiva, is present. In rare instances severe deterioration of the ophthalmopathy has been observed during thiouracil therapy. In such cases the thyrotoxicosis should be brought under control very gradually by relatively small doses of antithyroid drugs (propyl thiouracil, Tapazole) or by small repeated doses of I 131 or by external roentgen radiation.

The treatment of the ophthalmopathy is directed to the anterior pituitary. If at the time of treatment the patient is euthyroid or hypothyroid (regardless of whether this was preceded by thyrotoxicosis or not) desic

cated thyroid given to tolerance is frequently beneficial Results appear better, the earlier treatment is started Thyroid hormone probably acts by suppressing secretion of TSH (or an exophthalmos factor) from the anterior pituitary Estrogens in large doses have been used successfully in severe cases they also probably act by suppressing the anterior pituitary Roentgen ray irradiation to the pituitary was up to recently, used mostly as an adjunct treatment of questionable value combined with either thyroid or estrogen medication recently however, cases have been reported in which roentgen radiation to the pituitary alone yielded good results (15)

## 2 *Thyrotoxic Myopathy (141, 167 240)*

It has long been known that some muscle wasting is a common occurrence in thyrotoxicosis The increased (endogenous) creatinuria as well as the increased creatinuria following an exogenous creatine load (decreased creatine tolerance ) is probably the result of the muscle dysfunction induced by the excess circulating thyroid hormone In rare instances of thyrotoxicosis muscle wasting is very much more pronounced and may be the presenting feature of the disease these cases are referred to as thyrotoxic myopathy Doubt has been voiced as to whether this syndrome represents only a quantitative exaggeration of the more moderate muscle involvement commonly seen in many severe thyrotoxic patients it has been suggested that a qualitatively different but as yet unknown process is involved Good recovery is observed when the thyrotoxicosis is brought under control Diagnosis may be difficult inasmuch as the myopathy can dominate the picture to the point of masking the thyrotoxicosis careful history and examination and appropriate tests will aid in establishing the diagnosis When the presence of thyrotoxicosis and of myopathy has been established the cases have to be differentiated from myasthenia gravis The latter disease is occasionally associated with thyrotoxicosis Whether there is any causal relationship is as yet unknown Commonly the response to prostigmine is used to differentiate between myasthenia gravis associated with thyrotoxicosis and thyrotoxic myopathy The former does and the latter does not respond to the drug However the specificity of the response to prostigmine has been questioned and cases responding to a test dose of prostigmine have been reported as thyrotoxic myopathies This must remain a debate in terminology until other tests characteristic for myasthenia gravis are also employed in the study of these cases (response to curare electromyogram etc ) The review of clinical problems in neuromuscular physiology by Denny Brown is helpful in understanding these complex problems (46) In patients simultaneously afflicted with myasthenia gravis and with thyrotoxicosis the myopathy has been reported to show improvement deterioration or no change after

control of the thyrotoxicosis It is trite to state that more studies of this problem are needed

*Cases are on record of simultaneous occurrence of thyrotoxicosis and* periodic familial paralysis (141, 228) As far as can be judged from the reported cases the periodic paralysis appears to be improved but not abolished following control of the thyrotoxicosis

### 3 *Thyroid Crisis*

The exact nature of thyroid crisis or thyroid storm has yet to be completely unraveled The factors most commonly precipitating crisis are *(1) any operations performed upon untreated or incompletely controlled* thyrotoxicosis cases (surgical storm) and (2) severe infections as well as major upsets (medical storm) Whether these factors act as 'stressors' through the adrenal cortex (see below) is at present a matter for speculation *There seems however, little doubt that thyroid crisis represents* thyrotoxicosis at peak severity Why, in this condition the heat regulating processes of the body break down, with the skin becoming hot and dry rather than dripping with perspiration, is not understood It has *long been suggested that an acute adrenocortical insufficiency occurs* during thyroid storm and may in part be responsible for the fatal outcome Treatment with cortisone has been successful and in several cases the development of post operative 'surgical' storm was prevented by early *treatment with ACTH or with cortisone, when an emergency operation* became necessary in uncontrolled thyrotoxicosis (209)

### C CRETINISM

*Cretinism is clinically subdivided into endemic and sporadic cretinism* Prima facie this is a geographical classification of cretinism occurring in goiter areas (iodine deficiency areas) or in nongoitrous regions The endemic cretin is identified with the bearer of a degenerated thyroid usually *but not always goitrous incapable of manufacturing thyroid* hormone The sporadic cretin on the other hand is identified with a congenital defect a thyroid aplasia

Tracer studies with I 131 on cretins believed to have congenital thyroid aplasia have in some instances yielded some small uptake in the region of the thyroid indicating the presence of some thyroid tissue This confirms old histological reports in such cases in which small amounts of thyroid tissue rather than complete absence was reported It will be interesting to see whether studies including the response to TSH of a larger material will detect early cases of TSH hypopituitarism in sporadic cretins

The occurrence of goitrous cretins outside areas of goiter endemics has been described studies reported in some of these cases raise intriguing

problems regarding the pathophysiology The uptake of I 131 is rapid and of a high order (81 122 137 199) such as is usually seen in the thyro toxic gland and has been found in endemic iodine deficiency goiter (198) The utilization of the iodine by the thyroid gland appears to differ in different cases In Stanbury s cases the rapid purging of the gland by thiocyanate indicates that the iodide was not oxidized, and therefore not incorporated into organic compounds (199) The deficiency in these cases was similar to that induced by thiourea and imidazole compounds In contradistinction in McGirr s cases thiocyanate failed to purge the gland suggesting oxidation of iodide with subsequent organic binding This was further supported by the presence of protein precipitable I 131 in the blood It was assumed that this was an abnormal compound which was not identified (137) In Hamilton s cases analysis of thyroid tissue following administration of I 131 yielded radioactive fractions corresponding to thyroxine and diiodotyrosine these studies were done before chromatographic analysis of these compounds had been introduced In these cases it was assumed that the thyroid gland was capable of synthesizing thyroid hormone but not of releasing it (81)

Hurxthal has claimed that some goitrous cretins subsequently and without treatment become euthyroid or even hyperthyroid (94 96) These cases have not been reported in great detail and from the data recorded the diagnosis of cretinism may be questioned

It is now generally recognized that treatment of cretins even when started early and continued uninterruptedly frequently fails to bring about normal mental development mental development improves during therapy, but in many instances does not reach normality The nature of the changes underlying the lack of mental development is not known Recent observations indicate the presence of electroencephalographic abnormalities in untreated cretins (43 217) it has been suggested that the EEG pattern becomes normal under thyroid medication only in those cases in whom mental development is fully restored and not in those who remain mentally deficient in spite of treatment (217) This has not been borne out regularly in our as yet limited experience (47a)

On the basis of the concept of a reciprocal relationship of the pituitary tropic hormones with those of the secondary glands one would expect a high TSH secretion from the anterior pituitary in severe hypothyroidism including congenital thyroaplasia (an analogy could be drawn between the latter and ovarian agenesis associated with high FSH excretion) However this is not always the case In a study of cretins Paschkis and D Angelo found absence of TSH in the blood of an untreated cretin 11 years of age TSH becoming detectable only after thyroid medication This suggests that the anterior pituitary which has not been exposed to

thyroid hormone may be incapable of manufacturing and/or secreting TSH. In untreated cretinous babies the expected high TSH levels have been found. Studies regarding secretion of hormones other than TSH by such pituitaries will be of interest.

Familial incidence of sporadic cretinism has been reported (22) (familial incidence is of course common in endemic cretinism). On the other hand, the occurrence of cretinism in one of two identical twins has been reported, unfortunately very briefly and without detailed data (175).

## D. Goiter

Iodine deficiency is firmly established as the most common cause of goiter (see Chapter V by Salter, in Volume II of *The Hormones*). Interesting historical aspects are touched upon by Salter (177). Certain aspects of the pathophysiology of human iodine-deficiency goiter had not been studied with modern methods because iodine prophylaxis in the form of iodized table salt has been in general use for several decades in the goiter regions and goiter belts of Europe and North America. Recently such studies were carried out in an endemic goiter area in the Argentine Andes where the natural conditions of the iodine-deficiency goiters had heretofore never been interfered with by administration of extra iodide. The great avidity of these goiters for iodine was demonstrated by the very high uptake of tracer doses of I 131 (198). Further results of this study will be awaited with interest.

Colloid goiters clinically and pathologically indistinguishable from those occurring in iodine deficiency regions are observed sporadically in regions such as the Atlantic seaboard where iodine supply in water and food is adequate. Endemic occurrence of goiter in locations not deficient in iodine has long been recorded (e.g. Derbyshire goiter in England) (138). Following the observations of goitrogenic action of cabbage, brassica seeds, and other vegetable matter in rabbits and rat, Astwood isolated an active goitrogen *l*-5-2-vinyl thiooxazolidone from rutabaga and studied the goitrogenic action of a great number of vegetables (6, 7). The chemical composition as well as the goitrogenic action is similar to that of thiourea and its derivatives. The subject has been reviewed by Greer (75) and by Fertman *et al.* (54). Only very few observations in human beings are recorded in whom the presence of a goiter could with great probability be traced to the ingestion of large amounts of goitrogenic vegetables (55, 138). Means has coined the term "struma cibaria" for this type of goiter (138). Careful nutritional histories should be obtained in all goiter cases, especially outside iodine deficiency areas. However, it appears very doubtful to us that the majority of colloid goiters occurring in regions with adequate iodine supply can be explained on this basis.

The etiology of most cases is still obscure an endogenous deficiency in iodine utilization may be thought of hypothetically (76)

In the medical treatment of colloid goiter medication with desiccated thyroid has been revived (76) Thyroid medication evidently acts by suppressing endogenous TSH secretion The latter was previously increased causing enlargement of the thyroid because the normal size gland has for some reason been unable to produce adequate amounts of thyroid hormone

## F Thyroiditis

Whereas the etiology of acute nonsuppurative thyroiditis remains obscure in most instances an interesting physiopathologic aspect has been detected The uptake of tracer doses of I 131 is greatly diminished or absent No other manifestations of hypothyroidism are present PBI values are normal or even slightly elevated The reason for the depressed I 131 uptake is not evident administration of TSH will increase the uptake though to a lesser degree than in euthyroid individuals in thyrotoxicosis factitia or in hypopituitarism Curiously administration of TSH brings about an improvement at least a temporary one of the condition Various hypotheses have been offered in explanation of the low I 131 uptake and the response to TSH but none is entirely satisfactory (176)

Milder cases last only a few days but in more severe cases fever pain and swelling may continue for several weeks In the absence of any knowledge regarding the etiology (bacteriological cultures have been almost uniformly sterile) treatment is symptomatic Antibiotics have been without effect in almost all instances There have been reports of beneficial action of external radiation and of thiouracil therapy more recently cortisone or ACTH has been used successfully their beneficial action is probably due to the anti inflammatory action of the adrenal steroids (230)

### 1 *Subacute Thyroiditis (36a)*

Attention has been drawn to this form of thyroiditis which is not infrequently confused with Riedel s struma A history of an acute phase is frequently obtained but sometimes the onset is insidious and fever and severe general constitutional manifestations are absent The gland is enlarged and very hard it may or may not be tender Even in the absence of fever the sedimentation rate is elevated and the I 131 uptake quite as low as in acute thyroiditis Extensive fibrosis of the surrounding structures characteristic of Riedel s struma is absent The histological picture (surgical or needle biopsy) is characterized by pseudotubercles and

giant cells In contradistinction to Riedel s struma subacute thyroiditis is self limited and responds rather well to roentgen ray therapy, operations are unnecessary The etiology is as obscure as that of other forms of thyroiditis

## IV The Adrenal Gland

### A Adrenocortical Hypofunction

Chronic hypoadrenocorticalism in human beings is due to (1) Destruction of the adrenal cortex by *chronic disease (Addison's disease)*, (2) adrenalectomy performed for hypertension or advanced cancer of prostate or breast, (3) hypopituitarism resulting in lack of stimulation of the adrenal cortex by ACTH, (4) salt losing adrenogenital syndrome ' (see p 851) Acute adrenocortical insufficiency probably plays a role in Waterhouse Friderichsen syndrome (acute fulminant bacteremia mostly meningococcemia, with extensive hemorrhage into the adrenal cortices), and perhaps as a phenomenon of adrenocortical exhaustion in severe infection without hemorrhage into the adrenals

### B Addison s Disease

'Cortical atrophy so called and tuberculosis are the causes of Addison s disease in the vast majority of cases The term atrophy is a misnomer unfortunately ingrained in the clinical literature The anatomical changes in the adrenal cortex in these cases are the result of cytotoxic damage necrosis and fibrosis In a small number of cases histoplasmosis involving the adrenal cortex causes Addison s disease cases of amyloidosis (84 225) leucemic infiltration and extensive tumor metastases are rare occurrences (29) It is believed that severe clinical manifestations occur only after 90% of the adrenal cortex is destroyed

#### 1 *Diagnostic Procedures*

*The absence of response to ACTH has been widely employed*, with ACTH given either in a single intramuscular injection or by intravenous drip infusion The end points studied are the decrease in circulating eosinophils and the rise in urinary 17 ketosteroids and 11 oxysteroids after the more intensive stimulation by continuous infusion The eosinophil response to epinephrine however cannot be used as a test for adrenal responsiveness since it is very doubtful whether in man epinephrine in doses employed actually fires the pituitary adrenal system (213) The most significant tests are those based on the disturbance of electrolyte metabolism The response to NaCl deprivation combined with potassium administration (Cutler Power Wilder) still appears to be the most reliable and simple procedure if it is carried out with due safeguards for the possi

ble precipitation of a crisis (40) The deranged Na and K metabolism can be demonstrated by testing the composition of sweat (35 126) or saliva (57 226) These studies have yielded interesting results regarding hormonal control of electrolyte balance but they would at present appear less suitable than urine studies for diagnostic purposes Tests designed to demonstrate the abnormal electrolyte metabolism are the most significant not only because most of the clinical manifestations of Addison s disease are due to the abnormal electrolyte metabolism but also because only the salt hormone deficiency" is a constant finding in all cases of Addison s disease Disturbances of carbohydrate metabolism are present in the majority of cases in varying severity Severe hypoglycemia after prolonged fast and severe hypoglycemia following the hyperglycemia induced by intravenous glucose administration are indicative of the sugar hormone deficiency In many instances this deficiency is not severe enough to be demonstrable by these procedures Increased sensitivity to insulin ( hypoglycemia irresponsiveness ) is present in the majority of cases Because most Addison patients lapse into hypoglycemic coma when tested with intravenous administration of insulin by the standard procedures modifications of the test have been devised Thorn suggests giving insulin in increasing doses with the meals (214) Engel has devised an insulin glucose test in which the glucose administration prevents severe hypoglycemic manifestations (52) Cases of dissociated Addison s disease are of considerable interest in the presence of the classical picture with pigmentation hypotension weakness and demonstrable defect in Na and K metabolism the response to insulin is normal and low normal urinary excretion of glucocorticoids is present (152) Similarly the very low or absent 17 ketosteroid excretion is not always found in some instances values in the lower range of normal excretion have been reported (131) These dissociated cases cannot be adequately interpreted at this time Even if in man the different hormones are secreted by different zones of the cortex as has been suggested in the rat and mouse it is difficult to visualize a zonal destruction in the cortex No autopsy reports have come to our attention which would throw light on this problem if indeed the pathologist will hold the answer Also follow up studies of such patients will be desirable in order to see whether eventually the dissociated disturbance will in later stages of the disease embrace all hormones resulting in what may be called panhypocorticalism

## 2 *Addison s Disease and Pregnancy*

Before adrenocortical hormones became available for treatment of Addison s disease the occurrence of pregnancy in Addison patients was extremely rare and almost invariably fatal (25) With present therapy

several women have been carried through pregnancy successfully and have given birth to viable children. The most critical periods are the first trimester, during which profuse vomiting may precipitate crisis and the delivery with the incident stress and blood loss. Adrenalectomized pregnant rats have been known not only to tolerate pregnancy but to be maintained easier and better. This was ascribed to the desoxycorticosterone-like action of progesterone. However in human beings there is evidence that the placenta produces corticoids (and probably ACTH). These compounds are not produced in amounts sufficient for complete maintenance but may contribute to make maintenance after the third month easier (2, 99, 150).

### 3. Treatment of Addison's Disease

The use of desoxycorticosterone acetate and salt was discussed in Volume II of *The Hormones*. The compound is now available in "buccal tablets"; resorption is through the mucosa of the oral cavity into the major circulation. Dosage is of necessity less accurate than with administration by injection or pellet implantation; the degree of accuracy or inaccuracy depends entirely upon the intelligent and meticulous co-operation of the patient. Recently desoxycorticosterone trimethylacetate was introduced. This is a "long-acting" preparation permitting of maintenance by one intramuscular injection about every four weeks (215). The required dose of desoxycorticosterone and of salt must be determined empirically for every case; overdosage must be avoided because it causes hypertension, edema, and cardiac failure. Standardization is carried out with the use of body weight, blood pressure, hematocrit and serum electrolyte levels as criteria; one of the most valuable and sensitive effects is that on heart size as determined by roentgenograms. Cortisone is used as an adjunct to the treatment with desoxycorticosterone. Even when treatment with the latter succeeds in normalizing the patient by all objective criteria, additional therapy with small doses of cortisone (12.5 to 25 mg. per day) frequently adds to the sense of well-being. Whether this is due to the action on carbohydrate metabolism or rather to the cerebral "euphorizing" action of cortisone is not immediately apparent. Perhaps the fact may be significant in this respect that the abnormal electroencephalographic pattern of Addison's disease is corrected by cortisone therapy but not by the use of desoxycorticosterone. Perhaps the most important place of cortisone is in prevention of crisis. Patients can be instructed to take extra cortisone if they have a sore throat or any other febrile episode even before they report to their physician. Corticosterone (Compound B of Kendall) has been investigated as a therapeutic agent in Addison's disease but this steroid has not been available for large scale clinical testing (34).

The fact that glycyrrhizinic acid, the active ingredient of Liquorice is capable of correcting the electrolyte disturbance of Addison s di ease is at this time at least less of practical than of theoretical interest (24 71)

Addisonian crisis is treated by intravenous infusion of glucose saline plasma antibiotics and aqueous adrenocortical extract and by intra muscular injections of corti one and deoxycorticosterone acetate

## C Cushing s Syndrome

### 1 *Pathogenesis*

Cu hing s syndrome has been recognized as a form of hyperadreno corticalism as was described in Volume II page 692 The adrenals are bilaterally diffusely hyperplastic in some cases whereas in others an adrenocortical cancer is present In a small number of cases no morpho logical changes are found in the adrenals if no ovarian tumor is present an excessive function of adrenal of normal size and structure may be assumed This however is an assumption ince it is conceivable that normal amounts of hormone might be secreted by the glands and the picture of hyperfunction be simulated by abnormal metaboli m of the hormone The tumors of the ovary mentioned above are probably adrenal rest tumors such tumors are rare and most of those reported have caused adrenogenital syndrome rather than Cushing s the occurrence of the latter however has been establi hed

Cases of hyperplasia may be the link between the newer concept of Cushing s syndrome as hyperadrenocorticali m and Cushing s original the sis that these cases were due to basophilic hyperpituitari m Whether or not a ba ophilic adenoma is present excessive secretion of ACTH by the pituitary basophils would explain hyperplasia and hyperfunction of the adrenal cortex Older reports of increased ACTH levels in the blood of such patients are inconclusive because of the inadequacy of the methods employed With the use of Sayer s method increased levels of ACTH have now been demonstrated in some instances (23 70) but not in others (208a, 209a) Correlation with anatomical findings in adrenals and pi tuitary will be de irable If the findings of increased ACTH levels in the blood of Cushing cases with adrenocortical hyperplasia are confirmed in a larger number of cases a further problem arises We have reason to assume that hypersecretion of cortical hormones will suppress ACTH secretion from the anterior pituitary making endogenously caused adreno cortical hypersecretion self limited This would of course not apply to adrenocortical carcinomas which may be autonomous and continue to function at high levels in spite of the retrograde suppression of ACTH ecretion the contralateral adrenal in such cases is however atrophic be cause of this mechanism In cases of hyperplasia the question therefore

arises why *the excessive ACTH should persist, and not be suppressed by* the liberated adrenal steroids There is at present no answer, one could suggest the hypothesis that the control mechanism in such cases is 'set at a higher level, either in the pituitary or in the hypothalamus

The latter has been implicated in the pathogenesis of Cushing's syndrome (83), and whereas the evidence is far from conclusive the experimental data on hypothalamic control of ACTH secretion (26, 56 93, 134) should stimulate further study of clinical material

In a discussion of the pathogenesis mention should be made of the Cushing like pictures which have been observed (*1*) after therapeutic administration of large doses of cortisone (197) and (*2*) in certain forms of hepatic cirrhosis (21) From a purely therapeutic angle, the occurrence of Cushing like features in patients treated with cortisone is an undesirable '*side effect*" *However these cases are of considerable interest* inasmuch as they would seem at first sight to *nullify* the concepts of the pathophysiology of Cushing s syndrome as first proposed by Albright (1) and Kepler (110) According to this concept Cushing's syndrome would represent 'panhypercorticalism ' some of the manifestations would be the result of excessive secretion of deoxycorticosterone like salt hormone (hypertension cardiac enlargement, edema) with others to be ascribed to the excess sugar hormones' (diabetes obesity deossification purple striae) and yet others (acne hirsutism, plethora) to increased androgen secretion After all these manifestations were induced by excess cortisone *administration it appeared doubtful to some investigators* whether the concept of Cushing's syndrome as multihormone excess or panhyper corticalism ' could be upheld In view of the fact that "single hormone ' hypersecretion as well as hyposecretion appears to occur and in view of the fact that the adrenal gland can perform remarkable interchanges in the steroid structure the induction of Cushing's syndrome by cortisone administration does not appear conclusively to invalidate the previous concepts

The observation of Cushing like pictures in certain forms of hepatic cirrhosis is of interest This has been explained by assuming that the *diseased liver is deficient in inactivating or destroying endogenous* cortical hormones The urinary excretion of formaldehydogenic steroids was found increased in these cases—a finding compatible with the explanation (21) Hepatic inactivation of cortisone has been demonstrated *in vitro* and *in vivo*

To return to the pituitary basophils the only constant finding in all cases of Cushing s syndrome has been that of hyalinization of many basophils these changes are usually *referred to as Crooke s changes* ' The physiological significance of this finding was not apparent the hyaliniza

tion was interpreted as possibly indicative of excessive ACTH secretion, or as the late Dr Kepler suggested as an effect of the excessive adrenocortical secretion on the pituitary (109) The latter explanation may now be accepted in view of the fact that identical changes have been found in the pituitaries of subjects treated with ACTH or with cortisone for various conditions (66, 121)

## 2 *Hormone Secretion*

Following the development of methods for determining adrenocortical 11 17 hydroxy compounds in blood, high levels were found in cases of Cushing s syndrome (156) Urinary excretion studies have been carried out with various methods 17 Ketosteroids have been found to be within normal range, or moderately elevated (131) This would correlate well with the moderate degree of virilizing manifestations seen in Cushing s syndrome in contradistinction to the marked degree observed in the various forms of adrenogenital syndrome Corticoid excretion determined by bio assay, is elevated and this finding correlates well with isolation studies which have yielded large amounts of 17 hydroxycorticosterone (hydrocortisone) Reports of studies using either reducing methods or formaldehyde generation are less consistent excretion levels have been found high in some and within normal range in other cases most of these studies were carried out without $\beta$ glucuronidase hydrolysis On the other hand the enzyme appears to liberate substances not related to corticosteroids which react both in the reducing and formaldehydogenic procedures Even with technical refinements of the formaldehydogenic method (90 237) there are still considerable sources of error in this procedure With the use of methods based on the Porter Silber reaction (30 64 173 179 223) more consistent results are obtained even though (or perhaps because) this reaction determines a more limited range of compounds (17 hydroxy cortico teroids) Determination of the 17 hydroxycorticosteroids and especially of the enzyme liberated Porter Silber positive compounds (17 hydroxycorticosteroids) has yielded high values in cases of Cushing s syndrome (173 178) additional information can be obtained by separate determination of the free and the glucuronidase liberated 17 hydroxy steroids (33)

## 3 *Metabolic Changes*

The retention of sodium and loss of potassium in Cushing s syndrome is a well established fact (see Volume II) Eventually this may lead to a severe hypokalemic hypochloremic alkalosis To what extent the electrolyte changes are due to the action of $C_{11}$ oxygenated compounds alone or to a concomitant increased secretion of salt hormone is as yet unsettled

Studies for the presence of aldosterone[1] will shed light on this problem. An interesting abnormality of the electrolyte metabolism pertains to the response to desoxycorticosterone acetate. Whereas in the normal individual as well as in the Addisonian injection of this compound together with intravenous administration of sodium chloride is followed by sodium and chloride retention, sodium and chloride loss occurs under these conditions in cases of Cushing's syndrome (191, 192). There is at present no adequate explanation for the paradoxical action.

The disturbance of carbohydrate metabolism is clearly the result of excess secretion of $C_{11}$ oxygenated steroids ('sugar hormones'). As would be expected the resulting diabetes is relatively insulin resistant. *Differences of severity of impairment of carbohydrate metabolism, ranging from severe diabetes to 'latent' diabetes (impaired glucose tolerance), may in part at least be due to differences of β cell 'reserve capacity.'* The diabetogenic action of the $C_{11}$ oxygenated corticosteroids is due to increased gluconeogenesis and to decreased glucose utilization. An interesting aspect of disturbed carbohydrate metabolism in Cushing's syndrome has received some attention recently. It has been shown previously that in normal *individuals the fasting blood values for glucose* lactate and pyruvate were in the ratio of 100:10:1; this ratio is maintained following ingestion of glucose. In diabetics the rise of blood sugar during a glucose tolerance test is not associated with a proportional rise of lactate and pyruvate; these metabolites either fail to rise or this rise lags appreciably. In contradistinction in some cases of Cushing's syndrome the fasting levels of lactate and pyruvate were disproportionally high, and following glucose load the diabetic glucose curve was associated with high levels of lactate and pyruvate (89, 111). This has been interpreted to indicate that in this form of 'steroid diabetes' the islets respond to the glucose load with good insulin secretion in spite of the diabetic shape of the glucose tolerance curve. *High blood pyruvate levels following glucose* load have also been reported after administration of cortisone and ACTH (111). If the high blood levels indicate increased formation of pyruvate this could conceivably also have some bearing on the fat formation via acetate and help to explain the obesity observed both in Cushing's syndrome and following cortisone and ACTH therapy.

[1] A number of important papers regarding the physiologic and therapeutic action of aldosterone *have appeared after this manuscript was submitted.* (R. Gaunt *et al. Endocrinol.* 55: 236 (1954); F. T. G. Prunty *et al. Lancet* iii: 620 (1954); G. L. Farrell *et al. Proc. Soc. Exptl. Biol. Med.* 87: 141 (1954); J. A. Luetscher, Jr. *et al. J. Clin. Invest.* 33: 276 (1954); J. A. Luetscher, Jr. *et al. J. Clin. Endocrinol. and Metabolism* 14: 1086 (1954).

### 4 *Diagnosis*

There has been no significant new contribution with regard to diagnosis The frequency of occurrence of signs and symptoms as well as of abnormal laboratory data has been compiled by Plotz *et al* (163) Determination of urinary 17 ketosteroids is of limited diagnostic value because values are within normal range in many cases Determination of 17 hydroxysteroids promises to be of great significance

After the diagnosis of Cushing's syndrome is made it is obviously of greatest importance to determine presence or absence of a cortical tumor An interesting biological test is based upon the response to exogenous cortisone In the normal individual as well as in cases of adrenocortical hyperplasia administration of cortisone suppresses 17 ketosteroid excretion probably by suppressing ACTH secretion from the anterior pituitary with consequent suppression of secretion of the adrenal cortex In cases of adrenocortical cancer cortisone administration is not followed by a decrease of 17 ketosteroid excretion evidently because the adrenocortical cancer is autonomous and continues to function in the absence of ACTH stimuli (97) In a case of Cushing s syndrome which one of us (K E P) discussed in a CPC conference injection of a test dose of insulin was followed by profound and prolonged hypoglycemia usually seen in Addison s disease and in panhypopituitarism This was interpreted as indicating the presence of an autonomous adrenocortical cancer associated with the typical atrophy of the other adrenal

The biological tests (response to cortisone response to insulin) will have to be confirmed in a larger number of cases before the reliability and practical usefulness can be evaluated

At present direct evidence for the presence of a tumor must still be sought in every case To the older methods of roentgenography following iv urography and periadrenal air insufflation roentgenography following introduction of air into the retrorectal space (retroperitoneal pneumography) has been added (203) This procedure appears to be safe and has yielded good pictures in several cases As before exploratory operation has to be resorted to in doubtful cases

### 5 *Course of Disease*

The natural history of Cushing s syndrome has been critically re evaluated in a study by Plotz *et al* based on 33 observed cases and analysis of 189 cases reported in the literature (163) The rare occurrence of spontaneous remissions has been re emphasized by Knowlton (117) this may be important in trying to evaluate results of therapy

### G. Treatment

The treatment of choice in cases of tumor is clearly surgical removal of the tumor.

As mentioned in Volume II, treatment in cases due to adrenocortical hyperplasia had not been very satisfactory. Roentgen irradiation of the pituitary is now being carried out with much larger doses (3000 to 5000 tissue r) than were employed previously. Good results with remission of long duration have been obtained in two series of cases (51, 103). However, even these large radiation doses fail to bring about any improvement in some cases (33, 103). Partial hypophysectomy has been carried out in a few cases (128). B. Arner *et al.* report a good result following exposure and electrocoagulation of the pituitary (4). Beneficial results of subtotal and of total adrenalectomy have been reviewed (196). 2,2-*bis* (para-chlorophenyl)-1,1-dichlorethane (D.D.D.) has been found to cause atrophy of the zona fasciculata of the adrenals of dogs, rendering them insulin sensitive (146). It is also active in rats, increasing their insulin sensitivity and alleviating alloxan diabetes but without causing adrenocortical atrophy (27, 28). Similar observations have been made in rabbits (161). However, in two cases of Cushing's syndrome this drug failed to exert any effect on corticoid excretion and on carbohydrate metabolism (33, 182).

### D. The Adrenogenital Syndrome

As indicated in Volume II, the clinical manifestations are due to increased secretion of androgenic 17-ketosteroids from the adrenal cortex. The anatomical change in the latter is either diffuse bilateral hyperplasia or tumors of one cortex. The picture will differ depending on the age and sex of the individual, as summarized in Table I.

Excretion of 17-ketosteroids is increased above the levels normal for the respective age group. Higher values are found in tumor cases, but there is some overlap. *However, excretion of more than 150 mg./24 hours* indicates the presence of adrenocortical cancer. Tumor cases excrete proportionately more $\beta$-17-ketosteroids (mainly dehydroepiandrosterone) than do hyperplasia cases; in the latter $\alpha$- and $\beta$-17-ketosteroids are increased in approximately normal ratio, the $\beta$-fraction comprising up to 25% of the total. Separation into $\alpha$ and $\beta$ fraction can be carried out by Digitonin precipitation, or with the use of color reaction based on the Pettenkofen reaction. However, cases of adrenocortical cancer with low $\beta$ fractions have been recorded (62, 132) and in our experience also the differentiation of tumor and hyperplasia by determination of the $\alpha$–$\beta$ ratio is not reliable. Increased 17-ketosteroid values have recently also been found in blood in adrenogenital syndrome (61). Recently high urinary excretion values of pregnanetriol in adrenogenital syndrome have been

TABLE I

Synopsis Adrenogenital Syndrome

| Age of onset | Anatomy of adrenal | Clinical manifestations | | Remarks |
|---|---|---|---|---|
| | | Female | Male | |
| 1 Fetal | Diffuse hyperplasia | Female pseudohermaphrodite Persistent urogenital sinus large clitoris Later in addition features like #2 | Macrogenitosomia precox Manifestations not conspicuous at birth | Familial incidence in some instances Type 1 Effects of excess androgens only Type 2 Salt losing Effects of excess androgens plus salt hormone deficiency Type 3 Effects of excess androgens plus sugar hormone deficiency |
| 2 Childhood | Tumor or diffuse hyperplasia | Heterosexual precocious pseudo puberty Large clitoris Body and facial hair Deep voice Advanced bone age Rapid growth in length Amenorrhea Absent mammary tissue | Isosexual precocious pseudo puberty Large penis Small testes Body and facial hair Deep voice Advanced bone age Rapid growth in length Azoospermia | Tumor more frequent than hyperplasia Similar syndrome in females from virilizing ovarian tumors in males from Leydig cell tumors and teratomas |
| 3 Adult | Tumor or diffuse hyperplasia | Large clitoris deep voice Body and facial hair Amenorrhea Atrophy of breasts | | Same as #2 |

found (20, 132, 140), perhaps the high pregnanediol excretion reported in earlier studies (63, 221) actually was due to the presence of the triol in the precipitate. The possibility that the pregnanetriol is a metabolite of 17-hydroxyprogesterone, probably a precursor of hydrocortisone, has been discussed (20).

The pathogenesis of the diffuse hyperplasia poses a challenging problem. There is no evidence to suggest a primary hypothalamic or pituitary disorder causing excessive secretion of ACTH. However, high blood levels of ACTH have been reported (208). It is not immediately apparent how continuous stimulation by excessive ACTH causes excess androgen secretion only, and not panhypercorticalism as seen in Cushing's syndrome. A very interesting hypothesis has been proposed by F. Bartter *et al.* (14). According to this schema the primary abnormality of the adrenal cortex in these cases is the decreased ability to produce $C_{11}$ oxygenated steroids ("sugar hormones") or, with other words, a low target responsiveness of 11-oxysteroid production to ACTH. It has been subsequently suggested that the "block" might be at the conversion of 17-hydroxyprogesterone to hydrocortisone and that perhaps the high excretion of pregnanetriol (probably a metabolite of 17-hydroxyprogesterone) is due to this block (20). The low levels of circulating 11-oxysteroids take the brakes off the anterior pituitary's ACTH secretion, the larger amounts of ACTH now in circulation are adequate to maintain a fair 11-oxysteroid production but the 17-ketosteroid production now becomes excessive because the sensitivity of the processes responsible for their production is "normal"; this hypothesis assumes that the adrenal androgens are much weaker inhibitors of ACTH secretion than the $C_{11}$ oxysteroids. A number of observations in patients with adrenogenital syndrome would be explicable on the basis of Bartter's schema. ACTH administration fails to induce eosinopenia, nitrogen loss and increased urinary levels of reducing steroids in cases of adrenogenital syndrome. Also ACTH administration was not followed by Na retention or K loss (14, 124). Low blood levels of C11-17 oxygenated steroids have been reported in several cases (105); this is in apparent discrepancy with the normal urinary values of reducing or formaldehydogenic steroids found in other cases (14). The discrepancy may be more apparent than real since methods determining reducing and formaldehydogenic steroids are not entirely specific. On the other hand it is curious and not imediately understandable that individuals with low secretion of $C_{11}$-$C_{17}$ oxysteroids should have no manifestations referable to this deficiency. Furthermore studies indicate that these adrenals are incapable of increasing their 11-oxysteroid production under additional ACTH stimulation (14, 124), yet the individuals appear to tolerate infections and other "stresses" quite well. It would appear that

the oxysteroid deficiency differs in degree in different cases being relatively mild and subclinical in most instances in others severe deficiency with hypoglycemia has been observed (231 235) Hypertension has been recorded in a number of cases (236)

In some cases of adrenogenital syndrome a disturbance of electrolyte metabolism similar to that present in Addison s disease is found (12 36) In this type designated as the salt losing type of adrenogenital syndrome inability to retain sodium lead to Addisonian crisis From a clinical standpoint the most difficult cases are those occurring in very young infants in boys there may be little in the appearance of the patient to suggest virilizing adrenocortical hyperplasia because manifestations may not appear before the end of the first year of life An assay of urinary 17 ketosteroid excretion will permit of early diagnosis

In some cases an aplasia of the zona glomerulosa in addition to the hyperplasia of the inner zone of the cortex was found (18) In view of the fact that at least in rodents the zona glomerulosa is believed to elaborate salt hormone it was suggested that in these cases salt loss was due to a deficiency of salt hormones (234) Another explanation was suggested based on the observation that these patients responded to ACTH injection with sodium loss instead of with the sodium retention observed in normal individual It is possible that these adrenals elaborate an abnormal hormone which causes Na loss As yet such a compound has not been demonstrated The effect of this hypothetical factor has to be overcome with large doses of deoxycorticosterone acetate and salt until cortisone successfully suppresses its secretion (101 124)

The familial incidence of congenital adrenocortical hyperplasia causing pseudohermaphroditism in girls and macrogenitosomia precox in boys has been known for some time Bentinck and co-workers have analyzed all reported instances they point out that in no instance has the abnormality been found in more than one generation Generally they believe that the evidence available militates against the assumption of a genetic factor (16)

The differentiation of cases due to adrenocortical tumor from those resulting from diffuse hyperplasia is of course of greatest practical importance As mentioned above neither the urinary excretion levels of total 17 ketosteroids nor the ratio of urinary alpha and beta 17 ketosteroids give reliable information Whether pregnanetriol excretion in hyperplasia will prove a reliable differentiating feature from the tumor cases will have to await study of a larger material Except in the congenital form which is invariably due to hyperplasia a tumor must always be suspected The visualization of the adrenal glands on roentgenogram following air injections has been greatly improved by the use of retro-

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stance adrenergic in nature which passes along the portal vessels from the hypothalamus to the adenohypophysis (130)

Not only organic lesions but also various stress reactions both physical and emotional may influence this mechanism markedly and in at least several different ways to produce several syndromes of pituitary ovarian dysfunction

## A Genetic Abnormalities

### 1 *Ovarian Agenesis (Turner Syndrome (218) Turner Varney Syndrome)*

In this condition there is a congenital absence of the germ cells and failure of the follicular epithelium to develop, so that the ovary consists of a thin band of fibrous tissue lined by germinal epithelium lying in the mesosalpinx parallel to the Fallopian tube It is not known whether this condition is due to failure of production of the germinal cells or to failure of the germinal cells to migrate from their site of origin in the hind gut into the gonad

The question has been raised (232) as to whether all of these patients are in reality females as their external genital development denotes Since in the male the Mullerian system appears to regress under the influence of androgen from the developing testes is it not possible that many of the essentially agonadal individuals are genetic males On the other hand if this were so some other explanation would be required for the occasional male who presents a similar syndrome (p 865) Experimental observations made in young rabbit embryos by Jost (106) and in mice by Wells (229) support this hypothesis

Many other genetic and endocrine features in this syndrome require clarification The shortness of stature is hardly to be expected in view of the congenital hypogonadism There is a remarkable tendency for most of these people to range in height from 54 to 56 inches The skeletal proportions are usually normal rather than eunuchoid The bone age is generally within the normal range occasionally being moderately delayed Pubic hair is present in some but absent in other cases Webbing of the neck is often noted Among the other congenital defects sometimes seen are a receding chin hypertension coarctation of the aorta osteoporosis with resulting orthopedic abnormalities and polydactylism Some patients may show no congenital defects except for the hypogonadism and shortness of stature

Gonadotropin (FSH) excretion is uniformly elevated and indeed serves as an important point in differential diagnosis from pituitary infantilism

* A recent report by Wilkins *et al* *J Clin Endocrinol and Metabolism* 14 1?0 (1954) indicates that the chromosomal sex is male in most cases

peritoneal pneumography (203). Air is injected into the retrorectal space and spreads upward retroperitoneally outlining all retroperitoneal structures. This procedure appears to be technically simpler, and above all safer than the perirenal air insufflation. If for some reason adequate visualization is not obtained, surgical exploration is necessary.

Treatment in tumor cases consists of course in surgical removal of the tumor. Treatment of hyperplasia cases was very unsatisfactory up to the introduction of cortisone therapy (232, 233, 236). Treatment with cortisone suppresses the function of the adrenal cortex, evidently by suppressing ACTH secretion from the anterior pituitary. The inhibition of androgen production causes arrest of the virilization in females as well as of the rapid growth and bone maturation with premature closure of epiphysis. Development is normalized and a normal female pubertal development in girls with menarche and breast development takes place. Evidently exogenously administered cortisone adequately replaces the suppressed secretion of the other cortical hormones.

1. *Treatment of the Salt Losing Type* (86, 232)

Treatment with cortisone alone even with administration of extra salt, is usually inadequate for maintaining electrolyte metabolism. Desoxycorticosterone acetate (DCA) is used in combination with cortisone, the latter chiefly for suppressing androgenic 17-ketosteroid excretion, the former for correction of the inability to retain sodium. It has been found that large doses of desoxycorticosterone and NaCl are necessary. This has been tentatively considered supportive evidence for the theory that the salt loss is due not to a deficiency of salt hormone but to the production of a Na losing adrenal compound the action of which has to be overcome by DCA. The requirement for the latter is appreciably reduced after cortisone therapy has brought about suppression of adrenocortical activity.

## V. Ovary

Normal cyclic function of the ovaries with regard to both ovulation and hormone secretion involves not only a complex but also a very labile mechanism involving particularly (*1*) neurohumoral factors regulating the cyclic release of three gonadotropic substances (FSH, LH and LTH), (*2*) a precise sequential response of the germ cells to gonadotropic stimulation and (*3*) the cyclic secretion of steroid hormones by the granulosa and theca cells in proper amounts and at the proper time to favor a self-continuing *push-pull relationship between the gonad and the hypophysis*. The hypothalamus is undoubtedly an important relay station in the neurohumoral control of the pituitary-ovarian axis and present evidence suggests that gonadotropin release is mediated by a neurohumoral sub-

closure of the epiphyses will be hastened the secondary sex characters will develop as will the vagina and uterus and the vasomotor symptoms will be controlled but spontaneous ovarian function rarely follows the withdrawal of substitutional therapy

## C PITUITARY HYPOGONADISM

Pituitary hypogonadism in women may be associated with pituitary dwarfism (p 826) or may be a single gonadotropic deficiency The skeletal development may therefore range from marked dwarfism to eunuchoidism depending apparently upon the degree of growth hormone deficiency In some instances with moderate retardation of growth the clinical picture may resemble that of ovarian agenesis but can be distinguished from the latter by absence of FSH in the urine Other congenital defects are usually absent the bone age is more markedly retarded and the urinary 17 ketosteroids are lower (often less than 1 0 mg per 24 hours) (232) As indicated above the differential diagnosis from a primary ovarian deficiency is of importance because of the possibility of a response to gonadotropic therapy

## D NONNEOPLASTIC CYSTS OF THE OVARY

### 1 *Follicle Cysts of the Ovary*

The etiologic factors in follicular cystic disease of the ovary appear to be many and varied including genetic abnormalities inflammatory lesions of the ovary degenerative processes and endocrine factors Usually the physiologic disturbance results in a menstrual dysfunction and impairment of ovulation Estrogen secretion varies the follicular fluid in some of the large follicle cysts of the ovary contains small to moderate amounts of estrogenic activity whereas others are hormonally inactive Urinary excretion of estrogens also vary from diminished titers in most cases to excessive levels in occasional instances The endometrial pattern is likewise diverse ranging from an early proliferative type to one of cystic hyperplasia A secretory pattern with apparently normal ovulation may occur in the patient with one or a few follicle cysts of the hormonally inactive type in which the ovaries are otherwise normal

Little is known of the endocrine factors which may cause large follicle cysts of the ovary Occasionally these may be induced by too vigorous gonadotropic therapy (44) suggesting that a disturbance of the pituitary ovarian endocrine mechanism may be the etiologic factor in some cases In this respect it is of interest that the use of estrogens or androgens given in full dosage for several weeks may cause rapid regression of simple follicle cysts

in questionable case. Estrogens in the urine are very low. Despite the pattern of high gonadotropins and low estrogens, vasomotor symptoms do not occur in these patients as they so often do in acquired types of prepubertal hypogonadism. The 17-ketosteroid excretion varies considerably; in some cases it is normal and in others it is quite low.

As a rule, the diagnosis of this condition is not established with certainty until the postpubertal years when the hypogonadism becomes evident. However, in instances in which it is suspected on the basis of dwarfism together with other congenital stigmata, the diagnosis can be made with certainty if there is a high urinary gonadotropic excretion, a feature distinguishing this condition from pituitary and other types of dwarfism. Wilkins (232) has noted that urinary FSH was absent or low during early childhood but rose as puberty was approached. The administration of estrogens produces a gratifying development of the breasts, secondary sex characters, and growth of the vagina suitable for sexual purposes. This should be continued in cyclic fashion to maintain development, but in moderate dosage, since otherwise marked endometrial hyperplasia with *troublesome bleeding can occur.* We have not noted a stimulation of skeletal growth with estrogens, even when treatment has been started in late childhood.

### B. Infantile Ovaries with Eunuchoidism

In this syndrome, the ovaries are of the elongated ovoid (infantile) type *containing apparently normal numbers of primordial follicles.* These follicles fail to grow and mature despite adequate FSH secretion as evidenced by high urinary titers. The irresponsiveness of the ovary to gonadotropic stimulation is believed to be due to a genetic defect, although the possibility of a severe childhood infection affecting the ovaries must also be considered. These patients show the characteristic stigmata of prepubertal hypogonadism: they are relatively tall with eunuchoid proportions, the secondary sex characteristics and reproductive organs are immature, they have primary amenorrhea and hot flashes. The latter symptoms as well as the high urinary FSH titers distinguish them from the prepubertal eunuchoid hypogonad of pituitary origin. Furthermore, these patients rarely improve spontaneously, whereas the pituitary hypogonad may sometimes go into a delayed puberty in her late teens or early twenties and may eventually attain normal *reproductive function.* The latter may be encouraged or hastened by the cyclic administration of gonadotropic hormones with follicle-stimulating activity, whereas the primary *ovarian eunuchoid can be treated only symptomatically* with estrogens or with estrogens and progesterone in cyclic fashion. The

closure of the epiphyses will be hastened the secondary sex characters will develop as will the vagina and uterus and the vasomotor symptoms will be controlled but spontaneous ovarian function rarely follows the withdrawal of substitutional therapy

## C Pituitary Hypogonadism

Pituitary hypogonadism in women may be associated with pituitary dwarfism (p 826) or may be a single gonadotropic deficiency The skeletal development may therefore range from marked dwarfism to eunuchoidism depending apparently upon the degree of growth hormone deficiency In some instances with moderate retardation of growth the clinical picture may resemble that of ovarian agenesis but can be distinguished from the latter by absence of FSH in the urine Other congenital defects are usually absent the bone age is more markedly retarded and the urinary 17 ketosteroids are lower (often less than 1 0 mg per 24 hours) (232) As indicated above the differential diagnosis from a primary ovarian deficiency is of importance because of the possibility of a response to gonadotropic therapy

## D Nonneoplastic Cysts of the Ovary

### 1 *Follicle Cysts of the Ovary*

The etiologic factors in follicular cystic disease of the ovary appear to be many and varied including genetic abnormalities inflammatory lesions of the ovary degenerative processes and endocrine factors Usually the physiologic disturbance results in a menstrual dysfunction and impairment of ovulation Estrogen secretion varies the follicular fluid in some of the large follicle cysts of the ovary contains small to moderate amounts of estrogenic activity, whereas others are hormonally inactive Urinary excretion of estrogens also vary from diminished titers in most cases to excessive levels in occasional instances The endometrial pattern is likewise diverse ranging from an early proliferative type to one of cystic hyperplasia A secretory pattern with apparently normal ovulation may occur in the patient with one or a few follicle cysts of the hormonally inactive type in which the ovaries are otherwise normal

Little is known of the endocrine factors which may cause large follicle cysts of the ovary Occasionally these may be induced by too vigorous gonadotropic therapy (44) suggesting that a disturbance of the pituitary ovarian endocrine mechanism may be the etiologic factor in some cases In this respect it is of interest that the use of estrogens or androgens given in full dosage for several weeks may cause rapid regression of simple follicle cysts

## 2 The Stein Leventhal Syndrome

A particular type of microfollicular cystosis of the ovary which is of interest because of its endocrine stigmata is that seen in the Stein Leventhal syndrome (201, 202 123) This condition is characterized by a menstrual dysfunction (usually amenorrhea but sometimes irregular or excessive bleeding) infertility and hirsutism of varying degree, together with bilateral cystic enlargement of the ovaries The condition generally begins in the late teens or early twenties, usually after prior normal menstrual function Although the hirsutism may become quite marked in some cases, the clitoris is rarely appreciably enlarged or the voice deepened The 17 ketosteroid excretion is generally within the normal range occasionally being slightly elevated (up to 20 mg per 24 hours) A moderately high $\beta$ fraction (20% to 40%) is often noted Estrogens are present but usually in moderately low titers and the vaginal smear shows a persistent early follicular reaction FSH is often not demonstrable in the urine but high titers occur sporadically especially early in the condition

The appearance of the ovaries on exploration is quite characteristic They are enlarged flattened and pale and have a thickened gray capsule they resemble oysters The cut surface reveals many small cystic areas of varying size On histologic examination the significant abnormalities are the marked fibrous thickening of the ovarian capsule the presence of numerous follicle cysts, with a paucity of normal growing follicles and hyperplasia of the theca surrounding the cystic follicles The latter finding is not consistently present

A small type of Stein Leventhal ovaries is sometimes noted, in which the ovaries are only slightly enlarged and ovoid but have a markedly thickened capsule microfollicular cysts and theca hyperplasia These patients often have excessive and irregular bleeding rather than amenorrhea

The etiology of this condition is not known There is little to support an inflammatory basis for the excessive fibrosis of the capsule which could secondarily produce follicular cystosis A genetic dysfunction likewise seems improbable in view of the previous apparently normal menstrual history and absence of other stigmata in most cases It seems more likely that the condition is due to an endocrine dysfunction One wonders whether the hyperplastic theca may not be secreting increased amounts of an unusual steroid perhaps not excreted as a 17 ketosteroid which is responsible for the mild masculinization and inhibition of FSH In this regard, it is to be noted that hirsutism has been observed in association with ovarian hyperthecosis (162 167 204 211)

This condition may be differentiated (72) from an adrenogenital syndrome on the basis of lack of true masculinization and normal 17 ketoster

oids and from a Cushing's syndrome by lack of the typical habitus, purple striae and metabolic disturbances A masculinizing tumor of the ovary may also be differentiated by lack of true masculinization and the fact that the ovarian enlargement is bilateral and cystic '

Generous wedge resection of the ovaries frequently results in prompt and striking improvement of this syndrome, with return to normal ovulatory cycles in a high percentage of cases as indicated by basal temperature charts endometrial biopsies and improved fertility (202) This improvement in endocrine function following removal of ovarian tissue is difficult to explain

Hormonal therapy is not often of help in this condition In its early stages occasional cases benefit from cyclic progesterone therapy or treatment with cortisone (p 860)

## E Ovarian Deficiencies of Psychogenic Origin

It is well known that nervous or emotional disturbances may be followed by impairment of ovarian function resulting in anovulatory infertility secondary amenorrhea, dysfunctional uterine bleeding cystic mastitis or other manifestations of disturbed ovarian function The clinical findings together with hormonal and other endocrine studies of such patients indicate that there are several patterns or syndromes of ovarian hypofunction which may occur on a psychogenic basis (153 171 206)

*1* Hypogonadotropism Inhibition of pituitary gonadotropic secretion with secondary ovarian deficiency appears to be most common in such disturbances In its mildest form the menses may occur at fairly regular intervals but there is interference with normal ovulation causing relative infertility This may progress to dysfunctional bleeding or to secondary amenorrhea depending upon the degree of inhibition of the pituitary ovarian mechanism

*2* Inhibition of gonadotropins plus other pituitary hormones These patients generally have secondary amenorrhea with absence of gonadotropin excretion in the urine plus clinical and metabolic evidences of mild hypothyroidism In severe cases the clinical picture is that of anorexia nervosa with loss of weight tiredness hypotension and moderately decreased 17 ketosteroids In occasional instances the clinical manifestations may be so marked as to require differentiation from Simmonds disease (52)

*3* Normal FSH with ovarian deficiency This is the pattern seen in patients with so called hypothalamic amenorrhea as described by Klinefelter Albright and Griswold (116) It has been presumed by these authors that the ovarian hypofunction results from lack of LH secretion to synergize with FSH to stimulate the ovary The clinical

*manifestations are those of secondary amenorrhea with anovulatory infertility* In our experience, this group is not as commonly seen as those described under the first classification although the clinical picture is often the same

*4 Primary ovarian deficiency with high FSH* This syndrome is characterized clinically by secondary amenorrhea and infertility and usually accompanied by hot flashes and other menopausal symptoms It generally occurs after an acute psychic trauma and is often of long duration and highly resistant to treatment

From the endocrine patterns which have been described above, it would appear that psychogenic factors may influence ovarian function by way of the pituitary the hypothalamus, or the ovary directly Unfortunately, very little is known of the intermediary mechanisms by which psychogenic factors influence these structures

The therapeutic approach to psychogenic ovarian hypofunction is *often difficult In milder cases readjustment of environmental factors* and superficial psychotherapy may be all that is required For this reason, many such cases eventually undergo spontaneous correction On the other hand in many other instances the syndrome may be prolonged and may fail to respond even to intensive psychotherapy *Such patients are* also known to be very resistent to endocrine therapy of various types The use of gonadotropic hormones in patients with low FSH only rarely results in stimulation of ovarian function Preparations containing follicle stimulating activity should be tried in cyclic fashion in the milder cases In the more severe cases substitutional therapy with estrogen followed by estrogen and progesterone may bring on cyclic bleeding These patients seem to require much higher doses of estrogen and progesterone to obtain an endometrial response and bleeding than patients with other types of amenorrhea (67, 171) The repeated induction of cyclic bleeding is occasionally followed by resumption of spontaneous *ovarian function This approach with substitutional therapy is of course* the only plan of endocrine therapy indicated in patients with a primary ovarian deficiency

Low dosage irradiation to the pituitary or ovaries is not usually helpful in these cases of psychogenic ovarian hypofunction *(170) and is reserved* for cases of infertility in which all other methods of treatment have failed Better results are obtained in patients with a gonadotropic deficiency than in those with primary ovarian failure The effects of the irradiation therapy appear to be mediated by way of the ovary rather than the pituitary (170)

## F Metabolic Disturbances Affecting Ovarian Function

In some patients excessive gain of weight may result in ovarian hypofunction with secondary amenorrhea or oligomenorrhea This is usually due to inhibition of pituitary FSH secretion although the mechanism by which this occurs is not known It is remarkable to observe in some patients the manner in which menstrual function may return and then disappear again depending upon their gain or loss of weight by dieting

In this respect it is also to be noted that some young women may develop a very severe or resistant type of secondary amenorrhea following rapid loss of weight below their usual normal by severe dietary restriction (706) In these patients there is also an FSH deficiency Unfortunately in some cases ovarian function does not resume spontaneously following a return to normal diet and normal weight It is believed that diets deficient in protein and certain members of the B complex are particularly likely to cause ovarian dysfunction

Disturbances in liver function may indirectly affect the function of the ovaries because of failure of normal inactivation of the ovarian hormones Thus in hepatitis and in cirrhosis of the liver disturbances of ovarian function causing menorrhagia polymenorrhea or cystic mastitis are sometimes encountered (98)

## G Effect of Adrenal Hyperfunction on the Ovaries

It is of course well known that marked degrees of adrenal cortical hyperfunction will inhibit ovarian function by depressing the gonadotropins In the adrenogenital syndrome this may be attributed primarily to the androgens which are produced in excess In some cases there is also increased estrogen secretion by the adrenal (50) the effects of which are masked by the greater increase in the androgens However in some cases of Cushing s syndrome in which the 17 ketosteroids are not elevated and the total estrogen excretion is diminished the inhibition of gonadotropic function may be attributed to other steroids of the adrenal cortex

The histologic picture of the ovaries in patients with an adrenogenital syndrome due to an adrenal cortical tumor has been of particular interest to us In two such cases in which ovarian biopsies were obtained there was a striking resemblance of the histologic findings in the ovary to those seen in the Stein Leventhal syndrome Although the ovaries were not enlarged there was marked thickening of the capsule microfollicular cystosis and hyperplasia of the theca cells

Another type of ovarian dysfunction which appears to be related to a mild degree of adrenal cortical hyperfunction or dysfunction is seen in patients with mild hirsutism anovulatory infertility oligomenorrhea or

secondary amenorrhea and 17-ketosteroid excretion which is only slightly elevated above the normal, ranging from 15 to 22 mg. per 24 hours. Such patients will sometimes respond favorably to the administration of cortisone in a dosage of 25 to 50 mg. daily by mouth. Jones *et al.* (105) have drawn attention to the use of cortisone in stimulating greater follicular activity in patients of this type. The mechanism by which cortisone produces this favorable action would appear to be similar to that described by Wilkins in patients with congenital adrenal hyperplasia, namely the depression of adrenal androgens by cortisone with subsequent release of FSH.

## II. Ovarian Hyperfunction

The most common and the most perfect example of ovarian hyperfunction is that which is seen in early pregnancy, during which the corpus luteum is stimulated to produce maximal amounts of estrogen and progesterone by the rising titer of chorionic gonadotropin secreted by the cytotrophoblast. Even in states of anterior pituitary hyperfunction the hypophysis is not capable of secreting sufficient luteinizing and luteotropic hormone to maintain and stimulate the corpus luteum to a degree comparable to that seen in early gestation. There are, however, several syndromes in which this may occur to a lesser extent.

### I. Corpus Luteum Cysts

In some instances in which there has been excessive hemorrhage into a corpus luteum, this structure may become cystic and persist with functional activity beyond the usual two week period so regularly noted in the human cycle. Menstruation may be delayed because of a persistently high estrogen and progesterone level. There may be signs and symptoms of early pregnancy or the clinical picture may suggest an ectopic pregnancy. Also, in some cases, the pregnancy test may be weakly positive, suggesting increased secretion of LH and LTH.

Rare patients may often or regularly have a slightly prolonged corpus luteum phase, lasting for 16 to 18 days as indicated by basal temperature charts. This occurs occasionally in patients who are particularly anxious for pregnancy but who do not develop a true pseudocyesis syndrome.

### J. Pseudocyesis

The pseudocyesis syndrome is a striking example of the influence of psychogenic factors upon the pituitary ovarian mechanism to produce a state of ovarian hyperfunction. This condition occurs primarily in women who have a great desire to become pregnant but who have been unable to do so, or sometimes in women who have a great fear that they have become pregnant illegitimately or against their wishes. Endocrine studies (58, 59)

indicate that they have persistent corpus luteum function characterized by hyperhormonal amenorrhea associated with a secretory endometrium, persistently good levels of estrogen and pregnanediol plus such other evidences of ovarian hyperfunction as increased glandular tissue of the breasts pigmentation of the nipples softening of the cervix and slight enlargement of the uterus FSH in the urine is generally absent but assays for prolactin have been positive These findings explain the milky secretion which may often be expressed from the nipples of these patients It would seem that psychogenic factors have in some way interfered with the pituitary hypothalamic mechanism so as to favor continued secretion of LH and LTH and persistence of the corpus luteum The endocrine disturbance together with the other psychogenic manifestations accounts for the full blown syndrome of pseudocyesis Psychiatric studies on these patients indicate that the psychosis is a type of hysteria Psychotherapy is only of limited value in the treatment of this condition which also has a great tendency to be recurrent Testosterone is sometimes helpful apparently by inducing regression of the corpus luteum This is generally followed by a menstrual flow suggesting to the patient that she is no longer pregnant

## VI Placenta

The human placenta is a versatile and prodigious producer of both protein and steroid hormones There is well-documented evidence that the placenta secretes large amounts of chorionic gonadotropin estrogens and progestogens There is also good evidence suggesting that the placenta secretes glucocorticoids (220 100 150) Studies of the ketosteroids in the urine of pregnant women (48) show both qualitative and quantitative differences from those of nonpregnant women suggesting that the placenta produces other steroid precursors or that pregnancy induces alterations in steroid metabolism or both Evidence for the secretion of a protein hormone containing ACTH activity has also been presented (101)

There is increasing evidence (205 238) that the cytotrophoblast (Langhans layer) is the site of origin of chorionic gonadotropin and possibly of other protein hormones whereas histochemical studies suggest that the syncytial cells are the site of formation of the steroid hormones

Comparatively little is known concerning the functions of the placenta in gestation Chorionic gonadotropin appears in the blood and urine in rapidly rising titers in the early days of pregnancy and apparently maintains the corpus luteum of pregnancy During the last half of pregnancy the chorionic gonadotropin levels are much lower The function of this hormone during the placental phase is not known it has been suggested that it stimulates the production of estrogens and progesterone by the syncytial cells (190) The early appearance of chorionic gonadotropin in

blood and urine and its high concentration make it po ible to u e its assay as the basis for the more reliable pregnancy tests including the Aschheim Zondek Friedman, South African frog American male frog and rat-ovary hyperemia tests

Estrogens and progesterone are secreted in increa ing amounts as preg nancy progresses It has been generally belie ed that during the fir t trimester the bulk of these steroids are of corpus luteum origin and that the corpus luteum is es ential for the maintenance of pregnancy during this period More recent studies (118) would indicate that in the human being the corpus luteum is essential for a much briefer period of time and that removal of the corpus luteum during the fir t weeks of pregnancy does not necessarily result in termination of the pregnancy or diminution in preg nanediol excretion thus suggesting that the trophoblast may adequately take over this function very early

Estrogen and progesterone appe ar to be es ential for growth and quies cence of the uterine muscle during gestation (174) They also induce many other changes favorable for mother and fetus such as the further growth and development of the breasts softening of the cervix thickening of the vaginal epithelium and increa ed vaginal acidity With regard to some of these changes the action of the two hormones appears to be syner gistic whereas in others they are antagonistic

The intermediary metabolism of estrogen and progesterone during preg nancy is discus ed in Volume I of *The Hormones*

## A Pregnancy Complications

### 1 *Abortion*

Several patterns of hormonal imbalance have been observed (104) in patients with sporadic or habitual abortion The latter condition has been particularly useful for such studies since the hormonal excretion patterns have been studied from early gestation to the time of abortion In one type the titer of chorionic gonadotropin is low from the very early days of pregnancy and progressively falls Estrogen and pregnanediol excretion is also diminished In these pregnancies the fetus is often resorbed and there is reason to suspect a genetic defect In another type pregnanediol and estrogen levels are diminished in spite of normal titers of chorionic gonadotropin suggesting poor response by the corpus luteum or syncytio trophoblast In this group substitutional therapy with large doses of pro gesterone and estrogens appears to be particularly indicated In a third type abortion may occur repeatedly in spite of normal urinary levels of all three hormones suggesting a uterine factor or possibly a di turbance of other unknown hormonal factors Diminished thyroid function must be

considered it has been suggested (158) that failure of the serum protein bound iodine to rise during pregnancy may be of significance in this regard

2 *Hydatidiform Mole and Chorionepithelioma*

In many instances these conditions are associated with unusually high titers of chorionic gonadotropin which may be helpful in their diagnosis Since hydatidiform mole is most often suspected during early pregnancy when the titer is normally quite high a high serum chorionic gonadotropic titer will be indicative of hydatidiform mole only if it is several times higher than the peak values for normal pregnancy or if a moderately high level is sustained significantly beyond the period when the titer normally drops (12th to 16th weeks) A rising titer of chorionic gonadotropin following expulsion of a mole or after an abortion or normal delivery is presumptive evidence of chorionepithelioma provided the possibility of another pregnancy has been ruled out Patients who have expelled a mole or have had surgery for a chorionepithelioma should have repeated follow up assays over a long period of time in order to detect recurrence or metastases It should however, be emphasized that occasionally chorionepitheliomas produce little or no chorionic gonadotropin particularly in their early stages (212) The secretion of estrogens and progesterone by moles and chorion epitheliomas varies much more than that of chorionic gonadotropin the assay of the former is therefore less useful from a diagnostic standpoint Some of these tumors may secrete very large amounts of chorionic gonadotropic hormone with but little if any estrogens or progesterone whereas others may secrete considerable amounts of all three hormones (148)

3 *Toxemias of Pregnancy*

There are many reasons to suspect a placental factor possibly of hormonal nature of etiologic significance in the toxemias of pregnancy but satisfactory evidence for such an endocrine factor has not yet been obtained Results of hormonal excretion studies in toxemias of pregnancy are quite controversial The pattern reported by the Smiths (190) in many instances namely increased urinary chorionic gonadotropin and diminished urinary pregnanediol and estrogen as well as changes in the partition of the urinary estrogens has in part been confirmed by some workers On the other hand these alterations are often absent in patients with severe pregnancy toxemia (155 160, 193)

An increase in the urinary excretion of corticoids in toxemia of pregnancy has been reported (47 148 216) this finding has been tentatively correlated with the sodium and water retention observed in toxemia The increase above that occurring in normal pregnancy is probably of questionable significance especially in view of the methods employed in these studies

Even if such an increase will be confirmed with the use of more specific methods such as those based on the Porter-Silber reaction, the question will still remain what is cause, and what effect? The occurrence of a severe toxemia could well cause a non specific rise of adrenocortical secretion. It has been suggested that increased production of a mineral corticoid might play a role in this condition (147). This entire problem appears to be quite unsettled and further advances will depend in part on better assay methods for these steroids, particularly for the "mineralocorticoids." Similarly the question of the use of cortisone or ACTH in the treatment of late pregnancy toxemias is controversial (147, 211).

## VII. Testes

The hypophyseal testicular interrelationships have been discussed in Volume II. Several problems in this sphere remain unsolved. Although FSH is believed to be primarily responsible for stimulation of spermatogenesis, it is uncertain whether this occurs without the synergism of the other gonadotropins or of androgens. It is equally uncertain whether luteotropin (LTH, prolactin) plays any role in the male, perhaps as a synergist of LH (ICSH) in stimulating Leydig cell function. The question of a "second" testicular hormone awaits final answer. The existence of such a hormone has been postulated on the basis of analogy with the female. Interstitial tissue (and corpus luteum) in the female and interstitial tissue (Leydig cells) in the male producing progesterone and testosterone respectively is regulated by LH (ICSH). Ovarian follicles and estrogen secretion in the female and spermatogenic tubules in the male are governed by FSH. Is there a "tubular" hormone in the male corresponding to the follicular estrogen?

In addition to such general considerations, certain clinical observations suggest the existence of a tubular hormone. In Klinefelter's syndrome (Volume II, p. 679) hyalinization of the seminiferous tubes and intact Leydig cell function are associated with excessive urinary FSH excretion. This suggests that under normal circumstances the tubules secrete a hormone the absence of which releases pituitary FSH secretion. It has been proposed that this hormone might be an estrogen. Estrogens have been extracted from the testis of human beings as well as certain animal species and are believed by some to be secreted by the Sertoli cells. This assumption is based chiefly on the findings in Sertoli cell tumors of the testis which induce feminization in dogs and in man; from Sertoli cell tumors of dogs estrogens have been extracted. However, Nelson has questioned the origin of these testicular tumors from Sertoli cells and believes them to be Leydig cell tumors (144). If androgens and estrogens are according to

Nelson produced in the Leydig cell the postulated tubular hormone still goes begging

## A HYPOGONADISM (91)

Hypogonadism is discussed in Volume II of *The Hormones*

In view of the fact that in many instances nothing is known regarding the etiology of the various syndromes hypogonadism may be classified, on a symptomatological basis as follows Failure (*1*) of both tubular (spermatogenic) and Leydig cell function (*2*) of tubular function with intact Leydig cell function and (*3*) of Leydig cell function with intact tubular function All three types can at least theoretically be caused (*a*) by a primary testicular deficiency and (*b*) by a testicular deficiency secondary to pituitary gonadotropic failure

### 1 *Failure of Tubular and Leydig Cell Function*

This type is represented by the surgical castrate (eunuch) and by the eunuchoid in whom testicular failure is due to an anlage defect (functional castrate 86), or due to impaired blood supply incident to operations for cryptorchidism or hernia or due to bilateral orchitis

These cases may be indistinguishable in their physical appearance and in their functional deficiencies from cases of pituitary hypogonadotropism Differentiation of primary testicular eunuchoidism and eunuchoidism secondary to hypogonadotropism is readily accomplished by assay of urinary gonadotropins (FSH assay) which yields very high values in the former whereas gonadotropin excretion is absent in the latter

The castrates and the primary testicular eunuchoid are treated with testosterone In hypogonadotropic eunuchoids treatment with gonadotropic hormones (chorionic gonadotropin and FSH containing pituitary preparations) has been successful

Whether an anlage defect of the testis ever is associated with a syndrome analogous to that of ovarian agenesis (Turner's syndrome) is as yet uncertain As is discussed elsewhere (p 853) the absence of estrogen effects and of germ cells and the presence of high urinary gonadotropin (FSH) levels in ovarian agenesis are associated with features which are different from those of the prepubertal castrate shortness of stature frequent occurrence of other congenital malformations including webbing of the neck usually no delay of closure of the epiphyses None of the few cases reported as Turner's syndrome in the male (194) appear to us convincingly analogous to the female syndrome The suggestion has been made (p 853) that some patients with ovarian agenesis may in fact be genetic males and thus represent instances of testicular aplasia

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In addition to such general considerations certain clinical observations suggest the existence of a tubular hormone In Klinefelter's syndrome (Volume II p 679) hyalinization of the seminiferous tubes and intact Leydig cell function are associated with excessive urinary FSH excretion This suggests that under normal circumstances the tubules secrete a hormone the absence of which releases pituitary FSH secretion It has been proposed that this hormone might be an estrogen Estrogens have been extracted from the testis of human beings as well as certain animal species and are believed by some to be secreted by the Sertoli cells This assumption is based chiefly on the findings in Sertoli cell tumors of the testis which induce feminization in dogs and in man from Sertoli cell tumors of dogs estrogens have been extracted However Nelson has questioned the origin of these testicular tumors from Sertoli cells and believes them to be Leydig cell tumors (144) If androgens and estrogens are according to

(119 135 151) The evidence for the latter varied somewhat in individual cases eunuchoid body proportions sparseness or absence of pubic and axillary hair poor beard growth high pitched voice undersized penis as well as diminished urinary 17 ketosteroid values have been reported Testicular biopsy specimens showed diminished number and pycnotic appearance of Leydig cells and production of mature sperm in the tubules In some instances an ejaculate could not be obtained, in one case the fructose content of the ejaculate was very low (119), the fructose content of the semen is considered to be a measure of androgen secretion (120) In other cases the ejaculate was normal and fertility was claimed for one of these cases McCullagh has applied in his cases an assay method believed to show LH (ICSH) in urinary extract In three of his five cases LH was not demonstrable suggesting a pituitary deficiency limited to this one hormone Some response was obtained by treatment with chorionic gonadotropin this would add further support to the assumption of a pituitary deficiency (135)

## B Testosterone as a Gonadotropin

Several older studies are on record indicating that under narrowly circumscribed experimental conditions testosterone can exert a gonadotropic function In the male hypophysectomized rat treatment with testosterone initiated immediately after hypophysectomy will maintain spermatogenesis and prevent testicular atrophy but treatment with testosterone will not repair the previously atrophied testicle nor restore spermatogenesis once the posthypophysectomy changes have taken place (145 187)

There are for obvious reasons no observations in man which would duplicate the experimental conditions in the rat However in several cases of hypopituitarism sperm production occurred under treatment with testosterone (95 113) In contradistinction to the hypophysectomized animal the possibility cannot be ruled out in such a case that testosterone therapy induced the pituitary to secrete gonadotropins Kinsell rejects this possibility because no rise of urinary gonadotropin titer was observed during androgen therapy (113)

## C Testicular Tumors

Tumors of the testis will be discussed briefly only with regard to endocrine aspects Leydig cell tumors secrete large amounts of androgens and 17 ketosteroids and androsterone sulfate has been isolated from the urine In prepubertal boys these tumors cause precocious pseudopuberty (p 869) It has been pointed out that probably because of the protein anabolic action of the androgen secreted by the tumor tumor cachexia may be absent even in advanced cases with widespread metastasis (133) It is

*2. Impairment of Tubular Function with Intact Leydig Cell Function*

Typically the individuals are normally developed males, with normal secondary sex characteristics, normal libido and potency, and normal 17-ketosteroid excretion. The size of the testes varies from normal to markedly diminished. The sperm content of the ejaculate varies from oligospermia to azoospermia. Testicular biopsy specimens show normal Leydig cells; the germinal epithelium may be well developed, with spermatogenesis progressing to the spermatide stage ("spermatocytogenesis"), but with absent sperm maturation ("spermiogenesis"). In other cases the tubules are completely denuded with only Sertoli cells left. Pericanalicular fibrosis is present in varying degrees in most but not in all cases of the latter type. Urinary gonadotropin excretion is normal in some and very high in other cases, but a definite correlation of the FSH excretion pattern with the histological picture of the testis is as yet not possible. There is no evidence that any of these cases is due to a pituitary gonadotropic deficiency. The etiology is unknown.

It has been claimed that seminal failure can be recognized already in the histologic picture of the pubertal testis by marked retardation of tubular development (31).

The Klinefelter syndrome discussed in Volume II, characterized by hyalinization of the tubules, normal Leydig cells, gynecomastia and high FSH excretion, would appear to be a special entity within the group of seminal failure without Leydig cell failure (45, 116). Cases have been reported which show features of Klinefelter syndrome associated with varying degrees of Leydig cell failure (85, 87).

*Treatment.* Cases with azoospermia and extensive fibrosis and/or hyalinization of the tubules, as seen in the biopsy specimen, are obviously beyond any possibility of repair. However, results are also unsatisfactory in most instances of milder forms with oligospermia and maturation arrest. Various gonadotropic preparations have been used on a purely empirical basis. One would expect results, if any, from preparations containing adequate amounts of FSH such as pituitary gonadotropins or Synapoidin. *Chorionic gonadotropin* has also been used, and occasional good results have been claimed with any of the preparations. Treatment with large doses of testosterone has also been suggested; complete suppression of sperm production is reported to be followed by a rise of viable sperm cells in the ejaculate to values permitting fecundation ("rebound effect") (82). Confirmation has to be awaited.

*3. Leydig Cell Deficiency with Normal Tubular Function*

Recently three reports have appeared concerning patients who had spermatogenesis in the presence of evidence for impaired Leydig cell function

of the sex steroids These amounts are too small to induce somatic changes as evidenced by the absence of secondary sex characteristics in normal children and by the absence of detectable somatic changes in prepubertal castrates However the fact that the prepubertal castrate (80) as well as the child with ovarian agenesis (185) may secrete appreciable amounts of FSH indicates that the child's gonad is not wholly inactive The pubertal spurt of sex steroid secretion from the gonad is caused by a spurt of gonadotropin secretion from the anterior pituitary This can be shown by urinary gonadotropin assays and is evident from the fact that administration of exogenous gonadotropin causes sex hormone secretion and development of secondary sex characteristics years before the physiological onset of puberty

The anterior pituitary probably is triggered into gonadotropin release at the time of puberty by an hypothalamic mechanism presumably by the secretion of a neurohumeral agent which reaches the anterior pituitary by way of the portal venous system of the pituitary stalk Just as the gonad is capable of responding to gonadotropic stimulation long before the physiological onset of puberty the anterior pituitary of animals (181) and human beings (11) contains appreciable amounts of gonadotropic hormones without releasing them The release of gonadotropic hormones from the anterior pituitary through the hypothalamic mechanism has been demonstrated for the LH release in the rabbit (130) the fact that certain organic lesions of the hypothalamus in children cause precocious puberty suggests that a similar mechanism operates in the initiation of the pubertal chain of events in the human being

Precocious puberty is found in cases of hypothalamic lesions (tumors encephalitis hydrocephalus) perhaps the precocious puberty associated with polyostotic fibrous dysplasia (Albright's disease) is due to cerebral changes resulting from deformities of the base of the skull In the majority of cases of precocious puberty no organic lesion is present these cases are referred to as constitutional precocious puberty It may be assumed that for some reason the intrinsic triggering mechanism in the hypothalamus is set off abnormally early in life Because in precocious puberty both in that due to organic lesions and in the constitutional type the entire physiological *pubertal mechanism is duplicated* the preciocous development includes that of the gonad involving both its endocrine secretions and maturation of ova and sperm Pregnancy has been reported in a girl 5½ years old

Tumors or hyperplasia of the adrenal cortex tumors of the testis or ovary and teratomas containing testicular tissue cause precocious development of secondary sex characteristics either isosexual or heterosexual but not maturation of the gonad This type of precocity is called precocious pseudopuberty

interesting that the estrogen excretion may be increased simultaneously in such cases (222) This finding would add weight to the theory that the Leydig cell is the source of both androgens and estrogens (p 861) However, too few cases have been adequately and completely studied to permit any generalizations or final conclusions, unfortunately reports on this rare tumor continue to appear in the literature with little information other than a histologic study of the tumor

Tumors of the testis believed to originate from Sertoli cells, have been found in man (210) and more commonly in the dog (92) These tumors secrete estrogens and cause gynecomastia Nelson questions their origin from Sertoli cells and believes them to be a variety of the Leydig cell tumor (144) (see also p 864)

Certain malignant tumors of the testis contain chorion like tissue (chorio carcinoma) they are characterized by excretion of large amounts of gonado tropins which are biologically identical with chorionic gonadotropins of pregnancy Pregnancy tests' (Ascheim Zondek Friedman, rat ovary hyperemia test) carried out with urine of such cases are positive In creased urinary estrogen levels and the occurrence of gynecomastia in some cases suggest that the choriocarcinoma duplicates not only the gonado tropin production but also the estrogen production of normal chorionic tissue There is suggestive evidence that in some cases of choriocar cinoma of the testis there is increased urinary excretion not only of chorionic gonadotropin but also of FSH Because of the technical difficulties of assaying the two gonadotropins separately in a mixture of both the data should at present not be considered final The presence of large amounts of urinary FSH in such cases has been tentatively explained as the result of destruction of testicular tissue ( castration by tumor ) but in view of the fact that the same hormone pattern occurs in unilateral tumors with intact second testis this explanation is not wholly satisfactory Similar observations of increased urinary FSH levels have been reported in other testicular tumors Regarding the significance of this finding the same uncertainty has to be recorded as that mentioned above in the case of choriocarcinomas No essentially new facts have been reported recently a good review of the subject appeared in 1946 (219)

## VIII Puberty (149)

In our population the onset of puberty normally takes place at the age of about 11 to 12 years in girls and 12 to 13 years in boys From this mean age of onset there occurs considerable deviation with onset delayed up to 18 to 20 years or advanced to as early as 1 to 2 years of age (precocious puberty) *The onset of pubertal changes is brought about by the beginning of the secretions of estrogens and androgens respectively* The prepubertal gonad appears to be not completely inactive but to secrete small amounts

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Brown J B 4 (45) *49* 345 *383*
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Brownell G I 449 (684) *516*
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Brand F B 448 (88) *502*
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Butt W R 217 *231* 667 (49) *679*
Buttaro C A 164 (23) *170*
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Constantine A G 766 (878) *807*
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Chapman R I 742 (608) *801*
Chapman W H 759 (190) *790*
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Cheng C P 281 (59 60) 282 (61 62) *298* 719 (1231) 725 (146) *788 818*
Cheng D W 737 (760) 781 (760) *805*
Cheng P 673 (55) *679*
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Chesney A M 461 (112) *502*
Chesterman F C 753 (147) *788*
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Clark H E 30 (67) *49*
Clark I 728 (151) 729 (152) *789*
Clark I C Jr 616 618 (50 1) *656*
Clark L D 718 (153) *789*
Clark R F 497 (675) *516*
Clark W I le Gros 320 *380*
Clarke A P W 716 (157) *789*
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Clarke P M 375 376 *380*
Clary M L 670 (56) *679*
Clausen K H 468 (117) *502*
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Clawson T A 461 (112) *502*
Clayton B F 740 (154) 741 (155) *789*
Clayton J C 452 (118) *502*
Clayton R B 503 (30) *579*
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Clegg M T 344 *380*
Cleghorn R A 716 (157) 717 (156) *789*
Clemons H 467 (487) *511*
Cleroux A P 295 (281) *304*
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Curtis J M 591 (68) *656*
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Eya M 781 (1225) *818*
Eyer H 191 (43) *196*

F

Fager J 458 (468) 459 (468 469) 471 (469) 488 (466) *510 511* 742 (781) 782 (780 783) *806*
Fahey J I 749 (312) *793*
Fajans S S 271 (67) 273 (67) 291 (67) *298* 691 (831 832) 696 (172) 697 (170) 698 (170) 705 (170 172 177) 707 (177) 708 (170) 721 (172 177) 725 732 736 (168 313) 753 (177) 781 (168) *789 793 807* 742 (34) *870*
Falk M S 750 (314) *793*
Farah A 706 (425) 712 (425) *796*
Farber R K 243 (222) 254 (222) *303* 634 (152) *681* 733 734 (692) *804*
Farber S J 270 (81) *299* 395 (22) *426*
Farkas A 577 (88) *581*
Farkas L 577 (88) *581*
Farmer T H 221 (23) *231* 276 (85) 286 (85) 293 (85) *299*
Farnsworth E B 704 *793*
Farquhar M G 314 *381*
Farquharson R F 857 (53) *871*
Farrar B 729 (37) *787*
Farrar M W 579 (16) *579*
Farrell G I 254 (108) 288 (370) *300* 563 (163) *583 876*
Fatt P 114 115 126 *146 147*
Faustini R 78 (205) *87* 104 (92) *146*
Fauvet E 361 *381*
Favour C B 749 (1174) 770 (707) *804* *817*
Fawcett C H 14 *50*
Fawcett D M 441 (190) *442* (191) 462 (190) 473 (190) *504*
Fawcett D W 861 (238) *876*
Fawcett J K 756 (275) *792*
Featherstone R M 122 (218) *150*
Feder N 73 (204) 75 (455) *89 93*
Fedor E J 698 (1120) 708 (1122 1123) 712 (1121) 714 (1122 1123 1126) 722 (1125) *815*
Feil M S 702 (251) *791*
Feinberg S M 773 (1103) *815*
Feldberg W 101 104 (120) 106 (113 114 115) 115 119 (114 116) 136 137 (82) *146 147*
Feldberg W J 117 (36 37) *145*
Feldman J D 703 (316) 744 (316) 748 (317) *793*
Feldman W H 467 (192) *504*
Feldott G 477 (376) 482 (193) *504 508*
Feldstein E M 717 (179) 733 (179) *789*
Feller D D 187 (34) *196* 441 (709) 442 (709) *517*
Fellinger K 435 (89) 436 (89) *502*
Fels G 209 210 *231* 604 (289) *663*
Fettermann G H 455 (138) *503*
Felts J M 188 (85) *197*
Feng Y S L 739 (59) *805*
Fenwick M 288 (307) 289 (307) *305*
Ferguson D J 673 (81) 674 (81) *679*
Ferguson J K W 490 (530) *512*
Ferguson K A 289 (48) *298*
Fern J 671 (82) *679*
Ferner H 192 (47) *196*
Fernholz F 544 (332) *587*
Ferrari J 594 *656*
Ferri M G 28 (82 533) *50 56*
Ferstl A F 288 (109) *300*
Fertman M B 838 *871*
Fessard A 119 (121) *147*
Feurer M 546 (251) 558 (259) *585 586*
Fevold H L 369 *379*
Fickett W 533 (343) *588*
Fieber M H 850 (186) *875*
Fielding U 324 *385*
Fienberg R 766 (318) *793*
Fierce W I 221 222 (112) 223 *233*
Fieser L F 522 527 (92 96) 550 538

Elrick H 239 (97) 272 (55) 273 (55) *290 (95) 298 299*
Elson R 856 (901) *875*
Elster S K 763 (291) *792*
Elvehjem C A 497 (675) *516*
Ely F 400 (61) *427*
Ely R S 285 (380) *307* 694 (244) 745 (624) *791 802* 850 (109 208) *872 875*
Emmens C W 490 (665) *515* 668 *679*
Emerson G A 489 (182) *504*
Emerson K 841 (214) *875*
Emerson W J 672 (74) 674 (74) *679*
Emmes L E 490 (60) *501*
Emmett J W 703 (1232) 719 (1232) *818*
Enami M 60 (188 191 194) 61 (186 192) 62 (184 189) 63 (191) 78 (519) 81 (187) *87 94*
Enbiek H C 722 (1137) *816*
Enders I 59 (195) *87*
*Engback I 117 (80) 146*
Engel C R 566 (147) 567 (147 148) *582*
Engel F L 256 (98) 260 (339) 266 (99) *299 306* 495 (716) *517* 718 (878) 729 (97 296 297 301) 730 (297 298 299 982 983) 732 (982) 734 (299 300) 735 736 762 (984) *787 792 808 811* 841 (52) *874*
Engel G L 718 (302 303) *792 793*
Engel L L 545 (318) *587* 590 (84) 594 600 601 605 615 635 (85) 647 (257 258) 666 (250) 667 (78) 668 *657 662 679 683* 706 (1146) *816*
Engel M B 156 160 *171 172*
Engel M G 256 (98) *299*
Engelbach W 271 (100) *299*
Engelbreth Holm J 753 *793*
Engelman E L 470 (250) *506*
Engelman M 597 (301) 598 (301) *609*
Engfeldt B 158 163 *171 174*
Engle E T 693 (591) *801*
Engstrom A 436 (183) *504*
Engstrom W W 672 (71) *679* 712 (304) *793* 841 (131) 845 (131) *873*
Ennor A H 206 (76 77) *233*
Enrich S 164 (101) *172*
Entenman C 452 (708) 485 (184) *504 516* 753 (1011) *812*
Epstein S H 608 (201a 235 236) 609 (235) 641 (201 234) 642 (201a) 643 (235) *657 660 661*
Epstein D 782 (801) *808* 838 (90) *871*
Epstein M 591 (177) *660*
Epstein L R 724 (410) 725 (410) *796*
Eranko O 417 (62) *427*
Ercoli A 673 (80) *679*
Erickson A L 554 (47 294) 563 (47) *579 586*
*Erickson R I 552 (275) 554 (47) 563 (47) 579 586*
Ershoff B H 238 (101) *299* 337 *381* 453 (188) 459 (188 189) *504* 739 (305 306) *793*
Erspamer V 78 (197–199 200 201 202–204 205 207 516) 79 (196) 82 (202) *87 88 94* 100 (90 91) 103 104 (11) 108 135 136 *144 146 150*
Erulkar S D 187 (52) *196*
Erway W F 478 (452) *510*
Escamilla R 271 (25) *297*
Escamilla R F 271 *299*
*Eser S 192 (33) 196*
Esselier A I 746 (307) 747 (307 308) *793*
Estess A D 747 (945) *810*
Etheridge E M 716 (309) *793*
Etkin W 100 (94 95) *146*
Ettlinger M C 461 (36) *466* (36) *500* 838 (7) *870*
Ettorre R 548 (91) *581*
Eufinger H 488 (189) *504*
Eusman F E 781 (464) *797*
Evans H M 175 (7) *195* 202 (37 60 71) 204 (60) 209 (29) 210 (29) 214 (67) 216 (67) 217 (67) 218 (28) 219 220 221 (31) *231 232* 236 237 (405) 238 (139 250) 239 (22 107 117 151 152 247) 240 (151 362) 241 (21 104 204 249 361) 242 (103 151 152 248 398) 243 (24 151 152 277 262) 245 (261 262) 246 (103) 247 (139 232 24 ) 248 (24 117 247) 249 (2 6) 250 (276 278) 251 (23) 252 (167) 253 (24) 254 (232) 270 (103) 271 (25 297 350) 272 (117) 275 (103) *276* (359) 285 (130) 286 (130 150) 287 (398) 289 (227) *297 298 300 301 302 303 304 305 306 307* 342 369 *381 383 385* 434 (243) 492 (29 560 660) *500 505 513 515* 670 (183) *682* 692 (380) 731 (344) 735 (75) 754 (48) *786 794 795* 867 (187) *875*

Evans I D 770 (1015) 780 (1016) *81°*
Evans R D 438 (111) 456 (340) *50? 508*
Evans P M 554 (31) 556 (31) *679*
Evans A J 769 (310) *793*
Evan W F 493 (694 695) *516*
Evenari M 25 27 *50 53*
Everall J 451 (311) *793*
Everett G M 49? (667) *516*
Everett J W 330 333 (82-84) 334 (233) 368 371 *381 384 385* 671 (162 ?76) *681 683* 853 (130) 869 (130) *873*
Everett N B 346 *381*
Eversole W J 394 (74) *426 427* 712 (375) *795*
Evertzen A 177 (79) *197*
Ewer P F 398 399 (63 63) *42?*
Eya M 481 (1295) *818*
Ever H 191 (43) *196*

F

Fager J 458 (468) 459 (468 469) 471 (469) 488 (466) *510 511* 742 (781) *789* (780 783) *806*
Fahey J L 749 (312) *793*
Fajans S S 271 (67) 273 (67) 291 (65) *298* 691 (831 832) 696 (172) 697 (170) 698 (170) 705 (170 172 177) 07 (177) 708 (170) 721 (172 1 7) 725 732 736 (168 313) 753 (177) 781 (168) *789 793 807* 742 (34) *870*
Falk M S 770 (314) *793*
Farah A 706 (475) 71? (425) *796*
Farber R K 243 (?92) ?54 (?92) *03* 674 (152) *681* 733 734 (692) *804*
Farber S J 270 (81) *299* 305 (22) *426*
Farkas A 547 (88) *581*
Farkas L 577 (88) *581*
Farmer T H 221 (?3) *231* ?46 (85) 286 (85) 293 (85) *299*
Farnsworth F B 704 *793*
Farquhar M G 314 *381*
Farquharson R F 857 (53) *871*
Farrar B 729 (97) *787*
Farrar M W 539 (16) *5 9*
Farrell G I 284 (108) 288 (349) *300* 563 (16 ) *583 846*
Fatt P 114 115 126 *148 147*
Faustini R 78 (?05) 87 104 (9?) *146*
Fauvet E 361 *381*
Favour C B 749 (1174) 770 (707) *804 817*
Fawcett C H 14 *50*
Fawcett D M 441 (190) *44?* (191) 462 (190) 463 (190) *504*
Fawcett D W 561 (238) *876*
Fawcett J K 756 (275) *79?*
Featherstone R M 122 (248) *150*
Feder N 73 (294) 75 (455) *89 93*
Fedor E J 698 (1170) 708 (1122 1123) 712 (1124) 714 (1122 1123 1126) 722 (1125) *815*
Feil M S 702 (251) *791*
Feinberg S M 773 (1103) *815*
Feldberg W 101 104 (190) 106 (113 114 115) 115 119 (114 116) 136 1 7 (82) *146 4*
Feldberg W J 117 (36 3 ) *145*
Feldman J D 703 (316) 744 (316) 748 (317) *793*
Feldman W H 467 (192) *504*
Feldott G 477 (376) 482 (193) *504 508*
Feldstein I M 717 (179) 733 (173) *789*
Feller D D 187 (34) *196* 441 ( 09) 442 (409) *51*
Fellinger K 435 (89) 4 6 (89) *50?*
Fels G 209 210 *?31* 604 (289) *663*
Fettermann G H 455 (138) *503*
Felts J M 188 (85) *197*
Feng Y S I 739 (759) *805*
Fenwick M 288 (307) 289 (307) *305*
Ferguson D J 673 (81) 674 (81) *679*
Ferguson J K W 490 (530) *51?*
Ferguson K A 289 (48) *298*
Ferin J 671 (8?) *679*
Ferner H 192 (47) *196*
Fernholz F 544 (332) *58?*
Ferrari J 594 *656*
Ferri M G ?8 (82 333) *50 56*
Ferstl A E 288 (109) *300*
Fertman M B 838 *871*
Fessard A 119 (121) *147*
Feurer M 546 (?01) 558 (289) *585 586*
Fevold H L 369 *379*
Fickett W 533 (343) *588*
Fieber M H 830 (186) *875*
Fielding U 324 *385*
Fienberg R 766 (318) *793*
Fierce W I 221 2?? (112) 223 *?33*
Fieser L F 522 527 (92 96) 550 538

Ilrick H 239 (97) 272 (55) 273 (55) 290 (55) *298 299*
Ilson R 856 (701) *875*
Elster S K 763 (231) *792*
Ilvehjem C A 497 (675) *516*
Ely F 400 (61) *427*
Ely R S 285 (380) *307* 631 (244) 745 (624) *791 802* 850 (108 208) *872 875*
Emmens C W 490 (665) *515* 668 *679*
Emerson C A 489 (182) *504*
Emerson K 841 (214) *875*
Emerson W J 672 (74) 674 (74) *679*
Emmes L I 490 (60) *501*
Emmett J W 703 (1232) 719 (1232) *818*
Enami M 60 (188 191 194) 61 (186 192) 62 (184 180) 63 (191) 78 (519) 81 (187) *87 94*
Enback H C 722 (1137) *816*
Enders J 59 (195) *87*
Engback L 117 (80) *146*
Engel C R 566 (147) 567 (147 148) *582*
Engel F L 256 (98) 260 (339) 266 (99) *299 306* 495 (716) *517* 718 (878) 29 (97 296 297 301) 730 (297 298 299 982 983) 732 (982) 734 (95 300) 735 736 762 (984) *787 792 808 811* 841 (52) *871*
Engel G I 718 (302 303) *79 3*
Engel L L 545 (318) *587* 590 (81) 594 600 601 605 615 635 (85) 647 (257 258) 666 (250) 667 (78) 668 *657 66 679 683* 706 (1146) *816*
Engel M B 156 160 *1 1 1 2*
Engel M G 256 (98) *299*
Engelbach W 271 (100) *299*
Engelbreth Holm J 753 *793*
Engelman I P 470 (250) *500*
Engelman M 597 (301) 598 (301) *603*
Engfeldt B 158 163 *171 174*
Engle F T 693 (591) *801*
Engstrom A 436 (183) *504*
Engstrom W W 672 (79) *679* 712 (04) *793* 841 (131) 845 (131) *873*
Ennor A H 206 (46 74) *233*
Enrich S 164 (101) *170*
Entenman C 452 (708) 485 (184) *504 516* 753 (1011) *810*
Eppstein S H 608 (201 235 236) 609 (235) 641 (201 234) 642 (01) 643 (235) *657 660 661*
Epstein D 782 (851) *808 838* (55) *8 1*
Epstein M 505 (177) *660*
Epstein I R 724 (410) 725 (410) *796*
Eranko O 417 (62) *427*
Ercoli A 673 (80) *679*
Erickson A L 554 (47 291) 563 (47) *579 586*
Erickson R L 552 (275) 554 (47) 563 (47) *579 586*
Ershoff B H 235 (101) *299* 33 *381* 453 (188) 489 (186 187) *504* 733 (305 306) *793*
Erspamer V 78 (197-199 200 201 202-204 205 207 516) 79 (196) 82 (202) *87 88 94* 100 (90 91) 103 104 (11) 108 135 136 *144 146 150*
Erulkar S D 187 (52) *196*
Erway W F 478 (452) *510*
Escamilla R 271 (25) *297*
Escamilla R F 271 *299*
Eser S 192 (33) *196*
Esselher A F 746 (307) 747 (307 308) *793*
Etess A D 747 (945) *810*
Etheridge F M 716 (309) *793*
Etkin W 100 (94 95) *146*
Ettlinger M G 461 (36) *466* (36) *500* 538 (7) *8 0*
Ettorre R 548 (91) *581*
Eufinger H 488 (189) *504*
Eusman I P 781 (467) *797*
Evans H M 175 (7) *195* 202 (31 60 71) 204 (60) 209 (23) 210 (21) 214 (67) 216 (67) 217 (67) 218 (28) 219 220 221 (31) *231 232* 236 237 (405) 238 (159 250) 239 (22 107 117 151 152 247) 240 (151 362) 241 (21 104 204 249 361) 242 (103 151 152 248 398) 243 (24 151 152 297 362) 245 (261 262) 246 (103) 247 (139 237 247) 248 (24 117 247) 249 (296) 250 (226 228) 251 (23) 252 (167) 253 (24) 254 (232) 270 (103) 271 (25 221 350) 272 (117) 275 (103) 276 (309) 285 (130) 286 (130 150) 287 (398) 289 (227) *297 299 300 301 302 303 304 305 306 307* 342 363 *381 383 385* 434 (243) 492 (29 560 660) *500 505 513 515* 670 (183) *680* 692 (380) 731 (344) 735 (75) 754 (48) *786 794 795* 867 (187) *875*

Foglia V G 671 (86) *679*
Foldes F F 485 (201) *504*
Foley E J 753 (331 332) *795*
Folkers K 251 (84) 252 (78 9 87) 254 (7 47) 276 (7 9 87) *281 283 285* 270 (40 415) 286 (20) 294 (40) *297* *298 308* 558 (289) *580*
Folkow B *147*
Folley S J 169 *170 171* 243 (70) 270 (70) *298* 338 339 340 341 347 (54) 351 (91 107) 352 353 354 355 356 357 358 (30) 359 361 362 (93 94) 363 364 365 366 (91) 367 369 372 375 376 378 *379 380 381 382 385* 400 (39 65) 421 (25) *426 427* 704 (59) 731 (334 335) 732 (335) 734 (51) 784 *786 799*
Follis R H 701 (336 337 388) 754 (336 337 338) *795*
Follis R H Jr 487 (207) *504*
Fondal L 676 (236) *683*
Fonken G S 560 (203) *584*
Fontaine M 398 (65a) *427*
Fontant B J 757 (1069) *814*
Foote F W 499 (196) *504*
Forattini C 488 (203) *504*
Forbes A I 154 (3) *170* 703 (339) 718 (59) 783 (59) *786 794* 850 (14) *870*
Forbes T R 668 (87) *679 680*
Forchielli I 638 (96) 647 (96) 648 (96) *657*
Ford E 451 (15) 454 (15) *500* 782 (15) *785*
Ford W 81 (247) *88*
Ford Z W Jr 481 (699) *516*
Foreman N 696 (158) 781 (634) *789 809*
Forester R H 117 (261) *150*
Forker R F 551 (331) *587*
Forman C W 154 (97) *172*
Forsey R R 833 (142) *873*
Forsham P H 283 (296 399) 291 (233 309 391) 292 (115) 293 (114 115 308 309) 294 (114 309) *300 303 305 307* 419 (204) 495 (301) *504 507* 697 (341 1150) 698 (345) 709 (885) 718 (341 1150) 721 (1150) 732 (68 341) 733 (341) 744 (907) 745 (1150) 748 (514) 781 (346 513) *786 794 799* *809 816* 892 (125) 841 (214) *873 875*
Forsyth B T 706 (342) *794*
Fortier C 844 (56) *871*
Fortune I Y 434 (205) *504*
Foster C G 499 (510) *519*
Foster G I 239 *306* 443 (206) 475 (86) *509 504*
Foster R J 37 (83) *50*
Foster W C 437 (445) *510*
Foulks J 674 (88) *680*
Fourman P 707 (343) 721 (343) 753 (343) *794*
Fowler J L A 716 (157) *789*
Fox C I 772 *807*
Fox F 492 (207) *505*
Fraenkel Conrat H 239 (117) 248 (117) 272 (117) *300* 446 (208) *505* 731 (344) *794*
Fraenkel Conrat J 239 (117) 248 (117) 272 (117) *300* 485 (209) *505*
Frame E G 248 (118) 249 (118) *300*
Francis T 779 (12) *785*
Frank A H 338 340 *385*
Frankel S 727 (919 937) 728 (918) *810*
Franklin A E 443 (260) 449 (260) *506*
Franklin A L 442 (211 638 710) 446 (637) 462 (211) 464 (637) 469 (210 637) *505 515 517*
Franks W R 497 (53) *501*
Franksson C 102 (104) *146* 694 (381) *795*
Frantz R 733 (637) *802*
Fraps P M 333 338 340 *382 385* 671 (182) *680*
Fraser A M 393 (66) 401 *427 430*
Fraser R 208 *232* 470 (166) *504*
Frawley T F 291 (391) 292 (115) 293 (115) *300 307* 697 (1150) 698 (345) 713 718 (1150) 721 (1150) 745 (1150) *794 816* 841 (57) 842 (215) *871 875*
Fredga A 11 (84 85) *50* 536 (104) *581*
Fredericq E 176 (42) 177 (42) *196*
Frédéricq H 131 (124) *147*
Fredrickson D S 697 (1150) 718 (1150) 745 (1150) 781 (346) *794 816*
Free A A 452 (118) *502*
Freed S C 715 (944) *794 810*
Freed V H 5 (86) *50*
Freedberg A S 451 (217) 486 (72) *501* *506* 714 (660) *803*
Freedman H H 295 (66) *298* 783 (48) *794*

(152) 546 (89) 548 (91 99) 549 (90 99) 552 (91) 556 (91 98 100) 558 (97 152) 566 (96) *581 582* 604 (67 170) 615 (170) 632 (260b) *656 660*
Fieser M 522 527 (92) 530 552 (94) *581*
Fignolo W 480 (140) *508*
Finch C A 748 (514 1141) *799 816*
Finch S C 747 (319) *793*
Findlay C W 767 (320) *793*
Finerty J C 287 (159), *301* 671 (208) *682* 758 (321) *793*
Finestone A J 763 (1019) *812*
Fingal F 719 (322) *793*
Fingerman M 61 (75 525) 62 (76 77 78 79 80 82 83 421) *85 90 94* 116 (35) 117 (261) *145 150*
Fink K 443 (194) *504*
Fink R M 443 (194) *504*
Finkel A J 768 (1204) *818*
Finland M 270 (200 202) 281 (200) 291 (202) *302* 724 (616) 725 (616) 743 (617) 761 (617) 779 (614) 780 (612 613 615) *801 802*
Finnerty J 222 (5) *231* 256 (50) *237*
Finney D J 212 *231* 391 (30) 392 (30) 393 (30) *426*
Fiori C 366 *379*
Fiori W 373 *379 381*
Fischel I F 761 (324 326) 774 (88 323 324 325 326) 780 (901) *787 793 809*
Fischer A 2 (76 77) 3 (76 77) 28 (77) *49*
Fischer E 533 *581*
Fischer F I 104 (149) *147*
Fischer R 531 (102) *581*
Fischer P 78 (25) *84* 135 (14) *144*
Fischer W H 698 (473) *806*
Fish C A 265 (93) *299* 600 647 (89) *657 659*
Fish W A 449 (94) *502*
Fish W R 599 605 617 636 (144) 637 (144) 649 (141) *657 659*
Fisher A 779 (1023) *819*
Fisher C 321 (71) *381*
Fisher D A 492 (414 518 519 520) *509 512*
Fisher I R 437 (195) *504*
Fisher C 838 (55) *871*
Fisher C R 692 (977) 735 (977) 737 (234) *791 811*
Fisher J D 220 221 (51) 222 (102) *232 233* 277 (390), 292 (390) 294 (390) *307* 740 (1101) *815*
Fisher L, 754 (311) *793*
Fisher R J 252 (110) 253 (110) *300*
Fishman, J B 202 (115) 203 (115) 204 (92) *233 234* 236 252 (414) 265 (414) 271 (414) *308*
Fishman I 840 (29) *870*
Fishman L W 669 (85) *679*
Fishman W H 669 (85) 670 (83) 676 (84) 693 (101) 701 (534) *679 787 99*
Fitch D R 764 (959) *812*
Fitts W T, 715 (1214) *818*
*Fitzgerald J D I 771 (746) 805*
Fitzgerald P 499 (196) *504*
Fitzgerald P J 352 (244) *386*
Flanagan C K 729 (728) *804*
Flanagan J B 711 (327) *793*
Flataker L 763 (1216) *818*
Fleischman W 434 (197) 493 (743) 488 (198) 492 (197) *504 518* 653 (320) *664* 851 (234) *876*
Fleisher G A 569 (103) *581*
Fletcher A A 749 (328) *793*
Flink E B 286 (92) 291 (92) *299 691* (341) 718 (341) 732 (341) 733 (341) 745 (450) *793 794 797*
Flink M G 286 (92) 291 (92) *299*
Flint A H 461 *504*
Flock E V 455 (200) *504*
Florentin A A 286 (394) *307*
Flores I 67 (209 211 212 213) 78 (212) 79 (212) 81 (210) *88* 100 (122) 104 (122) 105 (122) 135 *141*
Florey Elisabeth 67 (212) 78 (212) 73 (212) *88* 100 (122) 104 (122) 105 (122) 135 *147*
Florkin M 79 (26) *84*
Florschutz P A 95 (316) *56*
Flowe B H 756 (949) *811*
Fluckiger F 81 (214) *88*
Fluharty R C 440 (443) 499 (551 557) *510 512 515*
Fluhmann C F 784 (330) *793*
Flux D S 365 (88) *381*
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Grunt J A 479 (268) *506* 673 (90) *680*
Grynkewich S F 653 (77) 834 (77) *872*
Gubler C J 748 (463) *797*
Gudernatsch J F 491 (269) *506*
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Hundley J M 75? (550) *800*
Hungerford G F 744 (551) 746 (552) *800*
Hunt C 719 (518) *799*
Hunter C D *656*
*Hunter J H 550 (236) 585*

Hunter O B 775 (553) *800*
Hunter S F 739 (554) *800*
Hurloch B 634 (54) 635 (54) *650*
Hursh J B 460 (390) *507*
Hurtz N 395 (29) *426*
Hurxthal L M 837 (94 96) 867 (95) *874*
Huston T M 332 (151) *385*
Hutchings I M 765 (760) *91*
Hutchinson J H 837 (137) *875*
Hutchison R 417 (391) *507*
Hutter O F 117 (166) *148*
Huxley A F 126 (167) *148*
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Hyman G A 740 (555 556) *800*
Hysom G 670 (137) 675 (137) *680*

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Ikonen M V 496 (327) *507*
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Ingalls T H 241 (184) *307*
Ingbar S H 459 (323) *507* 699 (597) 794 (616) 725 (558 616) 780 (615) 724 (616) 725 (558 616) *780* (615) *800* *801* *807*
Ingestad T 38 (131) *51*
Inhoffen H H 540 (344) 548 (171) *587* *588*
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Ingram J M A 14 (81) *50*
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Irwin J O 418 (111) *428*
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Itoiz J 449 (681) *516* 878 (198) 837 (198) 839 (198) *875*
Iversen K 834 (5) *870*
Ivon Sallman I 767 (585) 68 (585) *801*
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Ivy H B 788 (1095) *814*
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Jacob S 776 (869) *808*
Jacob W 407 (218) *505*
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Jaques I B 771 (788) *806*
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Johnson B B 290 (13) 292 (13) *297* 666 (12) *678* 702 (596) *801*
Johnson B C 737 (761) *739* (761) 1181) *791* *817*
Johnson B J 705 (177) 707 (177) 721 (177) 732 (177) 753 (177) *789*
*Johnson C 776 (79) 787*
Johnson C F 459 (426) *510* 831 (129) *873*
*Johnson G I 436 (324) 507*
Johnson H W 451 (328) 452 (328) 454 (328) *507*
*Johnson J A Jr 553 (340) 588*
Johnson P 770 (933) *810*
Johnson I C 452 (329) *507*
*Johnson R B 755 (104) 787*
Johnson R I 270 (238) *303* 349 *383* *384*
*Johnson R H 604 (107) 658*
Johnson R M 488 (330) *507*
Johnson S 493 (112) *498* 744 (484) *798*
Johnson S G 848 (103) *872*
Johnson T B 443 (331) *507*
Johnson W S 537 539 *588*
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Johnston S F 753 (140) *788*
Johnston T N 708 (272) *797*
Johnwick E B 833 (38) *871*
Joiner C I 724 (597) *804*
Joly I 76 (287) 82 (287) *89*
Joly P 68 (282) 70 (283) 76 (284 286 287) 82 (285) *89*
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Jones B M 71 (288) *89*
Jones D M 670 (262) *684*
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Jones G F S 784 (598) *801* 860 862 (104) *872*

Jones H B 485 (236) *505*
Jones H S 713 (703) *804* 841 (120) *8 3*
Jones H W 837 (122) *8 3*
Jones H W Jr 600 *659* 647 (123) *680*
Jones I C 239 (197) 2 8 (195 197) *301 30? 33? 38? 595 4 8*
Jones I S 767 (580 599) 768 (580) *801*
Jones J K N 667 (104) *680*
Jones J R 781 (64 192a) *786 818*
Jones M E 66 (116) *681*
Jones R N 592 (19) 549 (175) *580 583*
Jones T I 695 (431) *9?*
Jones W J 341 *383*
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Kaplan A 860 (58) *871*
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Kaplan N O 187 (72) *197* 737 (606) *801*
Kaplan S J 719 (947) *811*
Kappas A 698 (1000) *817*
Kapps A 830 (183) *875*
Kar A B 671 (129 130) 673 (131) *680* 783 (950) *811*
Kark A L 756 (1260) 759 (810) *807 819*
Kark R H 744 (84) *787*
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Karlson A G 777 (609) *801*
Karlson P 72 (290 291) *8? 89*
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Katsch G F 743 (411) 761 (411) *796*
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Katz A 501 (183) *583*
Katz B 114 115 196 *146 147* 7?2 (160) 751 (288) *789 792*
Katz L N 486 (139) *503* 716 (1089) *814*
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Ray S N 740 (469) *797*
Ray T W 436 (161) *503*
Raybuck H F 161 (119) *173*
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Read C H 693 (893) 738 (893) *809*
Reaven G 156 (6 7) *170*
Rebell G 778 (613) 779 (613) *802*
Reber F 581 (326) *587*
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Richards J B 287 (310) *305 809*
Richards J H 2?3 (135) *300*
Richardson F M 527 (?45) *585* 634 635 (?48) 636 (?48) *66?* 859 (50) *871*
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Richfield D F 754 (406) 770 (408) *796*
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Riniker B 536 (6a 264a) *578 585*
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Ritter H 277 (264) *304*
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Roberts R H ?0 (247) *54*
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Robinson J 132 133 134 (53) *146*
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Robinson T J 335 339 (224 226) *385* 671 (214) *687*
Robinson W D 727 (929) 732 (929) 765 (929) 766 (263) 781 (1225) *792 810 818*
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Robson J B 670 (211) *687*
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Roche M 291 (391) *307* 697 (341) 718 (341) 732 (63 341) 733 (341) 765 (1149) *786 794 816*
Roche P 777 (932) *810*
Rockenbach J 780 (1008) *813*
Rockwell F M 770 (407 933) *796 810*
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Sato H 497 (336) *507*
Sauer G 556 (150) *582*
Saunders F J *661* 714 (427) *796*
Saurino V R 780 (220) *790*
Savard K 289 (163) *301* 525 (291) *586 610 611 (31–33) 612 615* 619 (969) 623 (121 122) 626 634 (31–33) 635 (31–33) 645 (270) 648 (31) *655 656 657 658 662* 667 (225) *683*
Savit J 77 (501) *94*
Sawasa I 61 (425) *92*
Sawyer C H 322 3 0 333 (82–84) 334 (233) *381 384 385* 671 (162 226) *681 683 853* (130) 869 (130) *873*
Sawyer J 1 6 (6 7) *170*
Sawyer M K 137 (240) *149 430*
Sawyer P N 775 (57) *786*
Sawyer W H 137 (240) *149* 307 399 *398* (178) *429*
Sax K J 550 (6) 552 (6) 553 (6) 560 (6) *578*
*Sayers D 818*
Sayers G 166 *174* 207 (35) 220 221 (24 93) *231 233* 2 6 (331 332) 281 (328 331) 282 (61 62) 285 (328 330 331 380 381) 286 (44) 287 (310 329 382) 290 (28 331) 293 (135 136) *298 300 305* 307 375 376 (32 33) *380 391*

(180) *450* 6.0 (972) 683 (- 1) *66?* 692 (960) 701 703 (1229 1239) 715 (1239 1231 1239) 775 (116 960) 730 (969) 745 (960) 757 782 (1230) 788 *809 811 818* 843 (298a) 850 (908) *8 5*
Sayers M A 290 (88) *433* 946 (331 339) 281 (331) 285 (330) 286 (41) 290 (331) *298 305* 630 (972) *66?* 701 *811*
Scala F *450*
Schachner H 449 (638) 446 (637) 464 (634) 469 (637) *515*
Schachter M 104 (109) 137 (8 ) *146 148*
Schaefer A F 752 (5-9) *785*
Schaefer I I 721 (103) 7 5 (103) *787*
Schaeffer R L 271 (333 334) *305*
Schaffenburg C A 595 (199) *660* 740 (963) *811* 867 (135) *8 3*
Schaffer N K 238 (335) 247 (335) 248 (335) 272 (335) 275 ( 14) *30 305*
Schairer M A 755 (45) *786*
Schallek W 190 (300) *151*
Schaller F 68 (196) 77 (42, 4 8) *9?*
Scharfman W B 840 (295) *8 6*
Scharles F H 480 (639) *515*
Scharrer B 58 (431 439 434 440 445-449) 59 (439 445 449) 60 (418 419) 61 (440) 67 (437) 68 (431 43 433 434 435 436 437 440) 69 (434 437) 70 (434 438 439) 71 (439 434 440 448 449) 73 (439) 74 (442) 6 (437 449 449) 77 (449) 78 (430 431) 79 (448 449) 81 (431) 8? (436 443 444) *9 93* 98 105 107 113 (24? 24?) 138 139 *149 150* 331 *385*
Scharrer F 58 (445-449) 59 (445 449) 69 (448 449) 71 (448 449) 6 (449) 77 (449) 79 (448 449) *9? 93* 98 105 107 113 (247) 138 (246 247) 13? 14? *149 150* 331 *385* 391 (14) *4?6 4 9*
Scheer B T 66 ( 98 450 451 461) 67 (451) 80 (450) *89 93*
Scheer M A R 66 (450 451 461) 67 (451) 80 (450) *93*
Scheie H G 4 5 (834) *807*
Scheinberg I 491 (640 641) 493 (640 641) 494 (641) *515* 718 (964) 769 (964) *811*
Scheinberg S R 758 (594) 759 (594 595) 766 (964) 769 (594) *801 811*
Schenker V 606 (133) 693 (133) 6 4
(133 150 151 167 168) 625 (133 167) 646 (168) 648 (151) *658 659* 704 (965) *811*
Scherr M A 560 (203) *584*
Scherr G H 701 (966) *811*
Scheuing M 150 (253) *150*
Scheurmann R 25 ( 56) *54*
Schieve J F 718 (967) *811*
Schiffer F 734 (968) *811*
Schild K T 450 (441) *510*
Schiller E 768 (969) *811*
Schiller S 604 615 616 618 *66?* 674 (297) 676 (108) *680 683* 770 (480) 479 (301) 63 (66) *786 79? 798*
Schindler O 572 (308a) 573 (308a) 574 (308a) *587* 622 (286 287) *663* 687 (1079) *707 813*
Schlapp W 392 (10 ) *4?8*
Schlayer C 414 (870) *808*
Schmid H 544 (992) *586*
Schmid J 988 (109) *500*
Schmid K 450 *515*
Schmidlin J 552 (293) 557 (293 563 (325) 567 (23?) 571 (325) *585 586 587* 698 (1907) *818*
Schmidt C 410 ( 69) *518*
Schmidt I I 73 (454) 80 (452-454) 81 452) *93*
Schmidt I H 780 (940) *811*
Schmidt Thomé J 560 (41) *579*
Schneebeli G I 704 (257 941) *6? 791 811*
Schneider C I 14 (271) 37 (971) 38 (971) *54*
Schneider E 488 (643) *515*
Schneider J J 616 618 634 (276 2 ) 635 (276 ? ?) 637 638 (978) 646 ( 51) 647 (978 989) 648 (279 980 25 ) *663* 740 (807) *80?* 842 (150) 845 (33 149) 848 (33) 861 (150) *8?0 8 4*
Schneider R 164 (196) *1 5* 635 (171) *656 660*
Schneider W P 537 (172) 556 (98) *581 583*
Schneiderman H A 43 (109 110) *0* 73 (994 460) 75 (455 456 457 458 459 460) *89 93*
Schneierson S S 771 (1052) *813*
Schoch F P 743 (973) *811*
Schocken V 28 (257) *54*

Schoenheimer R 591 (251) *662*
Schoenewaldt, E F 534 (291) *586* 598 (302) *663*
Scholz C R 549 (75) 576 (209) *580* *584*
Schooley J I 211 *231* 316 *355*
Schoplick R R 560 (58) *871*
Schour I 169 (14 15 79) *170* *172*
Schrink A R 19 (259) 20 (258 259 260 323) *54* *56*
Schreiber I C 536 (201) *584*
Schrek R 703 (974 976) 741 (974-976) *811*
Schricker J A 692 (977) 735 (977) 737 (978) *811*
Schubert J 768 (1204) *818*
Schueler F W 122 (248) *150*
Schuler W 704 (751) 707 (229 979) *791* *805* *811*
Schulman I 467 (487) *511* 751 (980) *811*
Schulman J Jr 437 (611) 442 (611) *515*
Schultz R B 737 (981 1223) 738 (1221 1222) *811* *818*
Schulze I 459 (645) *515*
Schupbock S 164 (98) *172*
Schusdek A 188 (5) *195*
Schwabe C W 66 (461 461) 67 (461) *93*
Schwartz A 773 (1152) *810*
Schwartz H S 490 (51) 491 (51) *501*
Schwartz I I 270 (81) *299*
Schwartz R 483 (515) *512*
Schwartz S 452 (267) *506*
Schwartz T B 287 (329) *305* 718 (878) 730 (299 962 982 983) 732 (982) *792* *808* *811*
Schwartzman G 751 (1053) *813*
Schwarz H 166 *173*
Schwarz I I 335 (22) *426*
Schwarzer K 826 (180) 827 (180) *874*
Schweizer M 215 (336 337) 282 (336 338) *306*
Schwenk E 539 (134) 549 (205) 575 (135) 576 (135) *582* *586* 591 (253) 594 (324) *663* *664*
Schwert C W 414 *429*
Schwinck I 68 (469) 75 (463) *93*
Scotoni R Jr 558 *581*
Scott C C 459 (646) *515*
Scott F I 25 (170) 33 (169) *52*
Scott J C 399 (155) *429*
Scott J L Jr 260 (339) *300* 734 (300) 735 736 762 (984) *792* *811* 841 (52) *871*
Scott, K G 456 (274) 471 (567) 492 (520) *506* *512* *513*
Scott Watson H M 163 (51) *171*
Scow R O 239 (22) *29* 482 (644 648) *515* 674 (298) *683*
Scudamore H H 66 (464) *93*
Sealock R R *430* *431*
Seaman G R 80 (465) *93*
Searcy A M *560* *(203)* *584*
Searle C I 465 (649) 466 (649) *515*
Searle G I 187 (34) *196*
Searle N Z 436 (55) *501*
Seastone C V 761 (189) 776 (189) *790*
Seay P 607 (1121) 698 (1121) 792 *(1125)* *815*
Sebrell W H 737 (193) *790*
Sechet M 37 (135) *52*
Secker J 716 (985) *812*
Secklen J 840 (29) *870*
Seegal B C 715 (646 647) 716 (647) 769 (576) 771 (646) *799* *802*
Seeley R D 482 (384) *509*
Segal G 759 *803*
Segaloff A 211 *231* 340 (395) 343 571 375 380 *385* 606 631 647 (146) 648 (146) *659* 677 (100) *680* 722 (986) 736 *819* 856 (204) *875*
Segrè E 436 (129 397 650) *503* *50* *515*
Seidlin S M 436 (55) *501*
Seifter J 761 (991) 764 (988 988 989 990 991) 771 (990) 773 (990) *791* *810*
Selenkow H A 486 (30) *487* (30) 490 *(30)* *500*
Seligman A M 690 691 (212) *790*
Sell H M 5 (241) 42 (213a) *53* *54* *56*
Selle W A 430 (463) *510*
Sellers I A 481 (651 804) *515* *519* 758 *(992)* *812*
Selliber R 72 (466) *93*
Selm W 488 (203) *504*
Seltzer H S 271 (67) 273 (67) *298*
Selye H 243 (342) 246 (960) 268 (341) 269 (340 341 342 345) 270 (341 343 347) 281 (341 346) 295 (101 344) 302 303 *306* 361 367 *379* *386* 704 (995) 1001) 714 (1006 1007) 720 750 (499 1005) 756 (997 1004) 757 761

765 776 (998 1000) 777 (686) *798 803 812*
Sen S C 163 *173*
Sen S P 2 (961) *54*
Seneca H 75 (1008) *819*
Senger G 674 (88) *680*
Senenbach W 718 (1009 1010) *812*
Sequeira L 29 (963) *54*
Serchi C 594 *603*
Serini A 549 (296) *586*
Sereni I 77 (468) 79 (467) *93*
Sereni F 709 (954) 711 (954) *811*
Serres C Jr 574 (114) *581*
Severinghaus A F 310 *386* 869 (181) *874*
Sevringhaus F H 604 (2 5) 615 (275) *662*
Sgouris J T 358 359 (193) 360 361 362 *384 386*
Shacter B 753 (1011) *819*
Shafer W G 674 (229) *683*
Shaffer J M 773 (477) *797*
Shaffer P A 483 (652) *515*
Shaffhausen D D 366 *383*
Shaffner C S 460 (509) *507*
Shanaphy J F 700 (249) 775 (249 101 ) *791 812*
Shank R E 740 (986) 741 (986) *792*
Shannon J A 395 (183) *430*
Shapiro R 769 (1013) *812*
Shappirno D G 75 (469 470) *93*
Sharaf A A 668 *682*
Sharp E A 441 (450) 442 (450) *510*
Sharp J T 691 (1053) *813*
Sharpless G R 461 (653) *515*
Shaw E 136 *151*
Shaw G B 269 (74) 270 (75) *299*
Shaw J C 365 (240) 366 *380 386*
Shaw J H 237 (348) *306*
Shaw W 188 (121) *198*
Shay J C 489 (460) *510*
Shedlovsky T 215 (16 118) 218 (118) *231 234*
Sheahan M M 475 (418 654 776) 476 (418 777 778) *509 515 518*
Sheehan H L 189 (31) *196* 691 (890) 692 (1014) *807 812* 848 (182) *874*
Sheehan J C 561 576 *586*
Shelberg W E 537 (172) *583*
Sheldon W H 770 (1015) 777 (193) 780 (1016) *790 812* 845 (66) *871*
Sheline G F 830 (183) *875*
Shelley H 79 (95) 80 (95) *85* 119 (44) *145*
Shelton E K 271 (349 350 351) *300*
Shemano I 668 (230) *683*
Shepanek I A 767 (1017) *819*
Shepherd D M 103 (249) 103 *150*
Sheppard R 703 (843) 706 (584) *801 808*
Sherwood S I 136 *147*
Shetlar M R 156 *173*
Shick R M 766 (1018) *819*
Shimkin M B 243 (365) 245 (365) *306* 763 (1011) *819*
Shipley I G 544 (136) *582* 668 (111) *680* 702 (604) *801 870*
Shipley R A 194 *198* 252 (353) 253 (352 253) 275 (352) *306* 605 (79) 617 (81) 618 (79) 636 (144) 637 (144) 649 (144) *657 659* 703 (252) *791*
Shipner L B 260 (354) *306*
Shippel S *871*
Shipton M C 15 (12) 31 (12) *48*
Shock N W 693 (1054) *813*
Shoji K 29 (264) *54*
Shoppee C W 521 (301) 523 526 (303) 527 (299) *532* 540 ( 00) 544 (304 305) 558 (45 273) 559 (45 298) 564 (303) 565 (189 259) *579 583 585 586 587* 610 622 (247) *662*
Shorey J M 154 (104) *172*
Shorr E 187 (123) *198* 206 *233* 271 273 (355) *306* 483 (571) *513* 716 (1263) *819* 827 (184) *875*
Showacre J L 39 (965) *54*
Shubik P 751 (60a) 753 (60a) 762 (916) *786 810*
Shulman M H 716 (1244) *819*
Shuman C R 763 (1019) *819*
Shure N 773 (1184) *817*
Shwartzman G 734 (25) 779 (1020 1021 1022 1023) *785 812*
Sicé J 527 (118) *581*
Sichel R 768 (441) *797*
Sicuteri F 488 (655) *515*
Sidorkina M Ya 497 (656 657) 498 (656 657) *515*
Siebenmann R 765 (1024) *812*

Siecke V 747 (443) *797*
Siegel A N 828 (41) *871*
Siegel B M 746 (668) *803*
Siegel I 436 (55) *501*
Siegel S 521 (306), 525 (306) *587*
Siegel S M 39 40 (266) *54*
Siemienski J S 781 (61) *786*
Sifford R H *431*
Sikand R S 672 (179) *68?*
Silber R H 154 (136) *173* 700 (1219) 723 (1219) 724 (1025) 729 (1025) 734 (1025) 735 (1219) 750 (1219) 752 (1219) *813* *818*
Silberberg M 246 (356 357) 270 (358) *306*
Silberberg R 270 (358) *306*
*Silberger J 22 35 (267) 54*
Silberstein F 595 (105) *658*
Silides D N 202 *23?*
Siliprandi D 187 (121) *198*
Siliprandi N 187 (121) *198*
Silver H K 830 (186) 869 (185) *875*
Silver M 376 (241) 377 *386* 669 (231) *683*
Silver S 450 (658) *515* 830 (186) *875*
Silverman S H 714 (1212) *818* 851 (36 235) 852 (36 233) *871* *876*
*Silverstein R 753 (332) 793*
Silverstone G 553 (30) *579*
Silveus E 482 (725) *517*
Simay Kramar M 716 (656) *803*
Simcox S J 827 (172) *874*
Simkins S 476 (659) *515*
Simmons F A 865 (91) *872*
Simms H D 737 (792) *806*
Simond A E 236 (42) 270 *298*
Simonsen M 722 (1137) *816*
Simpson M F 202 (37 60 71) 204 (60) 209 (29) 210 (29) 214 (67) 216 (67) 217 (67) 218 (28) *231* *232* 236 237 (405) 238 (250) 239 (22 107 151 152 *247) 240 (151 362) 241 (21 104 204* 249 361) 242 (103 151 152 248 398) 243 (24 151 152 247 262) 244 (105 106 151 152 248) 245 (261 262) 246 (103) 247 (232 247) 248 (247) 250 (96) 251 (96) 254 (237) 270 (103) 271 (227) 275 (103) 276 (259) 287 (398) 289 (227) *297* *299* *300* *301* *302* *303* *304* *305* *306* *307* 342 369 *381* *483* 492 (70) 560 660) *500* *513* *515* 731 (314) *794* 867 (187) *875*
Simpson S 492 (661) *515*
Simpson S A 572 (121 507 308 508a) 573 (308a) 574 (307 308a) *58?* *587* 622 (286 287) *663* 669 (93?) *683* 687 (1029) 702 (1026) 707 (444 1131) *797* *813* *815*
Simpson S I 673 (233) *683*
Sinex F M 186 (145) 188 *198* 727 (1138) 728 (1030) *813* *816*
Singer B 702 *813*
Singer M 130 (252 253) *150* 437 (149) *503*
Singer M J 293 (244) *303* 696 (725) 755 (725) *804*
Singh G R *396* (29) *426*
Singh T 539 *588*
Sinkoff M W 243 (82) 256 (363) 265 (82 83) 266 (82) *299* *306* 373 *386* 726 (213 214 215 216) *790*
Sinn L 190 (135) 191 (135 136) *198*
Siperstein E 312 315 *386*
Siperstein M D 576 (46) *579*
Sirek O V 712 (1032) *813*
Sirota J H 155 *173*
Si son J H 709 (557) *800*
Si sons H A *767 (1033) 813*
Sita G E 552 (275) 554 (47) 563 (47) *579* *586*
Sjogren B 705 (714 1034) 707 (714) 708 (713 714) 709 (1031) 711 (714 1034) *804* *813* 818 (4) *870*
Skaggs R H 758 (143) *789*
Skanse B 440 (663) 441 (662) 499 (557) *513* *515*
Skoog F 14 (271) 18 19 (269) 22 28 (268) 35 (267) 37 (271) 38 (271) *53* *54*
Skowran S 471 (664) *515*
Sladek S A 492 (670) *516*
Slater R J 405 (175) *428*
Slater R R 45 (300) 46 (300) *55*
Slattery P A 243 (365) 245 (365) *306*
Slaunwhite W R Jr 594 (87) 613 (288) *657* *663* 667 (78) *679*
Slechta R F 671 (234) *683*
Slein M W 185 (75) *195*
Sles or A 607 (74 341) 709 (1035) 718

(311) 732 (311) 733 (341 741) *786* *794* *815*
Sloan L W 483 (508) *512*
Slocombe A G 192 (301) *151* 719 (518) 720 (1036) *799* *815*
Slocumb C H 705 (1080) 707 (1080) 708 (1080) 709 (1080) 721 (1080) 724 (1080) 732 (1080) 753 (1080) 765 (495 496 497 883 1186) 766 (883 1186) *798* *809* *814* *817*
Slosse J 481 (87) 486 (82) *507*
Slotta K H 604 *663*
Sluczewski A 672 (235) *683*
Smelser G K 254 (364) *300*
Smith A H 187 (45) *196*
Smith A L 763 (926) 776 (926) 778 (926 927) *810*
Smith A U 490 (665) *515* 693 (849) *808*
Smith B 81 (268 269) 89 398 (991) *498*
Smith C C 164 *173* 754 (115) *815*
Smith D C 435 (666 668) 476 (669) 497 (667 670) *515* *516*
Smith D E 672 (189) *683* 759 (1111) *815*
Smith D J 757 (1037) *815*
Smith D I 759 (1058) *815*
Smith D R 847 (203) 857 (203) *875*
Smith F 225 (91) *233*
Smith F A 489 (291) *506*
Smith E L 204 221 *230* *231* *233* 287 (2 329) 293 (2 135 136) *296* *300* *305* 750 (967) *811*
Smith F M 766 (763) *798*
Smith I 495 (184) *430* 750 (1046) *815*
Smith G F 6 (124) *51*
Smith G H 763 (1039) *815*
Smith G N 676 (200) 615 (200) *660*
Smith G V 861 (190) 863 *875*
Smith G V S 594 *663*
Smith H L 765 (497) *798*
Smith H M 490 (60) *501*
Smith J A 187 (41) 192 (36 38 40) *196* 260 (113) 261 (112) *300*
Smith J M 779 (1040) *815*
Smith M C 243 (365) 245 (365) *300*
Smith M I *430*
Smith M S 11 *54*
Smith O W 594 (290) *663* 861 (190) 863 *875*
Smith P K 239 *306* 491 (671) *515* 827 (188 189) *875*
Smith R A 342 (213) 343 *386*
Smith R C 770 (1242) *819*
Smith R H 185 (103 196) *198* 257 (301) 262 (301) *305* 478 (672 673) 479 (672 673) 480 (672 673) *516* 727 (903 1041) *809* *815*
Smith R I 437 (671) *516*
Smith R W 749 (484 1044) 768 (1043 1096) 784 (1042) *806* *815* *814*
Smith R W Jr 206 (90) *233* 271 (355) *306* 827 (184) *875*
Smith S C 497 (675) *516*
Smith S G 708 (1045) *815*
Smith S W *150*
Smith T C 676 (236) *683*
Smith V R 332 (172) *384*
Smith W W 759 (1046) *815*
Smull K 718 (946) *791*
Smyles E 590 (172) *660* 667 (153) *681*
Smyth C J 764 (509) *798*
Smyth F 492 (590) *512*
Smyth F S 492 (414) *509*
Smyth N P D 781 (1130) *815* 836 (909) *875*
Snair W D 243 (51) 262 (51) 265 (51) *298*
Snelling C E 751 (1047) *815*
Sniffen R C 865 (91) *879*
Snoddy C S Jr 559 (166) *683*
Snodgrass R F 71 (471) *93*
Snyder J G 695 (1048) *815*
Sobel E H 155 (138) 162 (138) *173*
Sobotka H 488 (676) *516* 846 (192) *875*
Sodig H G 5 7 *54*
Soffer L J 459 (677) 488 (676) *516* 705 (1050) 771 (1052) 782 (1049 1051) 783 (367 368 369) *794* *795* *815* 828 833 (127) 846 (191 192) *873* *875*
Soiva K 863 (148) *874*
Sokoloff L 691 (1053) *815*
Solandt D Y *498*
Solem J H 295 (368) *306*
Soley M H 436 (775) *506* 837 (81) *872*
Solganils R I 482 (678) *516*
Soloman B 436 (783) *518*
Solomon D H 693 (1054) 747 (1104) *815* *815*
Solomon S 602 698 *663*

Solomon S S 595 (127) 609 (127) *658*
Solotorovsky M 761 (1110) 774 (1110) 777 (1055) *813 851*
Soloway A H 529 (309) 571 (310) *587*
Soloway S 732 (68) *766*
Somers G F 452 (118) *507*
Somerville I F 667 (237) *683* 863 (193) *875*
Somerville W 697 (1057) 713 (1056 1057) *813*
Sommers S C 670 (238) *683* 770 (1058) *813*
Sondheimer F 521 (272) 579 (212) 537 (316) 538 (316) 544 (284) 46 (314) 550 552 (312) 554 (70 313) 558 (313) *580 584 586 587 588* 608 (179) *660*
Sonneborn T M 80 (472) *93*
Sonenberg M 352 *386* 456 (679 680 681) 473 (679) *516*
Sorkin M 526 *587*
Sorkin S Z 846 (192) *875*
Sorrentino J 829 (17) *870*
Soskin S 175 (127) *198* 260 (354) *300*
Sougin Mibashan R 865 (194) *875*
Soukop S W 376 (189) 377 (189) *384*
Soulavine A 672 (239) *683*
Sourkes T I 724 (1059 1060) *813*
Southam C M 79 (1061) *813*
Spadoline I 128 (138 255) *147 160*
Spain D C 743 (1065 1066) 747 (1064) 757 (1062) 763 (1067) 766 (1065 1066) *814*
Spain D M 2 0 (363) *306* 743 (779) 777 (1063) *806 814*
Spait T H 749 (1068) *814*
Sparrow F M 775 (1069) *814*
Spaul E A 417 (186) *430*
Specht N W 830 (195) *875*
Speeter M E 104 (756) *160*
Speirs R S 701 (1070) 703 (1072 1941) 746 (1070–1072 1941) 747 (848 1071) 752 (671) *803 808 814 819*
Spence A W 461 (437) *510*
Spence T 497 (392) *509*
Spencer A G 702 (1073) *814*
Spero C B 641 (201) *660*
Sperti I 257 (79) *299*
Spies J R 298 *233*
Spies T D 745 (1074) 751 (1075) *814*
Spingarn C I 710 (802) *807*
Spink W W 779 (2) *784*
Spinks A 497 (682) *516*
Spiro H M 756 (122) *796*
Spisni D 164 (132) *178*
Sprague R G 634 (198) *660* 696 (884) 697 (1079) 701 (1077 1078) 704 705 (1080) 707 (1076 1080) 708 (1080) 709 (1080) 714 (884) 721 (1080) 724 (1080) 725 (1079) 727 (1076) 732 (1080) 753 (1080) 765 (1080) *809 814* 843 (2094) 844 (110 194) 848 (196) *873 875*
Sprechler M 671 (104) *680* 741 (1081) *814* 841 (196) *876*
Spring F S 556 (37 38) *579*
Springer J J 635 (85) *657*
Sprott W E 453 (419 749) 454 (749) 475 (683 780) *509 516 518*
Squires W I 780 (970) *811*
Srere P A 576 (46) *579*
Stack Dunne M P 137 (294) *150* 219 221 (24) 222 (24) 224 (24) *231 233* 277 (57 89 370 371) 278 (57 370) 279 (88 370) 291 *298 299 306*
Stadie W C 181 (128) 185 (129 130) 186 (65 128 131 132 133) *197 198* 307 (372 373) *306* 7 6 (1083) 727 *814*
Staehelin D 291 (391) *307* 765 (1149) *816*
Stafford R O 643 (46 240) *678 683*
Stahl I 575 (135) 576 (135) *589*
Stalker H D 74 (115) *86*
Stamler J 716 (1089) 717 (871) *808 814*
Stanbury J B 441 (800) 449 (684) 469 (235 801) *505 516 519* 828 (198) 837 (198 199) 838 (198) *875*
Stand R 695 (1090) *814*
Stanley M M 440 (37) 441 (685) 449 (685) 465 (686 687) *500 516* 829 (9 200) 830 (9) 870 *875*
Staple I 649 (175) *660*
Staple T W 745 (798) *806*
Stare F J 187 (134) *198*
Starling W W 546 (250) *585*
Starr I 435 (558) 457 (558) 458 (688) 467 (105) *50 513 516*
Stasilli N R 459 (646) *516*
Statland H 459 (689) 495 (690) *516*
Staub A 190 191 *198* 239 (134) *300*

Stavraky G W 129 132 (145 146) *147*
Stead E A Jr 395 (149) 406 (149) *409* 491 (641) 493 (641) *515*
Stears J C 619 (139) *681*
Stebbins R B 278 (416) 281 (416) 295 (374) *306* *308* 665 (961) *684* 701 (1092 1218) 739 (806) *807* *814* *818*
Stecher R M 605 (310) 606 (310) *665*
Steelman S I 207 208 210 211 214 216 218 219 *231* *233* 457 (165) *504* 833 (49) *871*
Steen A S 769 (1093) *814*
Steeves F A 22 (276) *55*
Steeves T 3 (276) 29 (263) *54* *55*
Stefanini M 742 (1094) *814*
Steffensen I H 768 (1043 1095 1096) *813* *814*
Steger K 479 (691) *516*
*Stegner D 77 (473) 93*
Stehle R I 395 (192) 401 421 422 (189 191) 423 *430*
Steidle H 81 (474) *93*
Stein I F 856 (201 202) 857 (202) *875*
Stein J D Jr 251 (345) *306*
Stein K F 745 (1098) 747 (1097 1098) *814*
Stein M 331 (200a) *384*
Stein O 748 (317) *793*
Stein R H 766 (898) *807*
Stein Y 748 (317) *793*
Steiner A 486 (692) *516*
Steinback H L 847 (203) 852 (203) *8 5*
Steinbeck A W 288 (19) 292 (19) 293 (19) 294 (19) *297*
Steinberg H 840 (29) *870*
Steinmetz C G 768 (689) 779 (456) *797* *803*
Stephens C A L 727 (111 1099) 755 (104) *787* *814*
Stephens G C 62 (85 522) 63 (475) 66 (476) *85* *93* *94*
Stephens G J 64 (477) 66 (477) *93*
Stephenson F F M 540 (10) *578*
Stepto R C 740 (1100 1101) 766 (879) *809* *814* *815*
Sterling C 29 (277) *55*
Stern T N 733 (1254) *819*
Sternberg W H 342 371 *385* 856 (204) *875*
Sterne G D 458 (559) *513*
Sternheimer R 483 (693) *516*
Stetten D Jr 188 *198* 254 (376) *307* 482 (335) *507* 732 (68) *786*
Stevens A R 748 (1141) *816*
Stevens H 208 (94) *233*
Stevenson J A F 719 (947) *811*
Stevenson R 556 (371) *579*
Steward F C 23 (278) *55*
Stewart G N 715 (9 5) *810*
Stewart G S 157 16 *173*
Stewart H J 493 (694 695) *516*
Stewart J 695 (185) *789*
Stewart L H 861 (205) *875*
Stewart W C 397 *430*
Stickney J M 751 (1102) *815*
Stidworthy G 618 (157) 619 (157) *659*
Stier R A 743 (1103) *815*
Stimmel B F 599 647 (293) *663* 667 (241) *683*
Stitch S R 667 (124) *680*
Stobaugh R E 563 (238) *585*
Stock C C 699 (611) 751 (1113) 752 (119 1113) *788* *801* *815*
Stodola F H 560 (316) *584*
Stoerk H C 154 (136) 161 (27 135) *170* *173* 700 (1219) 715 (646 647) 716 (647) 723 (1219) 734 (1109) 735 (1219) 743 (1105) 750 (1219) 751 (1107) 752 (1219) 753 761 (1110) 770 (1106) 771 (646) 774 (58 325 1110) 777 (1055) *787* *792* *793* *809* *813* *815* *818*
Stokem M 609 (99) 610 (99) 611 (99) 616 (99) 633 (99) 635 (99) 636 (99) *657*
Stokes A M 190 (89) *197*
Stokes W M 535 536 *583* *587*
Stoll R 438 (69 70) *501*
Stoll W A 136 *150*
Stone D 625 626 (292) 628 (259 292) 640 (250) *662* *663*
Stone R E *745 (1074) 751 (1075) 814*
Storaasli J P 826 (102) *872*
Stork G 506 (73 267) *580* *586*
Stormont R T 293 (377) *307*
Stover J H Jr 117 (261) *150*
Stowe B B 2 (281) 6 (280 281) 7 (281) 29 (279) *55*
Strandh J R E 163 (50) *171*

Straube R I 672 (159) *682* 750 (1111) *815*
Strauss F 415 (71) *501*
Strauss M B 757 (1062) *814*
Strauss R 863 (212) *875*
Straw, R F *659* 667 (135) *681*
Street H I 23 *55*
Streeten D H P 698 (1151) 696 (1151) 697 (1151) 704 (1151) 721 (1151) 727 (1151) 745 (1148) 765 (1151) 766 (1151) 768 (1151) 777 (1151) *816*
Stricker P 347 351 *380*
Strickroot F L 271 (334) *305*
Striganova A R 164 *173*
Strisower B 485 (276) *505*
Strisower E H 157 (34) *196*
Stroud S W 599 *663*
Studer A 674 (242) *683*
Studer P I 635 (171) *660*
Stumpe W M 763 (21) *785*
Sturgis S H 670 (95) *680* 857 (206) 859 (206) *875*
Sturkie P D 452 (696) *516* 672 (243) *683*
Sturm F 752 (803 804) *807*
Sturtevant F M Jr 674 (244) *683*
Stutinsky F 60 (478) 68 (479) 75 (480) 81 (480) *93* 179 (262) *160*
Sudak F N 713 (1245) 716 (1245) *819*
Sue P 435 (388) *509*
Suguira K 751 (1112 1113) 752 (1113) *815*
Suiter I F 729 (728) *804*
Sullivan J 462 (38) *500*
Sulman F G 295 (278) *307* 423 (195) *480*
Sulzberger M B 754 (1114 1115) *815*
Summers C H R 544 (304 305) 558 (45 273) 559 (45) *579* *582* *587*
Summer A K 692 (1014) *812* 848 (182) *874*
Sunderman F W 831 (207) *875*
Suneson S 62 (119) *86*
Sung C 496 (697) *516*
Suomalainen P 81 (481 482) *93*
Sur R N 783 (350) *811*
Surányi I A 544 (42) *579*
Suranyl I 620 (30) *656*
Sure B 481 (699) 489 (698) *516* 781 (1116) *815*
Sussmann I N 747 (556) *809*
Sutherland L W 191 (142) 192 (144) 194 *198* 723 *815*
Sutherland K 740 (1101) 766 (879) *809* *815*
Sutton I I 851 (231) *876*
Sutton R F 583 (340) *588*
Svedberg T 448 (296) *507*
Sverdrup A 209 (79) *231*
Swahn B 486 (495) *507*
Swan J M 99 (180) *150* 404 (211) 409 (211 211a) *431*
Swanberg H 670 (3) *677*
Swann H G 690 (1118) *815*
Swanson C I 35 (284) *55*
Swanson I W 494 (700) *516*
Swartz I B 777 (717) *804*
Sweat M I 288 (379) *307* 600 616 617 (266) 645 (295) 648 (266 296) *662* *663* 694 (119) 696 (119) *815*
Swedin B 153 (95) *172*
Sweeney M J 243 (122) 248 (122) *300*
Swenson R F 442 (70) *516*
Swett W A 29 (264) *54*
Swift M N 672 (189) *682*
Swingle W W 694 (1121) 698 (1120 1121) 708 (1122 1123) 712 (1124) 714 (1122 1123 1126) 722 (1125) *815*
Swinyard C A 282 (61 62) *298*
Sydnor K I 285 (380 381) 287 (382) *307* 843 (208a) 850 (208) *875*
Sykes J F 366 *396* 667 (29) *678*
Syverton J T 753 (1198) 760 (1127) 779 (418) *796* *816* *817*
Szego C M 242 (383) 254 (383) *307* 669 (245) 670 (24 25 245) 676 (245) *678* *683* 728 (1203) 733 784 (1128) *815* *818*
Szerb J 720 (587) *801*
Szilagyi D F 781 (1130) *815* 836 (209) *875*
Szmuszkovicz J 539 (173) *585*

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Tagmann I 546 (257) *585* 601 (244) 610 615 *662* *663*
Tagnon R F 467 (184) *518*
Tainter M I 113 *150*
Tait J F 572 (121 307 308 308a) 573 (308a) 574 (307 308a) *582* *587* 672

(280 287) *663* 687 (1029) 707 (1026) 707 (444 1131) *797 813 815*
Takaoka M 72 (483) 80 (483) *93*
Takemura K H 552 (65) *580*
Takewaki K 61 (484) 64 (485 486) 66 (486) 67 (484) *93*
Taki I 79 (487) *93*
Tala P 208 *233*
Talalay I 673 (246) *683* 754 (1132) *815*
Talbot J 720 (284) *79_*
Talbot N B 154 (32) 155 162 (32 138) *170 173* 667 (247) *683* 713 (703) *804* 841 (126) *873*
Talbot T R 751 (898) *808*
Talbott J H 766 (828) *807*
Talesnik J 128 (163) *148*
Talmage R V 154 (41) 155 157 158 165 *172 173* 352 *383* 468 (702) *516* 710 711 (362) *794*
Tamaoki B 668 (121) 673 (122) *680*
Tambiah M S ? (18) 3 (18) 6 (18) *48*
Tang P C 8? (384) *307*
Tang Y W 3 28 (285) 29 (285) *55*
Tanzig S 47 *55*
Tarail R 05 (901) *790*
Tarr R *79?*
Tata J 454 (601) *514*
Tateno I 779 (632) *80*
Tatum A I 492 (703) *516*
Taub D 537 (346) 538 (346) *588*
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Taubenhaus M 704 (1135) 761 (1133) 76? (1013 1134 1135) *81? 815 816*
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Taylor A 434 (715) *517*
Taylor A B 843 (709a) *875*
Taylor B 704 (1135) 762 (1013 1135) *81 816*
Taylor D A H 551 (43) *579*
Taylor D B 119 (?64) 122 (?64) *150*
Taylor F H 484 (411) *509*
Taylor F N L 741 (454) *797*
Taylor H C 861 (48) *871*
Taylor H M 756 (949) *811*
Taylor J G 395 (11) *495*
Taylor J W 767 (628) *80*
Taylor N B G 116 (?08) *149*
Taylor N R W 286 (385) *307*
Taylor S I Jr 404 (176) 405 (176) *4?9 430*
Tchernoff G 12 (137 e f) 14 (137 e f) *51*
Teel H M 236 (386) 246 (387 388) 249 (388) 250 (387) 270 *307*
Teich S 545 (318) *587* 590 (173) 592 (173) 633 (173) *660*
Teicher R 754 (1136) *816*
Teilum G 7?? (1137) *816* 865 (910) *875*
Telfer M E 670 (248) *683*
Telfer W H 75 (488–49?) *93 94*
Telkka A 491 (367) *508* 782 (661) *803*
Templeton R D 157 (152 153) *174*
Teng C 186 (145) *198* 7?7 (1138) *816*
Teng C T 186 (104 145) *198*
ten Holt S P 843 (24) *8 0*
Tennenbaum M 106 (198) *148*
Tenney A 449 (21) *500* 782 (15) *785*
Teodoru C V 754 (25) *785*
Tepperman H M 166 (93) 168 *172 1 3* 735 736 *816*
Tepperman J 249 (90) 288 (589) *299 307* 479 (567) 495 (716) *513 517* 735 736 *816*
Terepka A R 154 (90) 164 (90) *17?*
Terpstra J 830 (19) *8 0*
Terpstra W 2 (286) 5 7 (286) *55*
Terroux K G 1 6 (48) *145*
Terry L L 766 (1140) *816*
Terry M L 154 (32) 155 (3?) 162 (3 ) *170*
Ter haakovec G A 761 (785) 763 (785) *806*
Tesar W C 247 (264) *304*
Teubner F G 5 (288) 33 (287) 42 (287) *55*
Tew M B 856 ( 11) 864 (211) *875*
Tewkesbury I B 443 (331) *507*
Teys eyre J 672 (233) *683*
Thalhimer W 747 (1064) *814*
Thatcher J S 164 (147) *173*
Thayer S A 546 (34) *579* 594 (68 148 176 318) 598 (19 215) *655 656 659 660 661 663 664*

Theis B M 481 (699) *516* 781 (1116) *815*
Therien M, 741 (766) *792*
Thibault C, *330 (57b) 380*
Thibault O 454 (370 371 717-720) *508 517*
Thiersch *J B* 748 (1141) *816*
Thimann K V *2* (281, 290) 4 (290 291) *6 (280 281 294 295) 7 (281 290 291)* 9 (291 293) 10 (291 293) 12 (290 291 293) *13 (291 293)* 15 (291 293 297) 16 (291) *20 (291a) 21* (105 106) 24 (298 299) 25 (293 297) 27 (295) *28 (289) 29 (279 290) 31 (171) 39* (60 111 112 112a 113 296 297a) 40 *(63 61) 41 (61-65) 43* 44 (113 295 301) 45 46 *(292 299 300) 48 49 50 59 55* 128 *150*
Thing E 137 *(266) 150* 418 423 *430*
Thomas R W, 608 (97a 98) 609 (97a) *642 (98) 65* 698 (550b) *794*
Thomas G B 205 (374) *306*
Thomas G H 554 (31) 556 (31) *579*
*Thomas J F* 87 (302) *65*
Thomas J R 758 (143) *788*
*Thomas J W* 495 (721) *517* 747 (131) *788*
*Thomas K L* 755 *(575) 800*
Thomas L 724 (711) 743 (1142) 753 (1143) 761 (1142) 771 (1144) 774 (777) 778 (777) *804 808 816*
Thomopoulos T 438 (722) *517*
*Thompson E C* 759 (1046) *813*
Thompson E O P 179 (116 117) 180 *198 227 253*
*Thompson J C* 485 (723) *517*
Thompson, J L 560 (203) *584*
Thompson J W 131 (292) *149*
Thompson L M 595 (127) 600 (127) *668*
*Thompson P K* 449 (724) 482 (725) *517*
Thompson R 863 (212) *875*
Thompson *R E 220* 222 (102) *283* 277 (390) 292 (390) 294 (390) *307*
Thompson W O 449 (724) 482 (725) *517*
Thomsen F 68 (493 497 498 502) 69 (496) 74 (491 496) 75 *94* 108 139 *150*
Thomsen M 68 *(499-502) 94* 108 139 (269) *150*
Thomson A P D 319 321 (1) 329 330 *379 386*
Thomson D L 232 (30) *99*
Thore S 79 (503) *94*
Thorborg J V 673 *(219) 683*
Thorn G W 283 (296 *392*) 291 (308 330 391) 293 (308 309) 294 (309) *305 307* 421 (84) 427 459 (204) 495 *(301) 504 607* 605 (86) 615 (86) *657* 666 (750) *685* 688 *(1151)* 696 *(472* 473 1151) 697 (74 341 1057 1150 1151) 704 (1151) 706 (571 1146) 707 (580) 713 (1057) 714 (472) 718 (441 522 *1150) 721 (1150 1151)* 724 (1151) 732 (341) 733 (74 341) 744 (902 1147) 745 (1145 1148 *1150*) 750 (443) 755 (455) 765 (1149 1151) 766 (1151) 768 (1151) 777 (1151) 781 *(316 513) 786 793 794 797 799 800 809 813* 840 (213) 841 (57 214) 842 (215) *871 875*
*Thorn J W* 284 (196) *302*
Thorn S W 845 (173) *874*
Thorogood I *193 199*
Thorp F 670 *(161) 681*
Thorsen R K 26 (811) *808*
Threlkeld A 439 (749) *517*
*Thresh R 2 (13) 48*
Thurlow J 31 (334) *48*
Thygeson I 779 (633) *807 816*
Tiengo M 397 *426*
Tierney N A 483 (726) *517*
*Tietze I* 176 (147 148) *199*
Timeris I 460 (74) *801*
*Tipton I H* 478 (728) *517*
Tipton S R 478 *(727 728) 517* 770 (1153) *816*
Tishkoff C H 443 (729) *517*
Tishler M 554 (47 294) 558 (331) 563 *(47) 571 (168) 579 583 586 587*
Tisdall F F 161 (61) *171*
Tiselius A 221 *(68) 232* 404 *409*
Tissières A 478 (730) *517*
Titman L C 22 (303) 47 *(303a) 56*
Titus F 104 (277) *150* 563 (249) *585* 608 *(232)* 620 (232) 642 (232) *661*
Toaff R 669 (276) *684*
Tobian L Jr 863 (210) *876*
Tobias C A 285 (130) 296 (130) *300* 697 (380) *795*

Tobias J M 77 (504) *94*
Toby C G 756 (827) *807*
Toennies G 178 (149) *199*
Toh C C 104 (120) 136 (48 120) *146 147*
Tokuyama I 218 (40) *232*
Toman J F P 119 (271) 125 (270 272) *150*
Tomasewski A J 540 (82 83) *581*
Tomich F G 490 (731) *517*
Tomkins G M 576 (46) *589*
Tompkins G 631 639 (298) 647 (298) 648 (298) 649 (298) *663*
Tompsett R 776 (684 685) 777 (683 685) *803*
Tong W 441 (714) 442 (714) 443 (711 713 732) 446 (713) 447 (734) 449 (712) *517*
Tonhazy N E 724 (14) 731 (1162 1163) *785 816*
Toolan H W 753 (1154) *816*
Toole E H 27 (36) *48*
Toole V K 27 (36) *48*
Toompas C A 741 (38) *785*
Topper A 837 (217) *876*
Torda C 720 (1155) *816*
Tornbean J A 671 (154) *681*
Tornblom N 154 (148) 162 *174*
Torrey J G 23 *55*
Tortoreto J N 285 (330) *305*
Touchstone J C 634 (248) 635 (248) 636 (248) *662*
Toverud S U 153 (112) 169 *173*
Towery T R 467 (784) *518*
Tracy A 672 (256) 674 (256) *683*
Traeger C T 696 (350) 766 (355) *794*
Traut H F 851 (16) *870*
Travis D F 66 (505) *94*
Treager H S 741 (1156) *816*
Trendelenburg U 133 (56) *145*
Trentin J J 347 *386*
Treshella P 843 (23) *870*
Trethewie E R 106 *150*
Treverrow V 450 (733) *517*
Trezzi F 47 *55*
Trikojus V M *517* 781 (749) *805*
Trimberger G W 331 354 (129) 335 (129) *382* 671 (106) *680*
Tripod J 707 (279) *791*
Trippett S 406 (171) 407 408 (171 210) *430 431*
Tristram G R 177 (150) *199*
Trotter W R 490 (266) *506*
Trounce J R 670 (216) *682*
Trunt A P 488 (172) *504*
Trubowitz S 745 (862) *808*
Trunnell J B 438 (735) *517* 839 (176) *874*
Tsatsas G 534 (258) *585*
Tschopp F 559 (279) *585* 698 (441 773) *797 806*
Tsui C 22 (242) *54*
Tubianan M 449 (736) 451 (736) *517*
Tudor Williams D P H 123 (168) 124 (168) *148*
Tuerkischer E 186 (151) *199* 756 (1157) *816*
Tui C 746 (1180) *817*
Tullner W W 672 (112) *680* 692 (506 507 977) 735 (977) 737 (234 978) 774 (505) *791 798 811*
Tully M E 550 (139) *589*
Tulsky A S 862 (118) *873*
Tum Suden C 241 (418) *308*
Tuppy H 179 (119 120) 180 *198* 404 412 *430*
Turchini J 60 (506) *94* 436 (737) *517*
Turfitt G E 620 *663*
Turiaf J 701 (1158) *816*
Turnbull J H 506 (39) *579*
Turnbull L 554 (294) *586*
Turner C D 58 (507) *94*
Turner C W 346 347 352 359 365 367 371 375 378 *379 384 386* 400 (200) *430* 445 (564 565) 449 (65) 450 (473) 452 (473) 453 (473) *501 511 513* 670 (205) *682*
Turner D L 528 (218 219) 529 (219) 561 (217 218 219) *584*
Turner H H 271 (393) *307* 853 (218) *876*
Turner J C 740 (555 556) *800*
Turner J F 40 (248) *54*
Turner J S 46 *55*
Turner L B 716 (1159) *816*
Turner R A 402 (201) 404 406 (51) *427 431*
Turner R B 575 (259) 595 (390) 599

(322) 561 565 (321) 571 (321 322) 574 (319) 575 (319) *585 587*
Turner T B 780 (1160) *816*
Tusini G 672 (2) 674 (2) *677*
Tuthill C H 278 (127 128) 281 (127) *300* 665 (94), *680*, 701 (378) *795*
Tuttle A H 439 (749) *517*
Tverdy G 192 (44) *196*
Twarog B M 100 (276) 101 (276) 105 120 (275) 126 (274 275) 133 (275) 135 *150*
Tweedy W R 154 (150) 155 157 161 (149 150) 169 (79) *172 174*
Twombly G A 868 (219) *876*
Twombly G H 597 598 (301 302) *655 668*
Tyler C 416 *427*
Tyler F H 284 (327) 286 (267 394) *287* (329) *304 305 307* 694 (815) 696 (815) 726 (115) 730 (962) 745 (957), *788 807 811* 843 (208a) 845 (156 178) *874 875*
Tyner D A 539 (342) *588*
Tyree E B 672 (180) *682* 759 (1111) *815*
Tyslowitz R 462 (38) *500*

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Udenfriend S 104 (277) *150 176 199*
Ueberwasser H 539 (339) *588*
Uhlenhuth E 491 (738 739) *517*
Ulberg L C 671 (251 251a) 673 (55) *679 683*
Ullrick W C 430 (740) *517*
Ulrich F 250 (395 396) *307*
Ulhafer P R 528 (218 219) 579 (219) 561 (218 219) *584*
Ulstrom R A 745 (447) *797*
Umbreit W W 724 (14 1161) 731 (1161–1163) *785 816*
Umeya T 58 (508) *94*
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